Three-way entanglement and three-qubit phase gate based on a coherent six-level atomic system

Chao Hang, Yun Li, Lei Ma,* and Guoxiang Huang†

Department of Physics, East China Normal University, Shanghai 200062, China

(Received 26 February 2006; revised manuscript received 16 March 2006; published 28 July 2006)

We analyze the nonlinear optical response of a six-level atomic system under a configuration of electromagnetically induced transparency. We show that the enhanced completely cross fifth-order nonlinearity generated in such a system can be used to produce efficient three-way entanglement and to realize a three-qubit quantum phase gate. We demonstrate that such phase gate can be transferred to an all-optical Toffoli gate, facilitating practical applications in quantum information and computation.

DOI: 10.1103/PhysRevA.74.012319 PACS number(s): 03.67.Lx, 42.65.−k, 42.50.Gy

I. INTRODUCTION

Photons are considered as promising candidates for carrying quantum information because of their high propagating speed and negligible decoherence [1]. Many proposals have come up for efficiently implementing all-optical quantum information processing and quantum computation, some of which are based on linear optics, and others are considered from nonlinear optical processes. As is well known, Kerr nonlinearity is crucial for producing photon-photon entanglement and for realizing two-qubit optical quantum gates. Similarly, higher-order optical nonlinearities can be used to produce an n-way ($n \geq 3$) entanglement and to realize a multiqubit quantum gate. However, optical quantum gates cannot be efficiently implemented based on conventional optical media. The reason is that either the optical nonlinearity produced in such media is very weak, or there is a very large optical absorption when working near resonant regime where nonlinear effect is strong.

In recent years, much attention has been paid to the study of electromagnetically induced transparency (EIT) in resonant atomic systems [2,3]. By means of the effect of quantum coherence and interference induced by a control field, the absorption of a weak probe field tuned to a strong one-photon resonance can be largely cancelled and hence an initially highly opaque optical medium can become transparent. The wave propagation in resonant optical media with EIT configuration possesses many striking features. One of them is the significant reduction of the group velocity of probe pulse. The other one is the giant enhancement of the Kerr nonlinearity [4,5]. In recent years, the physical systems for obtaining enhanced Kerr nonlinearity and a related large cross-phase modulation (CPM) by using EIT effect have been proposed, including "N" configuration [6,7], chain-Λ configuration [8], tripod configuration [9], and symmetric six-level configuration [10]. Based on the enhanced Kerr nonlinearity, two-qubit entanglement with photons and atoms [11–16] has been investigated and an all-optical two-qubit quantum phase gate (QPG) [17–20] has also been constructed recently by using different schemes. However, as far as we know, up to now only a few works [21] have explored higher-order (especially the fifth-order) optical nonlinearity and its applications to multiphoton entanglement and optical phase gates based on EIT effect.

In this work, we shall investigate a possible three-way entanglement and three-qubit phase gates based on a coherent six-level atomic system under an asymmetric EIT configuration. We first show that, due to the quantum interference induced by two strong control laser fields, the completely cross fifth-order optical susceptibilities [22] of the system can be greatly enhanced and at the same time the linear and other nonlinear susceptibilities are largely suppressed. This important property can be used to produce efficient three-way entanglement among three weak optical (i.e., probe, signal, and trigger) fields. We then explore the possibility of employing the enhanced CPM effect to devise a mechanism of polarization three-qubit quantum phase gate (QPG). The three-qubit QPG proposed here is rather robust and can be easily transferred to a universal three-qubit Toffoli gate. Notice that a Toffli gate can be constructed by some more basic quantum gates, but its physical realization in a compact way is needed to dramatically reduce the number of qubit and manipulations that are required to perform a given task. Although some studies of constructing Toffoli gates using different schemes exist [23–26], to the best of our knowledge the work presented here is a suggestion for a practical realization of a Toffoli gate in an all-optical way.

II. MODEL

We start with considering a lifetime broadened atomic system, where atoms with six levels [three lower (L) state levels $|1\rangle$, $|3\rangle$, $|5\rangle$, and three upper (U) state levels $|2\rangle$, $|4\rangle$, $|6\rangle$] interact with five laser fields (see Fig. 1). Such configuration can be realized in Zeeman-split alkali atoms by applying an external magnetic field B to split the degenerate levels in the lower and upper states. The lower (upper) state levels have the quantum number of total angular momentum $F=1$ ($F=2$) and different magnetic quantum numbers $m=-1,0,1$ ($m=0,1,2$) (e.g., the D_1 line of ^{87}Rb). The Zeeman shift of the sublevels in the lower and upper state is given by $\Delta_{L,U}=(\mu_B/\hbar)\Delta m g_{L,U}B$, where μ_B is the Bohr magneton and g_L (g_U) is the gyromagnetic factor of the lower (upper) levels. We assume that the transitions from $|2\rangle \leftrightarrow |3\rangle$ and $|4\rangle \leftrightarrow |5\rangle$ are driven by two strong, continuous-wave (cw)

*Electronic address: lma@phy.ecnu.edu.cn

†Electronic address: gxhuang@phy.ecnu.edu.cn

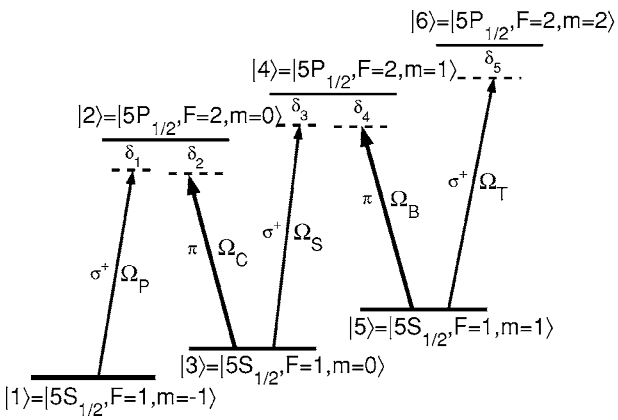

FIG. 1. The energy-level diagram and excitation scheme of a lifetime broadened six-level atomic system interacting with two strong, cw control fields of Rabi frequencies Ω_C and Ω_B, and three weak, pulsed (probe, signal, and trigger) fields of Rabi frequencies Ω_P, Ω_S, and Ω_T.

control laser fields of π polarization and with Rabi frequencies Ω_C and Ω_B, respectively. The transitions from $|1\rangle \leftrightarrow |2\rangle$, $|3\rangle \leftrightarrow |4\rangle$, and $|5\rangle \leftrightarrow |6\rangle$ are driven by three weak, pulsed laser fields of σ^+ polarization, called probe (with Rabi frequency Ω_P), signal (with Rabi frequency Ω_S), and trigger (with Rabi frequency Ω_T), respectively. Here the Rabi frequencies associated with the laser fields that drive the atomic transitions are defined as $\Omega_k = -D_{ij}\mathcal{E}_k/\hbar$, where $\mathcal{E}_k$ denotes the kth electric field envelope and D_{ij} is the relevant electric-dipole matrix element related to the transition $|i\rangle \leftrightarrow |j\rangle$. The detunings δ_i ($i=1$ to 6) are defined as $\delta_1 = (E_2-E_1)/\hbar - \omega_P$, $\delta_2 = (E_2-E_3)/\hbar - \omega_C$, $\delta_3 = (E_4-E_3)/\hbar - \omega_S$, $\delta_4 = (E_4-E_5)/\hbar - \omega_B$, and $\delta_5 = (E_6-E_5)/\hbar - \omega_T$, where E_i is the energy of the level $|i\rangle$ and ω_j ($j=P,C,S,B$, and T) is the frequency of the laser field with the Rabi frequency Ω_j. The evolution equations for the atomic probability amplitudes $a_j(t)$ ($j=1$ to 6) read

$$\dot{a}_1 = -\frac{\Gamma_1}{2}a_1 - i\Omega_P^* a_2, \tag{1a}$$

$$\dot{a}_2 = -\left(\frac{\Gamma_2}{2} + i\delta_1\right)a_2 - i\Omega_P a_1 - i\Omega_C a_3, \tag{1b}$$

$$\dot{a}_3 = -\left(\frac{\Gamma_3}{2} + i\delta_{12}\right)a_3 - i\Omega_C^* a_2 - i\Omega_S^* a_4, \tag{1c}$$

$$\dot{a}_4 = -\left(\frac{\Gamma_4}{2} + i\delta_{13}\right)a_4 - i\Omega_S a_3 - i\Omega_B a_5, \tag{1d}$$

$$\dot{a}_5 = -\left(\frac{\Gamma_5}{2} + i\delta_{14}\right)a_5 - i\Omega_B^* a_4 - i\Omega_T^* a_6, \tag{1e}$$

$$\dot{a}_6 = -\left(\frac{\Gamma_6}{2} + i\delta_{15}\right)a_6 - i\Omega_T a_5, \tag{1f}$$

where $\delta_{12} = \delta_1 - \delta_2$, $\delta_{13} = \delta_{12} + \delta_3$, $\delta_{14} = \delta_{13} - \delta_4$, and $\delta_{15} = \delta_{14} + \delta_5$. Γ_j denotes the decay rate of the state $|j\rangle$, introduced in a phenomenological manner for describing the effects of both spontaneous emission and dephasing [27].

III. ENHANCED COMPLETELY CROSS FIFTH-ORDER NONLINEARITY AND GROUP-VELOCITY MATCHING

We assume that the initial state of the system is in the ground state $|1\rangle$. Notice that if the intensity of the probe, signal, and trigger fields is much weaker than the intensity of the both control fields, due to the quantum coherence and interference (i.e., EIT) effect the depletion of the ground state $|1\rangle$ is not significant and hence one has $a_0 \approx 1$. This means that the pumping effect among different ground-state levels can be neglected during the evolution of the system because the population of atoms will remain nearly in the state $|1\rangle$. For solving Eq. (1) we assume that the typical temporal duration of the probe, signal, and trigger is long enough so that a steady state approximation can be employed. In order to get the expressions for nonlinear optical susceptibilities of the system we consider higher order contributions to a_0, which can be obtained by using the normalization condition $\Sigma_{i=1}^6 |a_i|^2 = 1$ [28]. We solve Eq. (1) under this condition and obtain the following expressions for the susceptibilities of three weak fields:

$$\begin{aligned}\chi_P &= -\frac{N_a|D_{12}|^2}{\hbar\epsilon_0\Omega_P}a_2a_1^* \\ &\simeq \chi_P^{(1)} + \chi_{PP}^{(3)}|\mathcal{E}_P|^2 + \chi_{PS}^{(3)}|\mathcal{E}_S|^2 + \chi_{PT}^{(3)}|\mathcal{E}_T|^2 + \chi_{PPP}^{(5)}|\mathcal{E}_P|^4 \\ &\quad + \chi_{PPS}^{(5)}|\mathcal{E}_P|^2|\mathcal{E}_S|^2 + \chi_{PPT}^{(5)}|\mathcal{E}_P|^2|\mathcal{E}_T|^2 + \chi_{PST}^{(5)}|\mathcal{E}_S|^2|\mathcal{E}_T|^2,\end{aligned} \tag{2a}$$

$$\chi_S = -\frac{N_a|D_{34}|^2}{\hbar\epsilon_0\Omega_S}a_4a_3^* \simeq \chi_{SP}^{(3)}|\mathcal{E}_P|^2 + \chi_{SPP}^{(5)}|\mathcal{E}_P|^4 + \chi_{SPT}^{(5)}|\mathcal{E}_P|^2|\mathcal{E}_T|^2, \tag{2b}$$

$$\chi_T = -\frac{N_a|D_{56}|^2}{\hbar\epsilon_0\Omega_T}a_6a_5^* \simeq \chi_{TPS}^{(5)}|\mathcal{E}_P|^2|\mathcal{E}_S|^2, \tag{2c}$$

where N_a is atomic density of the system and ϵ_0 is the vacuum dielectric constant. The explicit expressions of linear ($\chi^{(1)}$), third-order ($\chi^{(3)}$) and fifth-order susceptibilities ($\chi^{(5)}$) on the right-hand side of Eq. (2) have been given in the Appendix.

Based on the above result we now show that the susceptibilities associated with completely cross fifth-order nonlinearities can be greatly enhanced under EIT condition. This can be easily seen from the explicit expressions of $\chi^{(j)}$ ($j=1,3,5$) provided in the Appendix. It is obvious that, if we take $\delta_{12} = \delta_{14} = 0$, d_3 and d_5 depend only on Γ_3 and Γ_5, which describe spontaneous emission and dephasing processes of the states $|3\rangle$ and $|5\rangle$. Notice that such processes can be greatly suppressed if working with ultracold atoms. In this case the values of Γ_3 and Γ_5 can be made small (usually only few kHz or even smaller) [29]. Under such conditions the completely cross fifth-order susceptibilities, i.e., $\chi_{PST}^{(5)}$, $\chi_{SPT}^{(5)}$, and $\chi^{(5)}_{TPS}$, given, respectively, by (A8), (A11), and (A12), have giant values while other susceptibilities (including the linear, third- and fifth-order ones) are efficiently suppressed. Thus the system produces *only* completely cross fifth-order CPM effect among the probe, signal, and trigger fields. In addition, the imaginary parts of the enhanced completely cross susceptibilities are much smaller than their relevant real parts if one chooses $\delta_{15} \gg \Gamma_6/2$ and hence the absorption of the probe, signal, and trigger fields can be largely cancelled. The physical reason for the enhancement of the completely cross fifth-order susceptibilities, suppression of other susceptibilities and the cancellation of the absorption is due to the EIT condition [i.e., $|\Omega_i|^2$ $(i=P,S,T) \ll |\Omega_j|^2$ $(j=B,C)$, $\delta_{12}=\delta_{14}=0$] that results in quantum coherence and interference between the lower state levels.

Next we present the expressions of group velocities of the probe, signal, and trigger fields. The group velocity of a light pulse is given by $v_g = c/(1+n_g)$, where c is the light speed in vacuum and

$$n_g = \frac{1}{2}\,\mathrm{Re}(\chi) + \frac{\omega}{2}\frac{\partial\,\mathrm{Re}(\chi)}{\partial\omega} \tag{3}$$

is the group index. As we know, the group velocities of the probe, signal, and trigger must be small and comparable in order to achieve an effective CPM effect [30]. Unlike the six-level scheme used in Ref. [10], our present scheme is asymmetric and hence the group velocities of the probe, signal, and trigger fields are generally not equal. We assume the probe and signal fields work at the center of the EIT window, i.e., $\delta_{12}=\delta_{14}=0$, and neglect the dephasing rates Γ_1, Γ_3, and Γ_5, which are typically much smaller than all the other parameters. Under the EIT condition, we obtain the explicit expressions of the group velocities from Eqs. (2) and (A1)–(A12) for the probe, signal, and trigger fields,

$$v_g^P \simeq \frac{2\hbar\,\epsilon_0 c|\Omega_C|^2|\Omega_B|^2}{N_a|D_{12}|^2\omega_P\left(|\Omega_B|^2 - |\Omega_P|^2\dfrac{|\Omega_B|^2}{|\Omega_C|^2} + |\Omega_S|^2 + B + |\Omega_S|^2|\Omega_T|^2\beta\right)}, \tag{4a}$$

$$v_g^S \simeq \frac{2\hbar\,\epsilon_0 c|\Omega_C|^2|\Omega_B|^2}{N_a|D_{34}|^2\omega_S|\Omega_P|^2(1+|\Omega_T|^2\beta)}, \tag{4b}$$

$$v_g^T \simeq \frac{2\hbar\,\epsilon_0 c|\Omega_C|^2|\Omega_B|^2}{N_a|D_{56}|^2\omega_T|\Omega_P|^2|\Omega_S|^2\beta}, \tag{4c}$$

with

$$B = \left(1 - 3\frac{|\Omega_P|^2}{|\Omega_C|^2} - \frac{|\Omega_S|^2}{|\Omega_B|^2}\right)|\Omega_T|^2\tilde{\beta} - \frac{|\Omega_S|^2|\Omega_T|^2}{|\Omega_C|^2}\frac{\delta_1\delta_5 + \Gamma^2/4}{\delta_5^2 + \Gamma^2/4},$$

$$\tilde{\beta} = \frac{\delta_3\delta_5 + \Gamma^2/4}{\delta_5^2 + \Gamma^2/4}, \quad \beta = \frac{\delta_5^2 - \Gamma^2/4}{\delta_5^2 + \Gamma^2/4}{}_2. \tag{5}$$

For simplicity when obtaining the above results we have set $\Gamma_2=\Gamma_4=\Gamma_6=\Gamma$. We note that three velocities v_g^P, v_g^S, and v_g^T can be made both small and equal by properly adjusting the Rabi frequencies and detunings (see the example given below).

IV. THREE-WAY ENTANGLEMENT AND THREE-QUBIT PHASE GATE

We know that a significant three-body interaction is a key ingredient for the production of three-way entanglement and the construction of three-qubit QPG. In this section we demonstrate that such interaction can be realized in our system in terms of the giant completely cross fifth-order phase modulation induced by the EIT effect, by which an optical field acquires a large phase shift conditional to the state of the other two optical fields. A three-qubit QPG can be represented by the input-output relation $|\alpha\rangle_P|\beta\rangle_S|\gamma\rangle_T \to \exp(i\phi_{\alpha\beta\gamma})|\alpha\rangle_P|\beta\rangle_S|\gamma\rangle_T$, where $\alpha,\beta,\gamma=0,1$ denote three-qubit basis.

We choose two orthogonal light polarizations $|\sigma^-\rangle$ and $|\sigma^+\rangle$ to encode binary information for each qubit. We assume that the six-level system shown in Fig. 1 is implemented only when the probe, signal, and trigger have σ^+ polarization, which can be easily realized by choosing suitable atomic levels. For a σ^- polarized probe there is no sufficiently close excited state to which level $|1\rangle$ couples and no population in $|3\rangle$ and $|5\rangle$ to drive the signal and trigger transitions. So the probe, signal, and trigger only acquire the trivial vacuum phase shift $\phi_0^i = k_i L$ $(i=P,S,T;$ L denotes the length of the medium). When the probe and signal are σ^+ and σ^- polarized, the probe, subject to the EIT produced by the $|1\rangle$-$|2\rangle$-$|3\rangle$ levels Λ configuration, experiences a self-Kerr effect and acquires a nontrivial phase shift ϕ_Λ^P, while the signal and trigger acquire again the vacuum shifts ϕ_0^S and ϕ_0^T. For a σ^+, σ^+, and σ^- polarized probe, signal, and trigger, the first two fields will experience a cross-Kerr effect and acquire nontrivial phase shifts ϕ_M^P and ϕ_M^T, while the last acquire still the vacuum shift ϕ_0^T. Only when all three fields have the "right" polarization, they all acquire nontrivial phase shifts denoted by $\phi_{\rm all}^P$, $\phi_{\rm all}^S$, and $\phi_{\rm all}^T$.

Assuming that the input probe, signal, and trigger polarized single photon wave packets can be expressed as a superposition of the circularly polarized states [17–19], i.e., $|\psi_i\rangle = 1/\sqrt{2}|\sigma^-\rangle_i + 1/\sqrt{2}|\sigma^+\rangle_i$ $(i=P,S,T)$, where $|\sigma^\pm\rangle_i = \int d\omega \xi_i(\omega) a_\pm^\dagger(\omega)|0\rangle$ with $\xi_i(\omega)$ being a Gaussian frequency distribution of incident wave packets centered at frequency ω_i. The photon field operators undergo a transformation while propagating through the atomic medium of length L, i.e., $a_\pm(\omega) \to a_\pm(\omega)\exp[i\omega/c\int_0^L dz n_\pm(\omega,z)]$. Assuming that $n_\pm(\omega,z)$ (the real part of the refractive index) varies slowly over the bandwidth of the wave packet centered at ω_i, one gets $|\sigma^\pm\rangle_i \to \exp(-i\phi_\pm^i)|\sigma^\pm\rangle_i$, with $\phi_\pm^i = \omega/c\int_0^L dz n_\pm(\omega_i,z)$. Thus, the truth table for a polarization three-qubit QPG using our configuration reads

$$|\sigma^-\rangle_P|\sigma^\pm\rangle_S|\sigma^\pm\rangle_T \to \exp[-i(\phi_0^P+\phi_0^S+\phi_0^T)]|\sigma^-\rangle_P|\sigma^\pm\rangle_S|\sigma^\pm\rangle_T, \tag{6a}$$

$$|\sigma^+\rangle_P|\sigma^-\rangle_S|\sigma^\pm\rangle_T \to \exp[-i(\phi_\Lambda^P+\phi_0^S+\phi_0^T)]|\sigma^+\rangle_P|\sigma^-\rangle_S|\sigma^\pm\rangle_T, \tag{6b}$$

$$|\sigma^+\rangle_P|\sigma^+\rangle_S|\sigma^-\rangle_T \to \exp[-i(\phi_{\rm M}^P+\phi_{\rm M}^S+\phi_0^T)]|\sigma^+\rangle_P|\sigma^+\rangle_S|\sigma^-\rangle_T, \tag{6c}$$

$$|\sigma^+\rangle_P|\sigma^+\rangle_S|\sigma^+\rangle_T \to \exp[-i(\phi_{\rm all}^P+\phi_{\rm all}^S+\phi_{\rm all}^T)]|\sigma^+\rangle_P|\sigma^+\rangle_S|\sigma^+\rangle_T, \tag{6d}$$

with $\phi_\Lambda^P = k_P L(1+2\pi\chi_P^{(1)}) + \phi_{PP} + \phi_{PPP}$, $\phi_{\rm M}^P = \phi_\Lambda^P + \phi_{PS} + \phi_{PPS}$, $\phi_{\rm M}^S = \phi_0^S + \phi_{SP} + \phi_{SPP}$, $\phi_{\rm all}^P = \phi_{\rm M}^P + \phi_{PT} + \phi_{PPT} + \phi_{PST}$, $\phi_{\rm all}^S = \phi_{\rm M}^S + \phi_{SPT}$, and $\phi_{\rm all}^T = \phi_0^T + \phi_{TPS}$. The completely cross phase shifts are given by

$$\phi_{PS} = k_P L \frac{\pi^{3/2}\hbar^2|\Omega_S|^2}{4|D_{34}|^2}\,{\rm Re}(\chi_{PS}^{(3)})\frac{{\rm erf}(\xi_{PS})}{\xi_{PS}}, \tag{7a}$$

$$\phi_{PT} = k_P L \frac{\pi^{3/2}\hbar^2|\Omega_T|^2}{4|D_{56}|^2}\,{\rm Re}(\chi_{PT}^{(3)})\frac{{\rm erf}(\xi_{PT})}{\xi_{PT}}, \tag{7b}$$

$$\phi_{SP} = k_S L \frac{\pi^{3/2}\hbar^2|\Omega_P|^2}{4|D_{12}|^2}\,{\rm Re}(\chi_{SP}^{(3)})\frac{{\rm erf}(\xi_{SP})}{\xi_{SP}}, \tag{7c}$$

$$\phi_{PST} = k_P L \frac{\pi^{3/2}\hbar^4|\Omega_S|^2|\Omega_T|^2}{4|D_{34}|^2|D_{56}|^2}\,{\rm Re}(\chi_{PST}^{(5)})\frac{{\rm erf}(\xi_{PST})}{\xi_{PST}}, \tag{7d}$$

$$\phi_{SPT} = k_S L \frac{\pi^{3/2}\hbar^4|\Omega_P|^2|\Omega_T|^2}{4|D_{12}|^2|D_{56}|^2}\,{\rm Re}(\chi_{SPT}^{(5)})\frac{{\rm erf}(\xi_{SPT})}{\xi_{SPT}}, \tag{7e}$$

$$\phi_{TPS} = k_T L \frac{\pi^{3/2}\hbar^4|\Omega_P|^2|\Omega_S|^2}{4|D_{12}|^2|D_{34}|^2}\,{\rm Re}(\chi_{TPS}^{(5)})\frac{{\rm erf}(\xi_{TPS})}{\xi_{TPS}}, \tag{7f}$$

where $\xi_{ij} = \sqrt{2}L(1-v_g^i/v_g^j)/(\tau_j v_g^i)$ and $\xi_{ijk} = \sqrt{2}L[(1-v_g^i/v_g^j)^2/\tau_j^2 v_g^{i2} + (1-v_g^i/v_g^k)^2/\tau_k^2 v_g^{i2}]^{1/2}$ $(i,j,k=P,S,T)$ with τ_i being the time duration of the pulse. If group velocity matching is satisfied, i.e., $\xi \to 0$, the ${\rm erf}(\xi)/\xi$ reaches the maximum value $2/\sqrt{\pi}$.

A three-way entanglement can be calculated by "residual entanglement," which indicates the amount of entanglement among the probe, signal, and trigger that cannot be accounted for by the entanglements of arbitrary two weak fields. We stress that the residual entanglement will be greatly suppressed if one of the fields is fully entangled with another. Therefore, an excessive interaction only between two weak fields, which is valued by $\chi_{ij}^{(3)}, \chi_{iij}^{(5)}$, and $\chi_{ijj}^{(5)}$, is very harmful to the residual entanglement. Fortunately, in our scheme we can suppress such interaction which favors the residual entanglement. As in Ref. [31], the residual entanglement for a three-qubit pure state can be written as follows:

$$\zeta_{PST} = \mathcal{C}_{P(ST)}^2 - \mathcal{C}_{PS}^2 - \mathcal{C}_{PT}^2 = 2(\lambda_1^{PS}\lambda_2^{PS} + \lambda_1^{PT}\lambda_2^{PT}), \tag{8}$$

where λ_1^{PS} and λ_2^{PS} are, respectively, the square roots of two eigenvalues of $\rho_{PS}\tilde{\rho}_{PS}$, while λ_1^{PT} and λ_2^{PT} are defined in a similar way. The reduced density matrix $\rho_{PS} = {\rm Tr}_T(\rho_{PST})$ with ρ_{PST} being the density matrix of the output state, and $\tilde{\rho}_{PS} = \sigma_y^P \otimes \sigma_y^S \rho_{PS}^* \sigma_y^P \otimes \sigma_y^S$ with σ_y being the y component of the Pauli matrix.

We now provide a practical system working with a ultracold ^{87}Rb atomic gas, in which Doppler effect can be made small. Atoms are confined in a magneto-optical trap, with the pertinent lower levels $(5S_{1/2}, F=1)$ and upper levels $(5P_{1/2}, F=2)$ (see Fig. 1). The Zeeman shift of the sublevels in the lower and upper level can be adjusted by the intensity of an applied magnetic field. After taking $B \approx 340$ G $(g_L = -\frac{1}{2}$ and $g_U = -\frac{1}{6})$, we obtain $\Delta_L \simeq 2\pi \times 3 \times 10^8$ s^{-1} and $\Delta_U \simeq 2\pi \times 10^8$ s^{-1}, which are smaller than the hyperfine splitting of the lower levels $(6.8 \times 10^9$ s$^{-1})$ and upper levels $(8.1 \times 10^8$ s$^{-1})$. We take $\delta_1 = \delta_2 = 2.5 \times 10^8$ s^{-1}, $\delta_3 = \delta_4 = -6.0 \times 10^8$ s^{-1} (a perfect EIT regime for the probe and signal for $\delta_{12} = \delta_{14} = 0$), $\delta_5 = 2.5 \times 10^7$ s^{-1} [32], $\Omega_C = \Omega_B = 5.0 \times 10^7$ s^{-1}, $\Omega_P = 1.6 \times 10^7$ s^{-1}, $\Omega_S = 2.0 \times 10^7$ s^{-1}, $\Omega_T = 1.3 \times 10^7$ s^{-1}, $\Gamma = 0.5 \times 10^7$ s^{-1}, and $N_a = 10^{12}$ cm^{-3}. With these parameters, the probe, signal, and trigger have a mean amplitude of about one photon when these beams are tightly focused and has a time duration about 1.0×10^{-7} s. The intensity of the probe, signal and trigger are $I_P = 8.4 \times 10^{-4}$ W cm^{-2}, $I_S = 1.3 \times 10^{-3}$ W cm^{-2} and $I_T = 5.5 \times 10^{-4}$ W cm^{-2}. We remark that the intensity of a single 800 nm photon per nanosecond on the area of 1 μm^2 is $I_{ph} = 2.5 \times 10^{-2}$ W cm^{-2}. This shows that our scheme makes nonlinear phase shifts of the order of π be possible with a single photon. Furthermore, the system remains only the completely cross fifth-order susceptibilities (i.e., only $\chi_{PST}^{(5)}$, $\chi_{SPT}^{(5)}$, and $\chi_{TPS}^{(5)}$ are nonvanishing), and the imaginary parts of these susceptibilities are one order of magnitude smaller than their real parts and hence play no significant role. By using the CPM effect, the probe, signal, and trigger fields can acquire nontrivial nonlinear phase shifts when all of them have "right" polarization. In addition, the group velocities of these three optical fields can also be nearly matched, with the values given by $v_g^P \simeq 6.3 \times 10^3$ m/s, $v_g^S \simeq 5.1 \times 10^3$ m/s, and $v_g^T \simeq 1.0 \times 10^4$ m/s, respectively. Based on these results we may obtain a three-

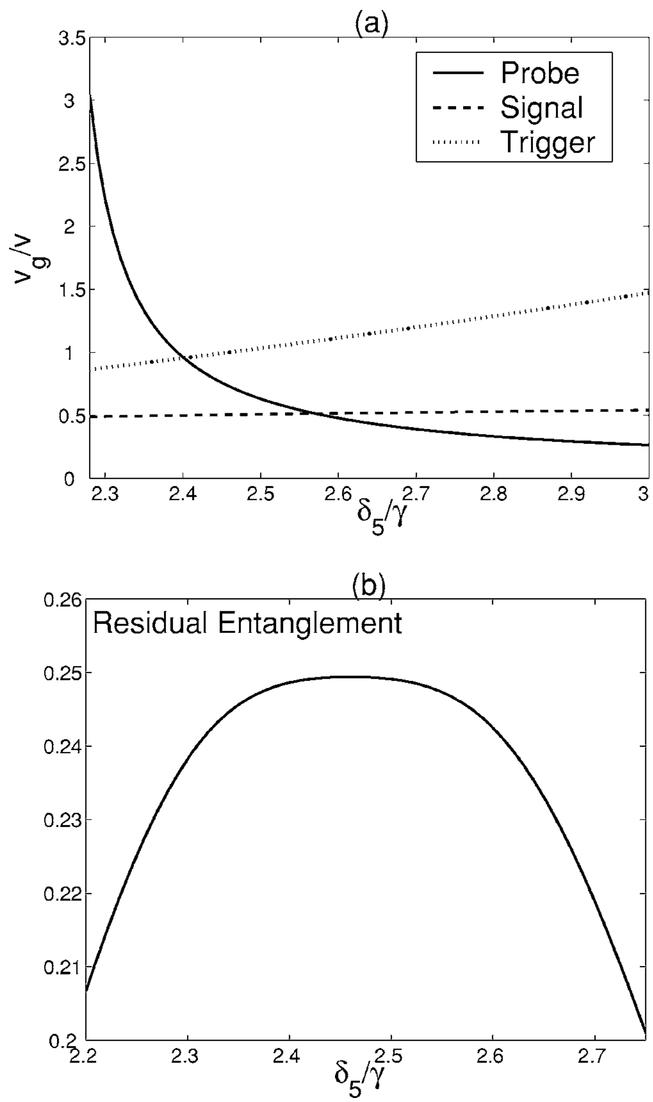

FIG. 2. (a) Dimensionless group velocity v_g^i/v of the probe, signal, and trigger fields versus dimensionless trigger detuning δ_5/γ for $v=10^4$ m/s. Solid line denotes the probe, dashed line denotes the signal, and dotted line denotes the trigger, respectively. (b): Residual entanglement versus δ_5/γ for $\gamma=10^7$ s^{-1}. The maximum occurs at $\delta_5/\gamma \simeq 2.5$ where the group velocity matching is satisfied. The parameters used in (a) and (b) are the same as in Sec. IV.

qubit phase gate, in which a total nonlinear phase shift up to π radians can be obtained for $L \simeq 0.22$ mm. By a detail calculation we find that the residual entanglement ζ_{PST} of the gate can be as large as 25% (the maximum of the residual entanglement).

Shown in Fig. 2(a) is the dimensionless group velocity v_g^i/v of the probe ($i=P$), signal ($i=S$) and trigger ($i=T$) fields versus the dimensionless trigger detuning δ_5/γ for $v=10^4$ m/s. The figure shows clearly that the group velocities of the three laser fields can be matched approximately in our asymmetric six-level system if one chooses $2.4 \leq \delta_5/\gamma \leq 2.5$. Figure 2(b) shows the calculating result on the residual entanglement versus δ_5/γ for $\gamma=10^7$ s^{-1}. Notice that the maximum of the residual entanglement locates at about $2.4 \leq \delta_5/\gamma \leq 2.5$, where the group velocity matching of the three laser fields occurs.

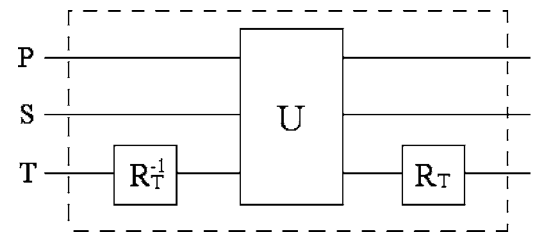

FIG. 3. The quantum circuit for realizing the Toffoli gate. U denotes the QPG with R_T being a single-qubit rotation.

The experimental demonstration of the quantum phase gate requires the measurement of total nonlinear phase shift. The fluctuations of light intensities and frequency detunings of the probe, signal, and trigger fields will result in errors of the nonlinear phase shift. Based on the parameters given above, our calculation shows that the light intensity (frequency) having fluctuations of 1% will yield an error less than 4% (2%) in the phase measurement. It is crucial to minimize the effect of relative detuning fluctuations, which can be achieved by taking all lasers tightly phase locked to each other. A similar result has also been reported in chain-lambda and tripod configurations which are used to realize two-qubit phase gates [17,18].

With the above parameters, we can realize an operation $\hat{U}=I-2|111\rangle\langle 111|$ when the total nonlinear phase shift gives $(2N+1)\pi$ radians ($N=0,1,\ldots$). By applying a single qubit rotation $\hat{R}_i$ to the trigger field with

$$\hat{R}_i(\theta,\varphi)=\begin{pmatrix} \cos\frac{\theta}{2} & ie^{-i\varphi}\sin\frac{\theta}{2} \\ -ie^{i\varphi}\sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{pmatrix}, \tag{9}$$

we can easily obtain the Toffoli gate by $\hat{U}_{\text{Toffoli}}=\hat{R}_T(\pi/2,\pi/2)\hat{U}\hat{R}_T^{-1}(\pi/2,\pi/2)$. The explicit operation is illustrated in Fig. 3.

V. CONCLUSION

To sum up, we have investigated the three-way entanglement and three-qubit phase gates based on a coherent six-level atomic system. We have shown that the completely cross fifth-order optical susceptibilities are greatly enhanced and the linear, third-order and other fifth-order nonlinear susceptibilities of the system are simultaneously suppressed due to the quantum interference effect induced by two strong cw control laser fields. Based on such important feature we have demonstrated that the system can produce efficient three-way entanglement among the weak probe, signal, and trigger laser fields. Different from the work in Ref. [21], here we have addressed a feasible method to fulfill the group velocity matching among the three weak optical pulses without using isotopes or solid quantum dots. In addition, we have explored the possibility of implementing a robust three-qubit QPG, which can be further transferred to a Toffoli gate by applying a single-qubit rotation. Our work is the first suggestion for constructing Toffoli gates in an all-optical way. The results provided in this study may be useful for guiding experimental realization of three-way entanglement and three-qubit phase gates and facilitating practical applications in quantum information and computation.

ACKNOWLEDGMENTS

The work was supported by the Key Development Program for Basic Research of China under Grants Nos. 2001CB309300 and 2005CB724508, and NSF-China under Grants Nos. 10434060 and 90403008.

APPENDIX: EXPLICIT EXPRESSIONS OF THE SUSCEPTIBILITIES IN EQ. (2)

The expressions of the susceptibilities in Eq. (2) are given by

$$\chi_P^{(1)} = \frac{N_a|D_{12}|^2}{\hbar\epsilon_0}\frac{d_3}{d_2d_3-|\Omega_C|^2}, \tag{A1}$$

$$\chi_{PP}^{(3)} = -\frac{N_a|D_{12}|^4}{\hbar^3\epsilon_0}\frac{d_3(|d_3|^2+|\Omega_C|^2)}{(d_2d_3-|\Omega_C|^2)|d_2d_3-|\Omega_C|^2|^2}, \tag{A2}$$

$$\chi_{PS}^{(3)} = -\frac{N_a|D_{12}|^2|D_{34}|^2}{\hbar^3\epsilon_0}\frac{d_5}{(d_4d_5-|\Omega_B|^2)(d_2d_3-|\Omega_C|^2)}, \tag{A3}$$

$$\chi_{PT}^{(3)} = -\frac{N_a|D_{12}|^2|D_{56}|^2}{\hbar^3\epsilon_0}\frac{d_3d_4}{d_6(d_4d_5-|\Omega_B|^2)(d_2d_3-|\Omega_C|^2)}, \tag{A4}$$

$$\chi_{PPP}^{(5)} = \frac{N_a|D_{12}|^6}{\hbar^5\epsilon_0}\frac{d_3(|d_3|^2+|\Omega_C|^2)^2}{(d_2d_3-|\Omega_C|^2)|d_2d_3-|\Omega_C|^2|^4}, \tag{A5}$$

$$\chi_{PPS}^{(5)} = \frac{N_a|D_{12}|^4|D_{34}|^2}{\hbar^5\epsilon_0}\left[\frac{d_5(|d_3|^2+|\Omega_C|^2)}{(d_4d_5-|\Omega_B|^2)(d_2d_3-|\Omega_C|^2)|d_2d_3-|\Omega_C|^2|^2} + \frac{d_3}{d_2d_3-|\Omega_C|^2}\left(\frac{d_3d_5^*}{|d_2d_3-|\Omega_C|^2|^2(d_4d_5-|\Omega_B|^2)^*} + \text{c.c.} - \frac{(|d_5|^2+|\Omega_B|^2)|\Omega_C|^2}{|(d_2d_3-|\Omega_C|^2)(d_4d_5-|\Omega_B|^2)|^2}\right)\right], \tag{A6}$$

$$\chi_{PPT}^{(5)} = \frac{N_a|D_{12}|^4|D_{56}|^2}{\hbar^5\epsilon_0}\left[\frac{d_3d_4(|d_3|^2+|\Omega_C|^2)}{d_6(d_4d_5-|\Omega_B|^2)(d_2d_3-|\Omega_C|^2)|d_2d_3-|\Omega_C|^2|^2} + \frac{d_3}{d_2d_3-|\Omega_C|^2}\left(\frac{(|d_3|^2+|\Omega_C|^2)d_4^*}{d_6^*|d_2d_3-|\Omega_C|^2|^2(d_4d_5-|\Omega_B|^2)^*} + \text{c.c.}\right)\right], \tag{A7}$$

$$\chi_{PST}^{(5)} = \frac{N_a|D_{12}|^2|D_{34}|^2|D_{56}|^2}{\hbar^5\epsilon_0}\frac{1}{d_6(d_4d_5-|\Omega_B|^2)(d_2d_3-|\Omega_C|^2)}, \tag{A8}$$

$$\chi_{SP}^{(3)} = \frac{N_a|D_{12}|^2|D_{34}|^2}{\hbar^3\epsilon_0}\frac{d_5|\Omega_C|^2}{(d_4d_5-|\Omega_B|^2)|d_2d_3-|\Omega_C|^2|^2}, \tag{A9}$$

$$\chi_{SPP}^{(5)} = -\frac{N_a|D_{12}|^4|D_{34}|^2}{\hbar^5\epsilon_0}\frac{d_5(|d_3|^2+|\Omega_C|^2)|\Omega_C|^2}{(d_4d_5-|\Omega_B|^2)|d_2d_3-|\Omega_C|^2|^4}, \tag{A10}$$

$$\chi_{SPT}^{(5)} = -\frac{N_a|D_{12}|^2|D_{34}|^2|D_{56}|^2}{\hbar^5\epsilon_0}\left(\frac{|\Omega_C|^2}{d_6(d_4d_5-|\Omega_B|^2)|d_2d_3-|\Omega_C|^2|^2} + \frac{d_4^*d_5|\Omega_C|^2}{d_6^*|d_4d_5-|\Omega_B|^2|^2|d_2d_3-|\Omega_C|^2|^2}\right), \tag{A11}$$

$$\chi_{TPS}^{(5)} = \frac{N_a|D_{12}|^2|D_{34}|^2|D_{56}|^2}{\hbar^5\epsilon_0}\frac{|\Omega_B|^2|\Omega_C|^2}{d_6|d_4d_5-|\Omega_B|^2|^2|d_2d_3-|\Omega_C|^2|^2}, \tag{A12}$$

where c.c. denotes corresponding complex conjugation term, $N_a=N/V$ is the density of the atomic gas. We have also made the definitions $d_2=\delta_1-i\Gamma_2/2$, $d_3=\delta_{12}-i\Gamma_3/2$, $d_4=\delta_{13}-i\Gamma_4/2$, $d_5=\delta_{14}-i\Gamma_5/2$, and $d_6=\delta_{15}-i\Gamma_6/2$.

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
[2] K. J. Boller, A. Imamoglu, and S. E. Harris, Phys. Rev. Lett. **66**, 2593 (1991).
[3] S. E. Harris, Phys. Today **50**, 36 (1997).
[4] M. D. Lukin, Rev. Mod. Phys. **75**, 457 (2003).
[5] M. Fleischhauer, A. Imamoglu, and J. P. Marangos, Rev. Mod. Phys. **77**, 633 (2005), and references therein.
[6] H. Schmidt and A. Imamoglu, Opt. Lett. **21**, 1936 (1996).
[7] H. Kang and Y. Zhu, Phys. Rev. Lett. **91**, 093601 (2003).
[8] A. D. Greentree, D. Richards, J. A. Vaccaro, A. V. Durrant, S. R. de Echaniz, D. Segal, and J. P. Marangos, Phys. Rev. A **67**, 023818 (2003).
[9] D. Petrosyan and Yu. P. Malakyan, Phys. Rev. A **70**, 023822 (2004).
[10] D. Petrosyan and G. Kurizki, Phys. Rev. A **65**, 033833 (2002).
[11] M. D. Lukin and P. R. Hemmer, Phys. Rev. Lett. **84**, 2818 (2000); M. D. Lukin, S. F. Yelin, and M. Fleischhauer, *ibid.* **84**, 4232 (2000).
[12] M. Paternostro, M. S. Kim, and B. S. Ham, Phys. Rev. A **67**, 023811 (2003).
[13] M. G. Payne and L. Deng, Phys. Rev. Lett. **91**, 123602 (2003).
[14] M. Kiffner and K.-P. Marzlin, Phys. Rev. A **71**, 033811 (2005).
[15] A. Peng, M. Johnsson, W. P. Bowen, P. K. Lam, H.-A. Bachor, and J. J. Hope, Phys. Rev. A **71**, 033809 (2005).
[16] I. Friedler, D. Petrosyan, M. Fleischhauer, and G. Kurizki, Phys. Rev. A **72**, 043803 (2005).
[17] C. Ottaviani, D. Vitali, M. Artoni, F. Cataliotti, and P. Tombesi, Phys. Rev. Lett. **90**, 197902 (2003).
[18] S. Rebić, D. Vitali, C. Ottaviani, P. Tombesi, M. Artoni, F. Cataliotti, and R. Corbalán, Phys. Rev. A **70**, 032317 (2004).
[19] A. Joshi and M. Xiao, Phys. Rev. A **72**, 062319 (2005).
[20] Y. Li, C. Hang, L. Ma, and G. Huang, Phys. Lett. A **354**, 1 (2006
[21] M. S. Zubairy, A. B. Matsko, and M. O. Scully, Phys. Rev. A **65**, 043804 (2002); A. B. Matsko, I. Novikova, G. R. Welch, and M. S. Zubairy, Opt. Lett. **28**, 96 (2003).
[22] The *completely cross* fifth-order nonlinear susceptibility here means a nonzero $\chi^{(5)}_{ijk}$, in which all three subscripts i, j, and k are different.
[23] J. K. Pachos and P. L. Knight, Phys. Rev. Lett. **91**, 107902 (2003).
[24] L.-M. Duan, B. Wang, and H. J. Kimble, Phys. Rev. A **72**, 032333 (2005).
[25] P. K. Gagnebin, S. R. Skinner, E. C. Behrman, J. E. Steck, Z. Zhou, and S. Han, Phys. Rev. A **72**, 042311 (2005).
[26] M. S. Tame, M. Paternostro, M. S. Kim, and V. Vedral, Phys. Rev. A **73**, 022309 (2006).
[27] In fact, Eq. (1) can be considered to be approximately equivalent to optical Bloch equations for density matrix by defining, e.g., $\Gamma_2=\Sigma_{E_j<E_2}\Gamma_{2j}+\gamma_{12}$, where Γ_{2j} is the decay rate of the atom from state $|2\rangle$ to state $|j\rangle$, $\gamma_{12}=\gamma_{11}+\gamma_{22}$ is the composite dephasing rate with γ_{jj} being the dephasing rate of the state $|j\rangle$ ($j=1,2$). For detail, see C. Ottaviani, S. Rebić, D. Vitali, and P. Tombesi, e-print physics/0510200 (unpublished).
[28] Note that this normalization condition is valid approximately due to the existence of dissipation in Eq. (1).
[29] H. J. Metcalf and P. van der Straten, *Laser Cooling and Trapping* (Springer-Verlag, New York, 1999).
[30] M. D. Lukin and A. Imamoğlu, Phys. Rev. Lett. **84**, 1419 (2000).
[31] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A **61**, 052306 (2000).
[32] $\delta_5 \gg \Gamma/2$ should be chosen in order to ensure a small absorption. In our parameters, we have taken $\delta_5=10\Gamma/2$.