Reversible quantum measurement with arbitrary spins

Hiroaki Terashima[1,2] and Masahito Ueda[1,2]
[1]*Department of Physics, Tokyo Institute of Technology, Tokyo 152-8551, Japan*
[2]*CREST, Japan Science and Technology Corporation (JST), Saitama 332-0012, Japan*
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We propose a physically reversible quantum measurement of an arbitrary spin-s system using a spin-j probe via an Ising interaction. In the case of a spin-1/2 system ($s=1/2$), we explicitly construct a reversing measurement and evaluate the degree of reversibility in terms of fidelity. The recovery of the measured state is pronounced when the probe has a high spin ($j>1/2$), because the fidelity changes drastically during the reversible measurement and the reversing measurement. We also show that the reversing measurement scheme for a spin-1/2 system can serve as an experimentally feasible approximate reversing measurement for a high-spin system ($s>1/2$). If the interaction is sufficiently weak, the reversing measurement can recover a cat state almost deterministically in spite of there being a large fidelity change.

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I. INTRODUCTION

Quantum measurements are widely believed to have intrinsic irreversibility, since they play different roles with respect to the past and future of the measured system [1]. With respect to the past, a quantum measurement verifies the predicted probabilities for possible outcomes. With respect to the future, a measurement brings about a new quantum state via nonunitary state reduction. However, as shown in Ref. [2], a quantum measurement is not necessarily irreversible. A quantum measurement is said to be logically reversible [2,3] if the premeasurement state can be calculated from the postmeasurement state and the outcome of the measurement. This means that all the information about the premeasurement state is preserved during the measurement. A quantum measurement is said to be physically reversible [3,4] if the premeasurement state can be recovered from the postmeasurement state by means of a second measurement, referred to as a reversing measurement, with a nonzero probability. In this case, not only is the information about the system preserved during the measurement process, but the original state can be restored by means of a physical process.

Some measurements are known to be logically reversible [2,5,6]. Royer [6] proposed a physically reversible quantum measurement of a spin-1/2 system using a spin-1/2 probe in an attempt to completely determine the unknown quantum state of a single system (see, however, erratum of Ref. [6]). In the context of quantum computation [7], the reversing measurement has been discussed for reducing the qubit overhead in quantum error correction [8] and for improving the probability of successful nonunitary gate operation in a nonunitary quantum circuit [9]. As an important step toward the experimental realization of a reversible measurement, a photodetection scheme that satisfies a necessary condition for logical reversibility ("sensitivity to vacuum fluctuations") has recently been demonstrated [10] using a stimulated parametric down-conversion process.

In this paper, we propose a scheme for making a physically reversible quantum measurement that is experimentally feasible in view of recent advances in experimental techniques [11,12]. Our model consists of two arbitrary spin systems (a measured system and a probe system) interacting via an Ising Hamiltonian. Since spin can describe diverse physical systems (e.g., the real spin of particles, collective two-level systems, Cooper pairs, interferometers, and Josephson junctions), our model can be used to implement both physically reversible measurements and reversing measurements in such diverse systems. We explicitly construct a reversing measurement for our model, in which quantitative analysis is performed in terms of fidelity [7]. When the probe system has a high spin, the fidelity changes drastically in both the reversible measurement and the reversing measurement. The high-spin probe thus makes the recovery of the measured state more pronounced than for the spin-1/2 model, though at the cost of decreasing the probability of successful recovery.

To clarify what kind of irreversibility is at issue, we here review a projective measurement [13], which is often used to describe measurement processes in quantum theory. Let $\hat{O}$ be a measured observable, whose eigenstate with eigenvalue m is denoted by $|m\rangle$. The observable $\hat{O}$ can then be decomposed as $\Sigma_m m\hat{P}_m$, where $\hat{P}_m$ is the projector $|m\rangle\langle m|$. From the completeness condition, the projectors $\{\hat{P}_m\}$ satisfy

$$\sum_m \hat{P}_m = \hat{I}, \tag{1}$$

with $\hat{I}$ being the identity operator. Suppose that the measured system is initially in a state $|\psi\rangle$. The projective measurement with respect to $\{\hat{P}_m\}$ yields an outcome m with probability

$$p_m = \langle\psi|\hat{P}_m|\psi\rangle \tag{2}$$

and then causes a state reduction of the measured system to

$$|\psi_m\rangle = \frac{1}{\sqrt{p_m}}\hat{P}_m|\psi\rangle. \tag{3}$$

Clearly, the projective measurement is irreversible in the sense that we cannot recover the premeasurement state $|\psi\rangle$ from the postmeasurement state $|\psi_m\rangle$, unless we *a priori* know the former state. This is because the information about the states orthogonal to $\hat{P}_m$ is completely lost during the measurement. One might think that any quantum measurement has this type of irreversibility, since quantum measurement entails a nonunitary state reduction associated with information readout. However, there exist quantum measurements that are logically reversible in spite of nonunitary state reduction [2,5,6].

To formulate the conditions for logical reversibility, we adopt a general formulation of quantum measurement [7,14], in which a quantum measurement is described by a set of measurement operators $\{\hat{M}_m\}$ that satisfies

$$\sum_m \hat{M}_m^\dagger \hat{M}_m = \hat{I}. \tag{4}$$

If the measured system is in a state $|\psi\rangle$, the general measurement with respect to $\{\hat{M}_m\}$ yields an outcome m with probability

$$p_m = \langle\psi|\hat{M}_m^\dagger \hat{M}_m|\psi\rangle \tag{5}$$

and then causes a state reduction of the measured system to

$$|\psi_m\rangle = \frac{1}{\sqrt{p_m}}\hat{M}_m|\psi\rangle. \tag{6}$$

Note that this state change depends on the outcome m. The general measurement can be simulated by a projective measurement with the help of a measurement probe, even though the projective measurement is a special case of the general measurement ($\hat{M}_m = \hat{P}_m$). The necessary and sufficient condition for logical reversibility is $\hat{M}_m|\psi\rangle \neq 0$ for any $|\psi\rangle$ in the Hilbert space [3]. In other words, the measurement must respond to any input state so that no possibility of the premeasurement state is excluded by any outcome of the measurement. For example, the usual photon counting [15] is logically irreversible because the detection of a photon excludes the possibility that the premeasurement state is the vacuum state. On the other hand, the necessary and sufficient condition for physical reversibility is that $\hat{M}_m$ has a bounded left inverse [3,4]. Thus physical reversibility implies logical reversibility, but not vice versa. An important special case is that of a finite-dimensional Hilbert space, where physical reversibility is equivalent to logical reversibility. However, in an infinite-dimensional Hilbert space, there exist logically reversible yet physically irreversible measurements [3] such as quantum counting [2].

A different type of reversibility is discussed in Refs. [16,17]. A quantum measurement is said to be unitarily reversible if the premeasurement state can be recovered by a reversing unitary operation on the postmeasurement state. In this case, although successful reversal occurs with unit probability owing to the unitarity, it is essential that the premeasurement state lie within a certain subspace of the entire Hilbert space. Since the subspace is chosen so that the probability of each measurement outcome is the same for all states in the subspace, no information about the premeasurement state can be obtained from the unitarily reversible quantum measurement [17].

This paper is organized as follows. Section II formulates a physically reversible quantum measurement of a spin-s system using a spin-j probe. Section III explicitly constructs the reversing measurement for the case of a measured system with $s=1/2$, focusing on the effect of a high-spin probe ($j > 1/2$). Section IV describes two approximate schemes of the reversing measurement for the case of measured systems with $s > 1/2$: one in which the measured system is initially in a two-dimensional subspace and the other in which the interaction is sufficiently weak. Section V discusses a possible experimental situation using an ensemble of atoms as a measured system and two-mode photons as a probe system. Section VI summarizes our results. Throughout this paper, we refer to the measured system and the probe system simply as *system* and *probe*, respectively.

II. REVERSIBLE SPIN MEASUREMENT

First, we formulate a quantum measurement of a spin-s system described by spin operators $\{\hat{S}_x, \hat{S}_y, \hat{S}_z\}$. These operators obey the commutation relations

$$[\hat{S}_i, \hat{S}_j] = i\epsilon_{ijk}\hbar\,\hat{S}_k, \tag{7}$$

where the indices i,j,k denote x,y,z and ϵ_{ijk} is the Levi-Civita symbol. The Hilbert space of this system is spanned by the eigenstates of $\hat{S}_z$,

$$\hat{S}_z|\sigma\rangle_q = \sigma\hbar\,|\sigma\rangle_q, \tag{8}$$

where $\sigma = s, s-1, \ldots, -s+1, -s$. Using these states, the state to be measured is written as

$$|\psi\rangle_q = \sum_\sigma c_\sigma|\sigma\rangle_q, \tag{9}$$

with the normalization condition

$$\sum_\sigma |c_\sigma|^2 = 1. \tag{10}$$

It should be emphasized that the coefficients $\{c_\sigma\}$ are unknown, since it is assumed that we have no *a priori* information about the measured state $|\psi\rangle_q$. The measured system is assumed to be in a pure state as in Eq. (9); a mixed initial state of the system makes no difference in constructing a reversing measurement.

To measure the spin state of the system, we introduce a probe with spin j. The probe is described by spin operators $\{\hat{J}_x, \hat{J}_y, \hat{J}_z\}$ satisfying the commutation relations

$$[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hbar\,\hat{J}_k. \tag{11}$$

The Hilbert space of this system is also spanned by the eigenstates of $\hat{J}_z$,

$$\hat{J}_z|m\rangle_p = m\hbar\,|m\rangle_p, \tag{12}$$

where $m = j, j-1, \ldots, -j+1, -j$.

We prepare the probe in a state

$$\begin{aligned}|\theta,\phi\rangle_p &= \exp\left(-\frac{i}{\hbar}\hat{J}_z\phi\right)\exp\left(-\frac{i}{\hbar}\hat{J}_y\theta\right)|j\rangle_p\\ &= \sum_m e^{-im\phi}d_{mj}^{(j)}(\theta)|m\rangle_p,\end{aligned} \tag{13}$$

where $d_{m'm}^{(j)}(\theta)$ is defined by [18]

$$d^{(j)}_{m'm}(\theta) \equiv {}_p\langle m'|\exp\left(-\frac{i}{\hbar}\hat{J}_y\theta\right)|m\rangle_p = \sum_{\substack{0\le k\le j+m \\ m-m'\le k\le j-m'}} \frac{\sqrt{(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!}}{(j+m-k)!\,k!\,(j-k-m')!\,(k-m+m')!}(-1)^{k-m+m'}\left(\cos\frac{\theta}{2}\right)^{2j-2k+m-m'}\left(\sin\frac{\theta}{2}\right)^{2k-m+m'}. \quad (14)$$

We assume that the interaction between the system and the probe is of an Ising type,

$$H = \alpha \hat{J}_z \hat{S}_z, \quad (15)$$

where α is a real constant. This $\hat{J}_z\hat{S}_z$-type interaction has direct relevance to experimental situations in Refs. [11,19–22]. The interaction between the system and probe gives rise to a unitary transformation,

$$\hat{U}_i = \exp\left(-\frac{2ig}{\hbar^2}\hat{J}_z\hat{S}_z\right), \quad (16)$$

on the combined system, where $g \equiv \alpha t\hbar/2$ is the effective strength of the interaction.

After the interaction, the unitary operator

$$\hat{U}_p = \exp\left(-\frac{i}{\hbar}\hat{J}_y\frac{\pi}{2}\right) \quad (17)$$

is applied to the probe. The state of the whole system then becomes

$$\hat{U}_p\hat{U}_i|\theta,\phi\rangle_p|\psi\rangle_q = \sum_{m',\sigma} a^{(j)}_{m'\sigma}(\theta,\phi)c_\sigma|m'\rangle_p|\sigma\rangle_q, \quad (18)$$

where $a^{(j)}_{m'\sigma}(\theta,\phi)$ is given by

$$a^{(j)}_{m'\sigma}(\theta,\phi) \equiv \sum_m e^{-im(2g\sigma+\phi)}d^{(j)}_{mj}(\theta)d^{(j)}_{m'm}\left(\frac{\pi}{2}\right) = \frac{1}{2^j}\sqrt{\frac{(2j)!}{(j+m')!\,(j-m')!}} \times \left(e^{-(i/2)(2g\sigma+\phi)}\cos\frac{\theta}{2} + e^{(i/2)(2g\sigma+\phi)}\sin\frac{\theta}{2}\right)^{j-m'} \times \left(e^{-(i/2)(2g\sigma+\phi)}\cos\frac{\theta}{2} - e^{(i/2)(2g\sigma+\phi)}\sin\frac{\theta}{2}\right)^{j+m'}. \quad (19)$$

Note that $|a^{(j)}_{m'\sigma}(\theta,\phi)|^2$ is a binomial distribution as a function of m' (see Fig. 1):

$$|a^{(j)}_{m'\sigma}(\theta,\phi)|^2 = \frac{(2j)!}{(j+m')!\,(j-m')!} \times \left[\frac{1+\chi_\sigma(\theta,\phi)}{2}\right]^{j-m'}\left[\frac{1-\chi_\sigma(\theta,\phi)}{2}\right]^{j+m'}, \quad (20)$$

where

$$\chi_\sigma(\theta,\phi) \equiv \sin\theta\cos(2g\sigma+\phi). \quad (21)$$

We thus obtain the normalization condition

$$\sum_{m'} |a^{(j)}_{m'\sigma}(\theta,\phi)|^2 = 1. \quad (22)$$

The mean and variance of this distribution are given by

$$\mu_\sigma(\theta,\phi) \equiv \sum_{m'} m'|a^{(j)}_{m'\sigma}(\theta,\phi)|^2 = -j\chi_\sigma(\theta,\phi) \quad (23)$$

and

$$\nu_\sigma(\theta,\phi) \equiv \sum_{m'} (m'-\mu_\sigma(\theta,\phi))^2|a^{(j)}_{m'\sigma}(\theta,\phi)|^2 = j\left[\frac{1-\chi_\sigma(\theta,\phi)^2}{2}\right], \quad (24)$$

respectively. The central limit theorem states that as j increases, the binomial distribution becomes close to a normal distribution with the mean and variance unaltered. Thus, for large j, we can approximate the distribution as

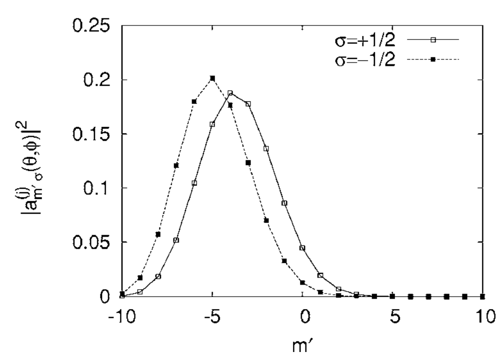

FIG. 1. $|a^{(j)}_{m'\sigma}(\theta,\phi)|^2$ as a function of m' ($\sigma=\pm1/2$, $j=10$, $g=0.25$, $\theta=\pi/6$, $\phi=\pi/6$).

$$|a_{m'\sigma}^{(j)}(\theta,\phi)|^2 \sim \frac{1}{\sqrt{2\pi\nu_\sigma(\theta,\phi)}} \exp\left[-\frac{[m'-\mu_\sigma(\theta,\phi)]^2}{2\nu_\sigma(\theta,\phi)}\right]. \tag{25}$$

We finally perform a projective measurement on the probe variable $\hat{J}_z/\hbar$ and obtain the measurement outcome m $(=j,j-1,\dots,-j+1,-j)$. Alternatively, we can perform the projective measurement of $-\hat{J}_x/\hbar$ without the unitary operator $\hat{U}_p$ in Eq. (17). Since the probability for outcome m is

$$p_m = \sum_\sigma |a_{m\sigma}^{(j)}(\theta,\phi)|^2 |c_\sigma|^2, \tag{26}$$

we can obtain information about the initial state (9) of the system from this measurement through the dependence of p_m on c_σ. However, if $|a_{m\sigma}^{(j)}(\theta,\phi)|^2$ does not depend on σ, the probability p_m does not depend on c_σ because of the normalization condition (10). Therefore, to obtain information about the measured state, the initial probe state $|\theta,\phi\rangle_p$ and the strength of the interaction g must satisfy

$$\sin\theta \neq 0,$$

$$\sin g \neq 0,$$

$$\sin[(2s-1)g+\phi] \neq 0, \tag{27}$$

according to Eq. (20), where the last condition is required if $s=1/2$ or if $s>1/2$ and $\cos g=0$. From Eqs. (10) and (22), it is easy to see that the total probability is

$$\sum_m p_m = 1. \tag{28}$$

Using Eq. (23), the expected value of m is given by

$$\bar{m} \equiv \sum_m m p_m = -j \sum_\sigma \chi_\sigma(\theta,\phi)|c_\sigma|^2. \tag{29}$$

The measurement process causes a nonunitary state reduction of the measured system. Corresponding to the outcome m, the state of the system becomes

$$|\psi_m\rangle_q = \frac{1}{\sqrt{p_m}} \sum_\sigma a_{m\sigma}^{(j)}(\theta,\phi) c_\sigma |\sigma\rangle_q \tag{30}$$

and its fidelity with the premeasurement state decreases to

$$F_m = |{}_q\langle\psi|\psi_m\rangle_q| = \frac{1}{\sqrt{p_m}} \left| \sum_\sigma a_{m\sigma}^{(j)}(\theta,\phi)|c_\sigma|^2 \right|. \tag{31}$$

We can describe this measurement process by a set of measurement operators, as in the general quantum measurement [7,14]. Let $\hat{T}_m(\theta,\phi)$ be the measurement operator for outcome m. Since the probability (26) and postmeasurement state (30) are expressed as

$$p_m = {}_q\langle\psi|\hat{T}_m^\dagger(\theta,\phi)\hat{T}_m(\theta,\phi)|\psi\rangle_q, \tag{32}$$

$$|\psi_m\rangle_q = \frac{1}{\sqrt{p_m}} \hat{T}_m(\theta,\phi)|\psi\rangle_q, \tag{33}$$

the explicit form of $\hat{T}_m(\theta,\phi)$ is given by

$$\hat{T}_m(\theta,\phi) = \sum_\sigma a_{m\sigma}^{(j)}(\theta,\phi)|\sigma\rangle_{qq}\langle\sigma|. \tag{34}$$

From Eq. (22), we can confirm that

$$\sum_m \hat{T}_m^\dagger(\theta,\phi)\hat{T}_m(\theta,\phi) = \hat{I}. \tag{35}$$

This measurement does not disturb the eigenstates of $\hat{S}_z$ owing to the commutation relation

$$[\hat{S}_z, \hat{T}_m(\theta,\phi)] = 0. \tag{36}$$

The measurement $\{\hat{T}_m(\theta,\phi)\}$ is logically reversible [2,3] if $\hat{T}_m(\theta,\phi)|\psi\rangle_q \neq 0$ for any $|\psi\rangle_q$ or, equivalently, if $a_{m\sigma}^{(j)}(\theta,\phi) \neq 0$ for any σ. This condition requires the initial probe state $|\theta,\phi\rangle_p$ and the strength of the interaction g to satisfy

$$\sin\theta \neq \pm 1 \quad \text{or} \quad \cos(2g\sigma+\phi) \neq \pm 1 \tag{37}$$

for $\sigma=s,s-1,\dots,-s+1,-s$. When these conditions are satisfied, the measurement $\{\hat{T}_m(\theta,\phi)\}$ is physically reversible as well, since $\hat{T}_m(\theta,\phi)$ has a bounded left inverse. This implies that there exists another measurement that can recover the *unknown* premeasurement state (9) from the postmeasurement state (30) with a nonzero probability. We explicitly construct such reversing measurements in the following sections. Note, however, that the measurement $\{\hat{T}_m(\theta,\phi)\}$ is not unitarily reversible [16,17] if condition (27) is satisfied. This is because we have obtained some information about the measured state from the measurement outcome via the probability that depends on the measured state [17]. Therefore, there is no unitary operation that can recover the premeasurement state from the postmeasurement state.

III. REVERSING MEASUREMENT ON A SPIN-1/2 SYSTEM ($s=1/2$)

A. Scheme

We consider a reversing measurement of a physically reversible measurement $\{\hat{T}_m(\theta,\phi)\}$ for the case of a measured system with $s=1/2$, where the measurement operator $\hat{T}_m(\theta,\phi)$ is in the basis $\{|1/2\rangle_q, |-1/2\rangle_q\}$ represented by a diagonal 2×2 matrix as

$${}_q\langle\sigma'|\hat{T}_m(\theta,\phi)|\sigma\rangle_q = \begin{pmatrix} a_{m,1/2}^{(j)}(\theta,\phi) & 0 \\ 0 & a_{m,-1/2}^{(j)}(\theta,\phi) \end{pmatrix}. \tag{38}$$

Suppose that a second measurement $\{\hat{T}_m(\pi-\theta,\pi-\phi)\}$ is performed on the postmeasurement state (30) and that an outcome m' $(=j,j-1,\dots,-j+1,-j)$ is obtained, as illustrated in Fig. 2. Using

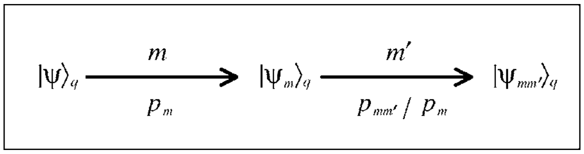

FIG. 2. Transitions of the measured state by successive measurements $\{\hat{T}_m(\theta,\phi)\}$ and $\{\hat{T}_m(\pi-\theta,\pi-\phi)\}$. The first measurement on the state $|\psi\rangle_q$ yields an outcome m $(=j,j-1,\ldots,-j+1,-j)$ with probability p_m, causing a state reduction to $|\psi_m\rangle_q$. The second measurement on $|\psi_m\rangle_q$ then yields an outcome m' with conditional probability $p_{mm'}/p_m$, causing a state reduction to $|\psi_{mm'}\rangle_q$.

$$a^{(j)}_{m'\sigma}(\pi-\theta,\pi-\phi)=e^{-im'\pi}a^{(j)}_{-m',-\sigma}(\theta,\phi), \tag{39}$$

the measurement operator $\hat{T}_{m'}(\pi-\theta,\pi-\phi)$ is represented by

$$ {}_q\langle\sigma'|\hat{T}_{m'}(\pi-\theta,\pi-\phi)|\sigma\rangle_q = e^{-im'\pi}\begin{pmatrix} a^{(j)}_{-m',-1/2}(\theta,\phi) & 0 \\ 0 & a^{(j)}_{-m',1/2}(\theta,\phi)\end{pmatrix}. \tag{40}$$

The state of the system thus becomes

$$|\psi_{mm'}\rangle_q=\frac{e^{-im'\pi}}{\sqrt{p_{mm'}}}\sum_{\sigma=\pm1/2}a^{(j)}_{-m',-\sigma}(\theta,\phi)a^{(j)}_{m\sigma}(\theta,\phi)c_\sigma|\sigma\rangle_q, \tag{41}$$

where

$$p_{mm'}=\sum_{\sigma=\pm1/2}|a^{(j)}_{-m',-\sigma}(\theta,\phi)a^{(j)}_{m\sigma}(\theta,\phi)|^2|c_\sigma|^2 \tag{42}$$

is the joint probability of obtaining the outcomes m for the first measurement and m' for the second measurement. The expected values of m and m' are given by

$$\bar{m}=\sum_{m,m'}mp_{mm'}=-j[\chi_{1/2}(\theta,\phi)|c_{1/2}|^2+\chi_{-1/2}(\theta,\phi)|c_{-1/2}|^2], \tag{43}$$

$$\bar{m}'=\sum_{m,m'}m'p_{mm'}=+j[\chi_{-1/2}(\theta,\phi)|c_{1/2}|^2+\chi_{1/2}(\theta,\phi)|c_{-1/2}|^2], \tag{44}$$

respectively. Therefore, as a function of m and m', the joint probability $p_{mm'}$ has two peaks at

$$(m,m')=(-j\chi_{\pm1/2}(\theta,\phi),+j\chi_{\mp1/2}(\theta,\phi)), \tag{45}$$

where the heights of the peaks depend on the values of $|c_{1/2}|^2$ and $|c_{-1/2}|^2$.

An interesting case of recovery of the measured state occurs when the outcome of the second measurement is the negative of the first one (i.e., $m'=-m$). Since $a^{(j)}_{m,-\sigma}(\theta,\phi)a^{(j)}_{m\sigma}(\theta,\phi)$ does not depend on σ $(=\pm1/2)$, the final state (41) with $m'=-m$ is identical to the original state (9) except for an overall phase factor,

$$|\psi_{m,-m}\rangle_q=e^{i\alpha}\sum_{\sigma=\pm1/2}c_\sigma|\sigma\rangle_q, \tag{46}$$

where

$$e^{i\alpha}\equiv e^{im\pi}\frac{a^{(j)}_{m,-1/2}(\theta,\phi)a^{(j)}_{m,1/2}(\theta,\phi)}{|a^{(j)}_{m,-1/2}(\theta,\phi)a^{(j)}_{m,1/2}(\theta,\phi)|}. \tag{47}$$

Therefore, the second measurement $\{\hat{T}_m(\pi-\theta,\pi-\phi)\}$ is a reversing measurement of the first measurement $\{\hat{T}_m(\theta,\phi)\}$. Here, the state recovery results from the identity

$$\hat{T}_{-m}(\pi-\theta,\pi-\phi)\hat{T}_m(\theta,\phi)=[e^{im\pi}a^{(j)}_{m,-1/2}(\theta,\phi)a^{(j)}_{m,1/2}(\theta,\phi)]\hat{I}, \tag{48}$$

which implies that $\hat{T}_{-m}(\pi-\theta,\pi-\phi)$ is proportional to the inverse of $\hat{T}_m(\theta,\phi)$. The total probability of state recovery is given by

$$q\equiv\sum_m p_{m,-m}=\sum_m|a^{(j)}_{m,-1/2}(\theta,\phi)a^{(j)}_{m,1/2}(\theta,\phi)|^2. \tag{49}$$

This is the overlap between the binomial distributions $|a^{(j)}_{m,1/2}(\theta,\phi)|^2$ and $|a^{(j)}_{m,-1/2}(\theta,\phi)|^2$ (see Fig. 1). The measured state can be recovered with high probability when these distributions overlap closely, although the case of complete overlap does not satisfy the condition (27). Note that when recovery occurs, we cannot obtain any information about the original state (9) from the measurement outcomes m and $-m$, since the joint probability $p_{m,-m}$ does not depend on c_σ.

If $m'\neq -m$, we can still expect that the original state is almost recovered as long as m' is close to $-m$. The extent to which the state of the system is recovered can be evaluated in terms of the fidelity between the original state (9) and the final state (41),

$$\begin{aligned}F_{mm'}&=|{}_q\langle\psi|\psi_{mm'}\rangle_q|\\&=\frac{1}{\sqrt{p_{mm'}}}\left|\sum_{\sigma=\pm1/2}a^{(j)}_{-m',-\sigma}(\theta,\phi)a^{(j)}_{m\sigma}(\theta,\phi)|c_\sigma|^2\right|\\&=\{|c_{1/2}|^4[e_+(\theta,\phi)]^{m'+m}+|c_{-1/2}|^4[e_-(\theta,\phi)]^{m'+m}\\&\quad+2|c_{1/2}|^2|c_{-1/2}|^2[e_+(\theta,\phi)e_-(\theta,\phi)]^{m'+m/2}\\&\quad\times\cos[(m'+m)f(\theta,\phi)]\}^{1/2}\\&\quad\times\{|c_{1/2}|^2[e_+(\theta,\phi)]^{m'+m}+|c_{-1/2}|^2[e_-(\theta,\phi)]^{m'+m}\}^{-1/2},\end{aligned} \tag{50}$$

where

$$e_\pm(\theta,\phi)\equiv\left(\frac{1-\chi_{\pm1/2}(\theta,\phi)}{1+\chi_{\pm1/2}(\theta,\phi)}\right), \tag{51}$$

$$\begin{aligned}f(\theta,\phi)\equiv\arg[1-\sin^2\theta(\cos^2\phi+\sin^2 g)\\+i\sin2\theta\cos\phi\sin g],\end{aligned} \tag{52}$$

and $\arg[\cdots]$ represents the argument of the complex number in the brackets. By definition, we obtain $F_{m,-m}=1$ as a result of the recovery (46). It is interesting that the fidelity $F_{mm'}$ depends on $m'+m$ but not on j or on $m'-m$. Expanding the

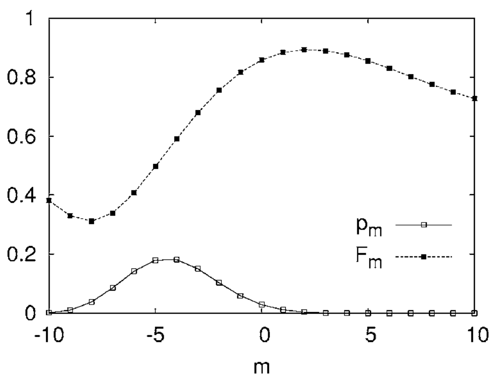

FIG. 3. Probability p_m and fidelity F_m of the first measurement as functions of the outcome m ($|c_{1/2}|^2=|c_{-1/2}|^2=1/2$, $j=10$, $g=0.25$, $\theta=\pi/6$, $\phi=\pi/6$).

fidelity $F_{mm'}$ to the second order in $m'+m$, we obtain

$$F_{mm'} \sim 1 - \frac{1}{5}|c_{1/2}|^2|c_{-1/2}|^2\left(\frac{m'+m}{\delta m(\theta,\phi)}\right)^2$$
$$\geq 1 - \frac{1}{20}\left(\frac{m'+m}{\delta m(\theta,\phi)}\right)^2, \quad (53)$$

where $\delta m(\theta,\phi)$ is defined by

$$\delta m(\theta,\phi) \equiv \sqrt{\frac{8}{5}}\left[\left(\ln\frac{e_+(\theta,\phi)}{e_-(\theta,\phi)}\right)^2 + 4f(\theta,\phi)^2\right]^{-1/2}. \quad (54)$$

The equality is satisfied when $|c_{1/2}|^2=|c_{-1/2}|^2=1/2$. If the outcomes m and m' satisfy

$$|m'+m| \leq \delta m(\theta,\phi), \quad (55)$$

the fidelity is greater than 0.95. In this case, we can say that more than 95% of the information about the measured state is recovered. The total probability of this approximate recovery is defined by

$$q' = \sum_{\substack{m,m' \\ F_{mm'}\geq 0.95}} p_{mm'}, \quad (56)$$

which depends weakly on c_σ.

As an example, we consider the case where $|c_{1/2}|^2=|c_{-1/2}|^2=1/2$, $j=10$, $g=0.25$, $\theta=\pi/6$, and $\phi=\pi/6$. This is the worst case for which the lower bound in Eq. (54) is achieved. Figure 3 shows the probability (26) and the fidelity (31) of the first measurement $\{\hat{T}_m(\pi/6,\pi/6)\}$ as functions of the outcome m. The average fidelity after the first measurement is $\Sigma_m p_m F_m=0.57$. To recover the fidelity lost by the first measurement, the second measurement $\{\hat{T}_m(5\pi/6, 5\pi/6)\}$ is performed. Figure 4 shows the probability (42) as a function of the outcomes m for the first measurement and m' for the second measurement. The two peaks (45) of the joint probability merge into a single peak located on the line of recovery ($m'=-m$), since $\chi_{1/2}(\theta,\phi)$ and $\chi_{-1/2}(\theta,\phi)$ are close to each other. This indicates that the highly probable events are concentrated near the line of recovery. In fact, the total probability of recovery, Eq. (49), becomes large due to the large overlap of $|a^{(j)}_{m,1/2}(\theta,\phi)|^2$ and $|a^{(j)}_{m,-1/2}(\theta,\phi)|^2$. In this example, we obtain $q=0.13$. The more tolerable is the error in terms of the fidelity, the larger is the probability of recovery. Figure 5 shows the fidelity (50) after the second measurement as a function of the outcomes m and m'. The average fidelity after the second measurement is $\Sigma_{mm'}p_{mm'}F_{mm'}=0.93$.The fidelity is larger than 0.95 provided that $|m'+m|$ is less than $\delta m(\theta,\phi)=2.3$ defined by Eq. (54). The total probability of approximate recovery, Eq. (56), is $q'=0.57$.

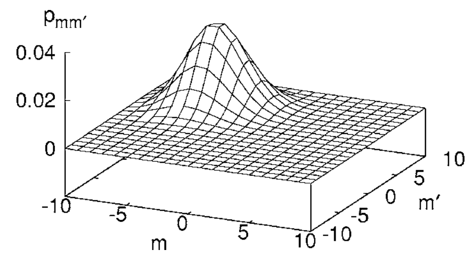

FIG. 4. Joint probability $p_{mm'}$ of the first and second measurements as a function of the outcomes m and m' ($|c_{1/2}|^2=|c_{-1/2}|^2=1/2$, $j=10$, $g=0.25$, $\theta=\pi/6$, $\phi=\pi/6$).

B. Information gain versus fidelity loss

As noted in the preceding subsection, we cannot obtain any information about the measured state if a successful recovery occurs by a reversing measurement. In other words, successful recovery obliterates the information obtained by the first measurement. Therefore, one might think that it is not worthwhile performing a reversing measurement. However, when the recovery is only partially successful, the reversing measurement can improve the fidelity together with providing further information. We show this here by a simple situation.

Suppose that the state of the system is known to be either $|a\rangle_q$ or $|b\rangle_q$ with equal probability, $p(a)=p(b)=1/2$, where we choose the two states as

$$|a\rangle_q = \cos\frac{\gamma}{2}|1/2\rangle_q + \sin\frac{\gamma}{2}|-1/2\rangle_q, \quad (57)$$

$$|b\rangle_q = -\sin\frac{\gamma}{2}|1/2\rangle_q + \cos\frac{\gamma}{2}|-1/2\rangle_q, \quad (58)$$

with γ being a real constant ($0<\gamma<\pi/2$). The Shannon entropy associated with the system is initially given by

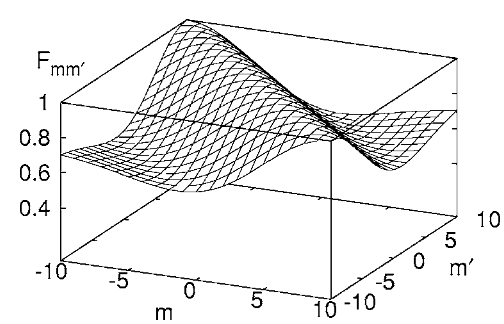

FIG. 5. Fidelity $F_{mm'}$ after the second measurement as a function of the outcomes m and m' ($|c_{1/2}|^2=|c_{-1/2}|^2=1/2$, $j=10$, $g=0.25$, $\theta=\pi/6$, $\phi=\pi/6$). $F_{mm'}$ depends only on $m'+m$ with $F_{m,-m}=1$.

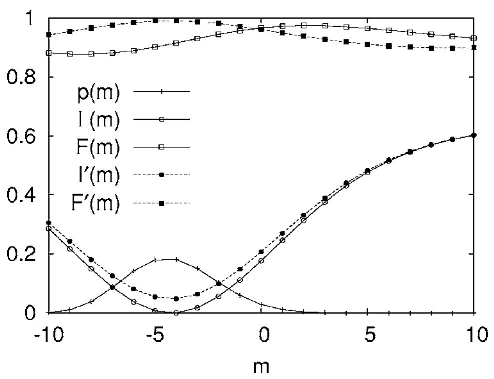

FIG. 6. Probability $p(m)$ of obtaining outcome m for the first measurement and the corresponding information gain $I(m)$ and fidelity $F(m)$, with $j=10$, $g=0.25$, $\theta=\pi/6$, $\phi=\pi/6$, and $\gamma=\pi/6$. Also shown are the expected information gain $I'(m)$ and expected fidelity $F'(m)$ after the reversing measurement, given that the outcome of the first measurement is m.

$$H_0 = -p(a)\log_2 p(a) - p(b)\log_2 p(b) = 1, \tag{59}$$

which is a measure of the lack of information about the system. We then perform the measurement $\{\hat{T}_m(\theta,\phi)\}$ in an attempt to obtain information about the system. If the input state of the system is $|a\rangle_q$, the measurement yields an outcome m with probability $p(m|a)$ and the postmeasurement state is given by $|a_m\rangle_q$ whose fidelity to $|a\rangle_q$ is $F(m,a)$. Here the probability $p(m|a)$, the postmeasurement state $|a_m\rangle_q$, and the fidelity $F(m,a)$ can be evaluated according to Eqs. (26), (30), and (31). Similarly, if the input state of the system is $|b\rangle_q$, the corresponding probability, the postmeasurement state, and the fidelity are given by $p(m|b)$, $|b_m\rangle_q$, and $F(m,b)$, respectively. The total probability for outcome m is $p(m)=p(m|a)p(a)+p(m|b)p(b)$. Suppose that we obtain the outcome m. Then Bayes' rule tells us that the probability that the input state is $|a\rangle_q$ [or $|b\rangle_q$] is given by $p(a|m)=p(m|a)p(a)/p(m)$ [or $p(b|m)=p(m|b)p(b)/p(m)$]. The Shannon entropy after the measurement with outcome m becomes

$$H(m) = -p(a|m)\log_2 p(a|m) - p(b|m)\log_2 p(b|m). \tag{60}$$

This means that the amount of information obtained from the outcome m is

$$I(m) = H_0 - H(m). \tag{61}$$

The average fidelity for a given outcome m is given by

$$F(m) = F(m,a)p(a|m) + F(m,b)p(b|m). \tag{62}$$

Figure 6 shows the probability for outcome $p(m)$, the information gain $I(m)$, and the fidelity $F(m)$ as functions of m for $j=10$, $g=0.25$, $\theta=\pi/6$, $\phi=\pi/6$, and $\gamma=\pi/6$. We find that an outcome that is realized with a high probability gives less information than one with a low probability.

After obtaining the outcome m for the measurement $\{\hat{T}_m(\theta,\phi)\}$, we perform the reversing measurement $\{\hat{T}_m(\pi-\theta,\pi-\phi)\}$ to recover the measured state. Let m' be the outcome of the reversing measurement. If the input state of the system is $|a\rangle_q$ before the first measurement $\{\hat{T}_m(\theta,\phi)\}$, the joint probability for a pair of outcomes (m,m') is given by $p(m,m'|a)$ and the corresponding postmeasurement state is $|a_{mm'}\rangle_q$ whose fidelity to $|a\rangle_q$ is $F(m,m',a)$. We can calculate the probability $p(m,m'|a)$, the postmeasurement state $|a_{mm'}\rangle_q$, and the fidelity $F(m,m',a)$ according to Eqs. (42), (41), and (50). Similarly, if the input state of the system is $|b\rangle_q$, the joint probability, the postmeasurement state, and the fidelity can be calculated to give $p(m,m'|b)$, $|b_{mm'}\rangle_q$, and $F(m,m',b)$, respectively. The total joint probability for a pair of outcomes (m,m') is $p(m,m')=p(m,m'|a)p(a)+p(m,m'|b)p(b)$. From the two outcomes (m,m'), we know that the input state is $|a\rangle_q$ with probability $p(a|m,m')=p(m,m'|a)p(a)/p(m,m')$ and is $|b\rangle_q$ with probability $p(b|m,m')=p(m,m'|b)p(b)/p(m,m')$. The Shannon entropy after the two measurements with outcomes (m,m') becomes

$$H(m,m') = -p(a|m,m')\log_2 p(a|m,m') - p(b|m,m')\log_2 p(b|m,m'). \tag{63}$$

The amount of obtained information is given by

$$I(m,m') = H_0 - H(m,m'), \tag{64}$$

and the fidelity becomes

$$F(m,m') = F(m,m',a)p(a|m,m') + F(m,m',b)p(b|m,m'). \tag{65}$$

When the two outcomes satisfy $m'=-m$, recovery is achieved by the reversing measurement, $F(m,-m,a)=F(m,-m,b)=1$. We cannot then obtain any information about the system because $p(m,-m|a)=p(m,-m|b)$—i.e., $I(m,-m)=0$ [note, however, that $I(m)>0$]. However, if $m'\sim -m$, we might expect a partial recovery should be achieved with some information loss. To check this, we consider the expectation value of the information to be obtained by performing the reversing measurement, given the outcome m of the first measurement with information $I(m)$. Since the conditional probability of obtaining outcome m' for the reversing measurement is $p(m'|m)=p(m,m')/p(m)$, the expectation value of the information is given by

$$I'(m) = \sum_{m'} p(m'|m)I(m,m'), \tag{66}$$

while the expectation value of the fidelity is given by

$$F'(m) = \sum_{m'} p(m'|m)F(m,m'). \tag{67}$$

The expectation value of the information gain $I'(m)$ and that of the fidelity $F'(m)$ are shown in Fig. 6 as functions of m for $j=10$, $g=0.25$, $\theta=\pi/6$, $\phi=\pi/6$, and $\gamma=\pi/6$. Note that $F'(m)>F(m)$ and $I'(m)>I(m)$ for several outcomes. This implies that the reversing measurement can achieve both a partial recovery of the quantum state and *further information gain* rather than information loss. The same statement holds true after the average over m is taken, that is, the

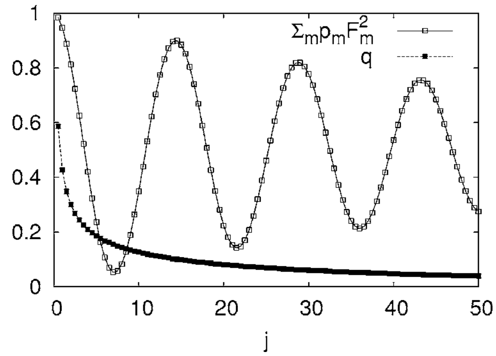

FIG. 7. Average squared fidelity after the first measurement $\Sigma_m p_m F_m^2$ and total probability of recovery q as functions of j ($|c_{1/2}|^2=|c_{-1/2}|^2=1/2$, $j=10$, $g=0.25$, $\theta=\pi/6$, $\phi=\pi/6$). Although the strength of the interaction g is much smaller ($\sim 10^{-8}$) in the real situation discussed in Sec. V, it can be enhanced by a cavity-assisted interaction or by collective enhancement via large j and s.

reversing measurement can achieve both $\Sigma_m p(m)F'(m) > \Sigma_m p(m)F(m)$ and $\Sigma_m p(m)I'(m) > \Sigma_m p(m)I(m)$.

C. Effect of probe spin

We discuss here the effect of a high-spin probe ($j>1/2$). In this case, the recovery of the measured state emerges more clearly because of the large change in the fidelity during the measurements. To simplify the calculations, we consider here the average squared fidelity after the first measurement, given by

$$\sum_m p_m F_m^2 = |c_{1/2}|^4 + |c_{-1/2}|^4 + 2|c_{1/2}|^2|c_{-1/2}|^2 h(\theta)^j \cos[jk(\theta)], \tag{68}$$

with

$$h(\theta) \equiv 1 - \sin^2\theta \sin^2 g, \tag{69}$$

$$k(\theta) \equiv 2\arg[\cos g - i \sin g \cos\theta]. \tag{70}$$

Figure 7 shows the average squared fidelity as a function of j, exhibiting a damped oscillation with period $2\pi/|k(\theta)|$. The oscillation results from $\arg[a_{m\sigma}^{(j)}(\theta,\phi)]$, which changes the relative phase between the states $|1/2\rangle_q$ and $|-1/2\rangle_q$. When the probe has a high spin ($j>1/2$), a large fidelity can be lost as a result of the first measurement. In particular, the fidelity loss becomes maximal at $j\sim\pi/|k(\theta)|$. Nevertheless, such a large fidelity loss can be recovered as a result of the second measurement, as discussed in the preceding subsection.

Of course, as a trade-off, the total probability of recovery, Eq. (49), becomes small, as shown in Fig. 7. For large j, the central limit theorem (25) gives an exponential decay of the probability of recovery,

$$q \sim \frac{1}{\sqrt{2\pi j v(\theta,\phi)}} \exp\left[-\frac{j[\chi_{1/2}(\theta,\phi) - \chi_{-1/2}(\theta,\phi)]^2}{2v(\theta,\phi)}\right], \tag{71}$$

where

$$v(\theta,\phi) \equiv 1 - \frac{1}{2}[\chi_{1/2}(\theta,\phi)^2 + \chi_{-1/2}(\theta,\phi)^2]. \tag{72}$$

This decay results from the fact that as j increases, the two peaks (45) of the joint probability split away from the line of recovery ($m'=-m$) and therefore the probability on the line decreases exponentially. Similarly, the total probability of approximate recovery, Eq. (56), also decreases exponentially as j increases, since the increase of j cannot expand the width (54) for approximate recovery. Due to the decrease in the probability of recovery, the average squared fidelity after the second measurement also decreases as

$$\sum_{m,m'} p_{mm'} F_{mm'}^2 = |c_{1/2}|^4 + |c_{-1/2}|^4 + 2|c_{1/2}|^2|c_{-1/2}|^2 h(\theta)^{2j}. \tag{73}$$

This fidelity does not oscillate, unlike the case in Eq. (68), because the change in the relative phase during the first measurement is on average canceled by that during the second measurement.

D. Quantum fluctuation of probe spin

So far, the spin j of the probe has been assumed to be a definite value. However, some physical systems are described by indefinite spin. For example, a two-mode laser is regarded as a spin system with indefinite spin because of quantum fluctuations in the number of photons (see Sec. V). We here show that even when the spin of the probe is affected by quantum fluctuations, the measurement $\{\hat{T}_m(\pi-\theta,\pi-\phi)\}$ remains a reversing measurement of the measurement $\{\hat{T}_m(\theta,\phi)\}$.

When the probe spin j fluctuates quantum mechanically, the initial probe state (13) is replaced with

$$|\theta,\phi\rangle_p = \exp\left(-\frac{i}{\hbar}\hat{J}_z\phi\right)\exp\left(-\frac{i}{\hbar}\hat{J}_y\theta\right)\sum_j b_j|j\rangle_p, \tag{74}$$

where $j=0,1/2,1,3/2,\ldots$ and the coefficients $\{b_j\}$ satisfy the normalization condition $\Sigma_j|b_j|^2=1$. Note that a measurement yielding an outcome m $(=0,\pm1/2,\pm1,\pm3/2,\ldots)$ eliminates probe states with $j\neq|m|,|m|+1,|m|+2,\ldots$, since

$$\sum_j \sum_{m=-j}^{j} = \sum_m {\sum_{j\geq|m|}}', \tag{75}$$

where the prime indicates summation over j such that $j-|m|$ is a non-negative integer. The measurement operators (38) and (40) are thus replaced with

$${}_q\langle\sigma'|\hat{T}_m(\theta,\phi)|\sigma\rangle_q = {\sum_{j\geq|m|}}' b_j \begin{pmatrix} a_{m,1/2}^{(j)}(\theta,\phi) & 0 \\ 0 & a_{m,-1/2}^{(j)}(\theta,\phi) \end{pmatrix} \tag{76}$$

and

$$
{}_q\langle\sigma'|\hat{T}_{m'}(\pi-\theta,\pi-\phi)|\sigma\rangle_q = e^{-im'\pi}\sum_{j\geq|m'|}{}'b_j\begin{pmatrix} a^{(j)}_{-m',-1/2}(\theta,\phi) & 0 \\ 0 & a^{(j)}_{-m',1/2}(\theta,\phi)\end{pmatrix}, \tag{77}
$$

respectively. It is easy to see that $\hat{T}_{-m}(\pi-\theta,\pi-\phi)\hat{T}_m(\theta,\phi)$ is proportional to the identity operator,

$$
\hat{T}_{-m}(\pi-\theta,\pi-\phi)\hat{T}_m(\theta,\phi) = \left[e^{im\pi}\sum_{j\geq|m|}{}'\sum_{j'\geq|m|}{}'b_jb_{j'}a^{(j)}_{m,-1/2}(\theta,\phi)a^{(j')}_{m,1/2}(\theta,\phi)\right]\hat{I}. \tag{78}
$$

Consequently, the measurement $\{\hat{T}_m(\pi-\theta,\pi-\phi)\}$ is still a reversing measurement of the measurement $\{\hat{T}_m(\theta,\phi)\}$ in the presence of quantum fluctuations. In contrast, the measurement $\{\hat{T}_m(\pi-\theta,\pi-\phi)\}$ is no longer a reversing measurement of the measurement $\{\hat{T}_m(\theta,\phi)\}$ if the probe spin is affected by classical fluctuations that replace the probe state (13) with a mixed state.

This tolerance for quantum fluctuation of the probe spin is emphasized when we consider the measurement $\{\hat{T}_m(\pi-\theta,-\phi)\}$. This is another reversing measurement of the measurement $\{\hat{T}_m(\theta,\phi)\}$, since

$$
a^{(j)}_{m'\sigma}(\pi-\theta,-\phi) = (-1)^{j+m'}a^{(j)}_{m',-\sigma}(\theta,\phi) \tag{79}
$$

holds, rather than Eq. (39). The measured state is recovered if the outcome of the second measurement is the same as that of the first ($m'=m$). As long as the spin j of the probe has a definite value, this reversing measurement is equivalent to the measurement $\{\hat{T}_m(\pi-\theta,\pi-\phi)\}$. However, when the probe spin is affected by quantum fluctuation, the measurement $\{\hat{T}_m(\pi-\theta,-\phi)\}$ is no longer a reversing measurement of the measurement $\{\hat{T}_m(\theta,\phi)\}$ due to the j-dependent factor $(-1)^{j+m'}$ in Eq. (79).

IV. REVERSING MEASUREMENT ON A HIGH-SPIN SYSTEM ($s>1/2$)

We next consider a reversing measurement of a physically reversible measurement $\{\hat{T}_m(\theta,\phi)\}$ for the case of measured systems with $s>1/2$. Provided that the condition (37) is satisfied, the physical reversibility implies the existence of a reversing measurement [3,4]. More specifically, for a first measurement with outcome m, we consider a second measurement $\{\hat{R}^{(m)}_0,\hat{R}^{(m)}_1\}$ with two possible outcomes—say, 0 and 1—such that

$$
\hat{R}^{(m)}_0 = \kappa_m\sum_{\sigma=-s}^{s}[a^{(j)}_{m\sigma}(\theta,\phi)]^{-1}|\sigma\rangle_{qq}\langle\sigma|, \tag{80}
$$

$$
\hat{R}^{(m)}_1 = \sqrt{\hat{I}-\hat{R}^{(m)\dagger}_0\hat{R}^{(m)}_0}, \tag{81}
$$

where κ_m is a nonzero constant. If this measurement yields the outcome 0, the original state of the system is restored because

$$
\hat{R}^{(m)}_0\hat{T}_m(\theta,\phi) = \kappa_m\hat{I}, \tag{82}
$$

as seen from Eq. (34). Unfortunately, the physical implementation of this measurement is not obvious. Instead, we consider an approximate reversing measurement that has a clear physical implementation using the measurement $\{\hat{T}_m(\pi-\theta,\pi-\phi)\}$. Unlike the case of $s=1/2$, the measurement $\{\hat{T}_m(\pi-\theta,\pi-\phi)\}$ is not an exact reversing measurement, since $\hat{T}_{-m}(\pi-\theta,\pi-\phi)$ is not proportional to the inverse of $\hat{T}_m(\theta,\phi)$. Contrary to Eq. (48), we have

$$
\hat{T}_{-m}(\pi-\theta,\pi-\phi)\hat{T}_m(\theta,\phi)\not\propto\hat{I}. \tag{83}
$$

Nevertheless, there are two physical situations in which the measurement $\{\hat{T}_m(\pi-\theta,\pi-\phi)\}$ serves approximately as a reversing measurement for the original measurement $\{\hat{T}_m(\theta,\phi)\}$: (i) the measured state can be confined to a two-dimensional subspace or (ii) the interaction between the system and probe is sufficiently weak. In this section, we describe these approximate schemes for the reversing measurement.

A. Two-dimensional subspace model

We assume that the initial state of the measured system with spin s is in a two-dimensional subspace spanned by $\{|\tilde{\sigma}\rangle,|-\tilde{\sigma}\rangle\}$, where $\tilde{\sigma}\hbar$ is any one of the nonzero eigenvalues of $\hat{S}_z$. That is, we know *a priori* that

$$
|\psi\rangle_q = \sum_{\sigma=\pm\tilde{\sigma}}c_\sigma|\sigma\rangle_q, \tag{84}
$$

instead of the general state (9). Since the measurement operator is diagonal, as in Eq. (34), the state of the system remains in this subspace after the measurement. The measurement operators $\hat{T}_m(\theta,\phi)$ and $\hat{T}_{m'}(\pi-\theta,\pi-\phi)$ are thus represented by 2×2 matrices within this subspace. These matrices are identical to those in the $s=1/2$ case [see Eqs. (38) and (40)] with the strength of the interaction given by

$$
g' = 2g\tilde{\sigma}. \tag{85}
$$

Consequently, the measurement $\{\hat{T}_m(\pi-\theta,\pi-\phi)\}$ is a reversing measurement of the measurement $\{\hat{T}_m(\theta,\phi)\}$ when the initial state of the system is confined to the two-dimensional subspace.

The analysis of fidelity in this model is the same as that in the case where $s=1/2$ in the preceding section, provided that the renormalized strength of the interaction (85) is used. The remaining problem is preparing the system in the two-dimensional subspace. In order to prepare the state (84), we here use the scheme in Ref. [21], which was originally proposed to realize a squeezed spin state [23]. The system is first prepared in the state

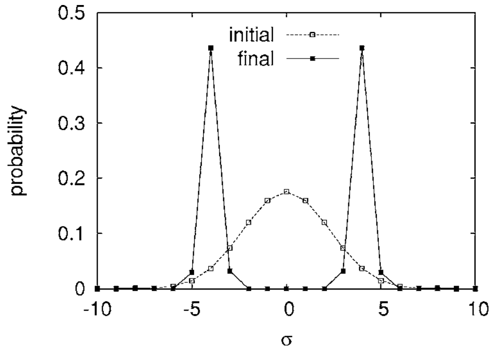

FIG. 8. Initial spin distribution $|c'_\sigma|^2$ and final spin distribution $\rho_m(\sigma)$ as functions of σ ($j=s=10$, $g=0.25$, $m=5$). $\rho_m(\sigma)$ has a pair of highest peaks at $\sigma=\pm 4$ (the other peaks are too small to be seen on the scale of this figure). The probability p'_m in Eq. (88) is calculated to be 0.016.

$$|\psi'\rangle_q = \exp\left(-\frac{i}{\hbar}\hat{S}_z\varphi\right)\exp\left(-\frac{i}{\hbar}\hat{S}_y\frac{\pi}{2}\right)|s\rangle_q = \sum_\sigma c'_\sigma|\sigma\rangle_q, \tag{86}$$

where

$$c'_\sigma \equiv e^{-i\sigma\varphi}d^{(s)}_{\sigma s}\left(\frac{\pi}{2}\right) = e^{-i\sigma\varphi}\frac{1}{2^s}\sqrt{\frac{(2s)!}{(s+\sigma)!\,(s-\sigma)!}}. \tag{87}$$

This is a coherent spin state [24] and is an eigenstate of the spin component $\hat{S}_\varphi=\hat{S}_x\cos\varphi+\hat{S}_y\sin\varphi$ with eigenvalue $s\hbar$. Performing the measurement $\{\hat{T}_m(\pi/2,0)\}$ on this state yields an outcome m with probability

$$p'_m = \sum_\sigma \left|a^{(j)}_{m\sigma}\left(\frac{\pi}{2},0\right)\right|^2|c'_\sigma|^2 \tag{88}$$

and then causes state reduction to

$$|\psi'_m\rangle_q = \frac{1}{\sqrt{p'_m}}\sum_\sigma a^{(j)}_{m\sigma}\left(\frac{\pi}{2},0\right)c'_\sigma|\sigma\rangle_q. \tag{89}$$

The spin distribution of this state $\rho_m(\sigma)\equiv|{}_q\langle\sigma|\psi'_m\rangle_q|^2$ is given by

$$\begin{aligned}\rho_m(\sigma) &= \frac{1}{p'_m}\left|a^{(j)}_{m\sigma}\left(\frac{\pi}{2},0\right)\right|^2|c'_\sigma|^2\\ &= \left[\frac{1}{p'_m}\frac{(2j)!}{(j+m)!\,(j-m)!}\right]\left[\frac{1}{2^{2s}}\frac{(2s)!}{(s+\sigma)!\,(s-\sigma)!}\right]\\ &\quad\times[\cos^2(g\sigma)]^{j-m}[\sin^2(g\sigma)]^{j+m}.\end{aligned} \tag{90}$$

Clearly, this distribution satisfies $\rho_m(0)=0$ (if $j\neq -m$) and $\rho_m(\sigma)=\rho_m(-\sigma)$, and is damped by the second binomial factor for large $|\sigma|$. These facts imply that when $j\neq -m$, the spin distribution has a pair of highest peaks at $\sigma=\pm\tilde{\sigma}_m$ (see Fig. 8), where $\tilde{\sigma}_m$ is evaluated as

$$\tilde{\sigma}_m \sim \frac{1}{g}\arctan\sqrt{\frac{j+m}{j-m}} \tag{91}$$

if $g\ll\pi/2<gs$. The state (89) can thus be approximated as

$$|\psi'_m\rangle_q \sim \frac{1}{\sqrt{2}}[e^{-i\tilde{\sigma}_m\varphi}|\tilde{\sigma}_m\rangle_q + (-1)^{j+m}e^{i\tilde{\sigma}_m\varphi}|-\tilde{\sigma}_m\rangle_q], \tag{92}$$

where the relative phase is determined from the identity $a^{(j)}_{m,-\sigma}(\pi/2,0)=(-1)^{j+m}a^{(j)}_{m\sigma}(\pi/2,0)$. According to Eq. (90), this is a good approximation for large j. Finally, by performing a further measurement $\{\hat{T}_m(\theta',\phi')\}$ on this state, we can prepare a state in the form of

$$|\psi\rangle_q = \sum_{\sigma=\pm\tilde{\sigma}_m} c_\sigma|\sigma\rangle_q, \tag{93}$$

where the coefficients depend on the angles (θ',ϕ') and the outcome.

B. Weak-interaction model

We next consider another physical situation for the approximate reversing measurement $\{\hat{T}_m(\pi-\theta,\pi-\phi)\}$. We assume that the interaction is so weak that the measurement operators can be expanded in powers of g. We then obtain

$$\hat{T}_{-m}(\pi-\theta,\pi-\phi)\hat{T}_m(\theta,\phi) \sim [e^{im\pi}a^{(j)}_{m,0}(\theta,\phi)^2]\hat{I} + O(g^2). \tag{94}$$

This means that the measurement $\{\hat{T}_m(\pi-\theta,\pi-\phi)\}$ is a reversing measurement of the measurement $\{\hat{T}_m(\theta,\phi)\}$ to an accuracy of the order of g.

As shown below, the second-order term, which is neglected in Eq. (94), does not affect the fidelity up to the order of g^3. For the two successive measurements $\{\hat{T}_m(\theta,\phi)\}$ and $\{\hat{T}_m(\pi-\theta,\pi-\phi)\}$, we define the joint probability, the final state, and the fidelity, as in the case of $s=1/2$, by

$$p_{mm'} = \sum_\sigma |a^{(j)}_{-m',-\sigma}(\theta,\phi)a^{(j)}_{m\sigma}(\theta,\phi)|^2|c_\sigma|^2, \tag{95}$$

$$|\psi_{mm'}\rangle_q = \frac{e^{-im'\pi}}{\sqrt{p_{mm'}}}\sum_\sigma a^{(j)}_{-m',-\sigma}(\theta,\phi)a^{(j)}_{m\sigma}(\theta,\phi)c_\sigma|\sigma\rangle_q, \tag{96}$$

and

$$F_{mm'} = \frac{1}{\sqrt{p_{mm'}}}\left|\sum_\sigma a^{(j)}_{-m',-\sigma}(\theta,\phi)a^{(j)}_{m\sigma}(\theta,\phi)|c_\sigma|^2\right|, \tag{97}$$

respectively, using the relation (39). Expanding the fidelity up to the second order in g, we obtain

$$\begin{aligned}F_{mm'} &\sim 1-\frac{1}{20}\left[\frac{\overline{\sigma^2}-(\bar{\sigma})^2}{s^2}\right]\left(\frac{m'+m}{\widetilde{\delta m}(\theta,\phi)}\right)^2\\ &\geq 1-\frac{1}{20}\left(\frac{m'+m}{\widetilde{\delta m}(\theta,\phi)}\right)^2,\end{aligned} \tag{98}$$

where

$$\bar{\sigma} \equiv \sum_\sigma \sigma|c_\sigma|^2, \quad \overline{\sigma^2} \equiv \sum_\sigma \sigma^2|c_\sigma|^2, \tag{99}$$

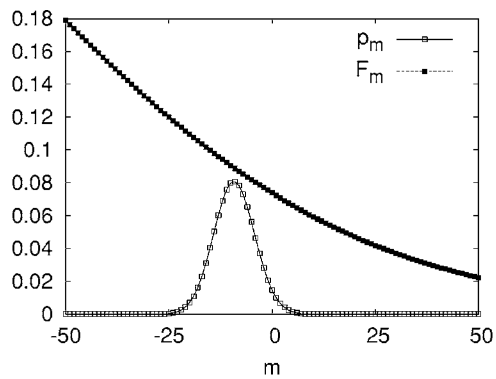

FIG. 9. Probability p_m and fidelity F_m of the first measurement on the state $|S_x=s\hbar\rangle_q$ as functions of the outcome m ($j=50$, $s=10$, $g=0.01$, $\theta=\pi/12$, $\phi=\pi/4$).

$$\widetilde{\delta m}(\theta,\phi) \equiv \frac{1}{2\sqrt{10}s}\left(\frac{\sqrt{1-\sin^2\theta\cos^2\phi}}{|g\sin\theta|}\right). \qquad (100)$$

Consequently, we find that when the two outcomes cancel each other ($m'=-m$), the information about the original state is restored to within the accuracy of g^3, because $F_{m,-m}\sim 1+O(g^4)$. The measurement $\{\hat{T}_m(\pi-\theta,\pi-\phi)\}$ is thus a reversing measurement of the measurement $\{\hat{T}_m(\theta,\phi)\}$ if the fourth-order term in g can be neglected. Evaluating the fourth-order term, we obtain the condition for the strength of the interaction as

$$g^4 \ll \frac{1}{s^4j^2}\left(\frac{1-|\sin\theta\cos\phi|}{\sqrt{2}\sin\theta}\right)^2. \qquad (101)$$

As in the case of $s=1/2$, we define the total probability of approximate recovery by

$$q' = \sum_{\substack{m,m' \\ F_{mm'}\geq 0.95}} p_{mm'}, \qquad (102)$$

where a sufficient condition for $F_{mm'}\geq 0.95$ is given by

$$|m'+m| \leq \widetilde{\delta m}(\theta,\phi). \qquad (103)$$

As an example, we consider the case where $j=50$, $s=10$, $g=0.01$, $\theta=\pi/12$, and $\phi=\pi/4$. The measured system is assumed to be in a coherent spin state

$$|\psi\rangle_q = |S_x = s\hbar\rangle_q = \exp\left(-\frac{i}{\hbar}\hat{S}_y\frac{\pi}{2}\right)|s\rangle_q, \qquad (104)$$

which is the eigenstate of $\hat{S}_x$ with eigenvalue $s\hbar$. Figure 9 shows the probability (26) and the fidelity (31) of the first measurement $\{\hat{T}_m(\pi/12,\pi/4)\}$ as functions of the outcome m. The average fidelity after the first measurement is $\Sigma_m p_m F_m=0.089$. The second measurement $\{\hat{T}_m(11\pi/12, 3\pi/4)\}$ is then performed. Figure 10 shows the probability (95) as a function of the outcomes m for the first measurement and m' for the second measurement. Figure 11 shows the fidelity (97) after the second measurement as a function of the outcomes m and m'. Although the fidelity $F_{mm'}$ may

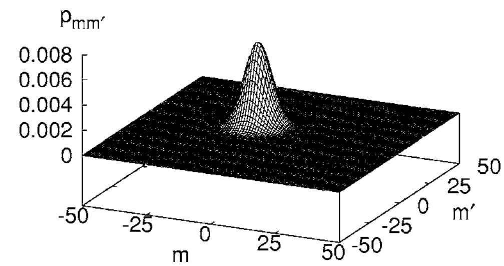

FIG. 10. Joint probability $p_{mm'}$ of the first and second measurements on the state $|S_x=s\hbar\rangle_q$ as a function of the outcomes m and m' ($j=50$, $s=10$, $g=0.01$, $\theta=\pi/12$, $\phi=\pi/4$).

depend on j and on $m'-m$ if $s>1/2$, it approximately depends only on $m'+m$ [see Eq. (98)], owing to the weak-interaction condition (101). The average fidelity after the second measurement is $\Sigma_{mm'}p_{mm'}F_{mm'}=0.997$. The width (100) and the total probability of approximate recovery (102) are given by $\widetilde{\delta m}(\theta,\phi)=6.0$ and $q'=0.999\,92$, respectively. Surprisingly, the measured state can be recovered almost deterministically, though a large portion of the fidelity is lost upon the first measurement, as shown in Fig. 9. This unexpectedly large reversibility is discussed in detail in the next subsection.

C. Reversibility in the weak-interaction model

The weak-interaction model exhibited near-deterministic recovery of a coherent spin state (104). The reasons for this considerable reversibility are that the measurements $\{\hat{T}_m(\theta,\phi)\}$ and $\{\hat{T}_m(\pi-\theta,\pi-\phi)\}$ commute with the spin z component, as shown in Eq. (36), and that the interaction between the system and probe is weak. Such a measurement does not greatly disturb a state with a small variance of the spin z component,

$$\langle\Delta\hat{S}_z^2\rangle \equiv [\overline{\sigma^2}-(\bar{\sigma})^2]\,\hbar^2. \qquad (105)$$

In fact, when the variance is small, the average fidelity after the second measurement is large, as in

$$\sum_{m,m'} p_{mm'}F_{mm'} \sim 1-2g^2j[\overline{\sigma^2}-(\bar{\sigma})^2]\sin^2\theta, \qquad (106)$$

to the second order in g. The coherent spin state (104) can thus be recovered near-deterministically because of its small

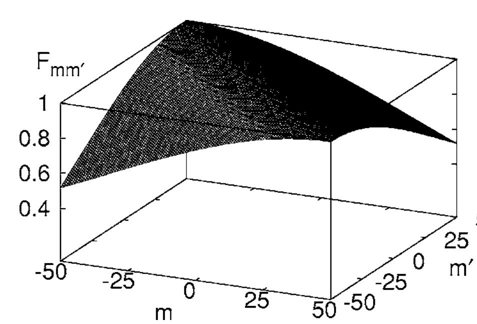

FIG. 11. Fidelity $F_{mm'}$ after the two measurements on the state $|S_x=s\hbar\rangle_q$ as a function of the outcomes m and m' ($j=50$, $s=10$, $g=0.01$, $\theta=\pi/12$, $\phi=\pi/4$).

variance of $s\hbar^2/2$, not on the order of $s^2\hbar^2$. Therefore, a considerable reversibility is obtained for other spin states as long as their variances are small. For example, a Schrödinger cat state between the eigenstates of $\hat{S}_x$ with eigenvalues $+s\hbar$ and $-s\hbar$,

$$|\psi\rangle_q = c_+|S_x = +s\hbar\rangle_q + c_-|S_x = -s\hbar\rangle_q, \qquad (107)$$

has the same variance as state (104) and can thus be recovered in a near-deterministic way *without any knowledge about c_+ or c_-*. In contrast, a cat state between the eigenstates of $\hat{S}_z$ with eigenvalues $+s\hbar$ and $-s\hbar$,

$$|\psi\rangle_q = c_+|s\rangle_q + c_-|-s\rangle_q, \qquad (108)$$

has a large variance, on the order of $s^2\hbar^2$, which decreases the probability of approximate recovery, Eq. (102). For the previous example ($j=50$, $s=10$, $g=0.01$, $\theta=\pi/12$, and $\phi=\pi/4$), the probability of approximate recovery for the cat state (107) gives $q'=0.999\,92$ independent of c_+ and c_-, while it is $q'=0.59$ for the cat state (108) in the worst case ($|c_+|^2=|c_-|^2=1/2$), which is still high.

The above discussion is based on the fact that the *joint* measurement $\{\hat{T}_m(\theta,\phi)\}$ and $\{\hat{T}_m(\pi-\theta,\pi-\phi)\}$ changes the measured state little. One might think therefore that the measured state is changed little throughout the whole measurement process. It should, however, be recalled that the first measurement $\{\hat{T}_m(\theta,\phi)\}$ can change the measured state substantially (see Fig. 9) through the high spin j of the probe. The average fidelity after the first measurement is given by

$$\sum_m p_m F_m \sim 1 - g^2 j[\overline{\sigma^2} - (\bar{\sigma})^2](\sin^2\theta + 2j\cos^2\theta) \qquad (109)$$

to the second order in g. As j increases, this average fidelity becomes small, even if the strength of the interaction g is decreased as $g^2\sim 1/j$, in accordance with the weak-interaction condition (101). [Of course, Eq. (109) is not valid when j is so large that the second term becomes comparable to 1.] The term of order g^2j^2 originates from $\arg[a^{(j)}_{m\sigma}(\theta,\phi)]$ rather than $|a^{(j)}_{m\sigma}(\theta,\phi)|$; the former changes the relative phases between the states $\{|\sigma\rangle_q\}$, while the latter changes the spin distribution $|{}_q\langle\sigma|\psi\rangle_q|^2$. If $a^{(j)}_{m\sigma}(\theta,\phi)$ were $|a^{(j)}_{m\sigma}(\theta,\phi)|$, thereby leaving the relative phases invariant, the average fidelity would be given by

$$1 - g^2 j[\overline{\sigma^2} - (\bar{\sigma})^2]\frac{\sin^2\theta\sin^2\phi}{1-\sin^2\theta\cos^2\phi}, \qquad (110)$$

which includes no term of order g^2j^2. On the other hand, the change in the measured state by the joint measurement $\{\hat{T}_m(\theta,\phi)\}$ and $\{\hat{T}_m(\pi-\theta,\pi-\phi)\}$ remains small, since the effect of the second measurement can also be amplified by the high-spin probe so as to cancel that of the first measurement. The average fidelity after the second measurement thus has no term of order g^2j^2, as in Eq. (106). As a result, in spite of the near-deterministic recovery by the weak measurements, the change in fidelity can be drastic due to the action of the high-spin probe.

V. POSSIBLE EXPERIMENTAL SITUATION

Finally, we describe a possible experimental situation for our reversible spin measurement. Consider an ensemble of atoms as a measured system. Each atom possesses a doubly degenerate ground state, which is regarded as a spin-1/2 system. Provided that the initial state and dynamics are totally symmetric under the interchange of atoms, the ensemble of atoms can be described by the total spin operator

$$\hat{\mathbf{S}} = \sum_{i=1}^{N_a} \hat{\mathbf{s}}^{(i)}, \qquad (111)$$

where $\hat{\mathbf{s}}^{(i)}$ is the spin operator of the ith atom and N_a is the number of atoms. In this case, the spin of the system is given by $s=N_a/2$. In addition, we consider the polarization of $2j$ photons as a probe system. This system can also be described by the spin operators [18]

$$\hat{J}_x \equiv \frac{\hbar}{2}(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1),$$

$$\hat{J}_y \equiv \frac{\hbar}{2i}(\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_1),$$

$$\hat{J}_z \equiv \frac{\hbar}{2}(\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2), \qquad (112)$$

where $\hat{a}_\lambda$ is the annihilation operator for the photon of mode λ (1=horizontal, 2=vertical). These operators obey the commutation relations (11) because

$$[\hat{a}_\lambda,\hat{a}^\dagger_{\lambda'}] = \delta_{\lambda\lambda'}, \quad [\hat{a}_\lambda,\hat{a}_{\lambda'}] = 0. \qquad (113)$$

The total spin of this probe is given by $j=(N_1+N_2)/2$, where N_λ is the number of photons with mode λ. The probe state $|m\rangle_p$ corresponds to the number state $|N_1=j+m, N_2=j-m\rangle$ of photons. The initial state (13) can then be simply prepared, since the operators $\exp(-i\hat{J}_y\theta/\hbar)$ and $\exp(-i\hat{J}_z\phi/\hbar)$ correspond to the half-wave plate $\exp[-\theta/2(\hat{a}_1^\dagger\hat{a}_2-\hat{a}_2^\dagger\hat{a}_1)]$ and the phase shifter $\exp[-i\phi/2(\hat{a}_1^\dagger\hat{a}_1-\hat{a}_2^\dagger\hat{a}_2)]$, respectively. The interaction (16) can be realized by using the paramagnetic Faraday rotation [11,19–21]. The unitary operator (17) corresponds to a half-wave plate, and the projective measurement of the probe variable $\hat{J}_z$ is achieved by two photodetectors for the two modes. In this way, we can implement the reversible spin measurement.

For the purpose of a quantitative understanding, we follow the estimation in Ref. [21]. For an ensemble of $N_a\sim 10^8$ cesium atoms trapped and cooled by laser beams and with the two-mode photons being laser beams with average photon number $N_\lambda\sim 10^8$, the spins s and j are both on the order of 10^8, while the strength of the interaction g is about 10^{-8}. Then, the weak-interaction condition (101) is satisfied for a very small angle $\theta\sim 10^{-8}$. This means that, with a half-wave plate rotated by a very small angle, we can apply the weak-interaction model of the reversible and reversing measurements for a high-spin system. Since the width (100) is on the order of 10^8, the probability of approximate recovery is expected to be large. Conversely, when θ is large, $N_a\sqrt{N_\lambda}$

should be on the order of 10^8 to satisfy the weak-interaction condition.

VI. CONCLUSIONS

We have proposed a physically reversible quantum measurement on a spin-s system using a spin-j probe, along with an experimentally feasible interaction that can experimentally realize reversibility in quantum measurements. The physical reversibility resulting from the reversing measurement allows the unknown premeasurement state to be recovered from the postmeasurement state. For a spin-1/2 system ($s=1/2$), we have analyzed an exact reversing measurement using fidelity as a measure of recovery, giving a criterion for more than 95% recovery of the measured state. We have found that a high-spin probe ($j>1/2$) drastically changes fidelity during the reversible and reversing measurements, and thus enhances the recovery of the quantum state, though reducing the probability of success. On the other hand, for a high-spin system ($s>1/2$), we have investigated an approximate reversing measurement instead of an exact one, in view of physical implementation. We have then shown that the reversing measurement for a spin-1/2 system is an approximate reversing measurement for a high-spin system ($s>1/2$) when the measured system is initially in a two-dimensional subspace or when the interaction is sufficiently weak. Notably, in the weak-interaction case, even a cat state can be recovered near-deterministically in spite of there being a large change in fidelity.

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