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Perturbation hydrogen-atom spectrum in deformed space with minimal length

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We studied energy spectrum for the hydrogen atom with deformed Heisenberg algebra leading to the minimal length. We developed the correct perturbation theory free of divergences. It gives a possibility to calculate analytically in the three-dimensional case the corrections to *s* levels of the hydrogen atom caused by the minimal length. Comparing our results with the experimental data from precision hydrogen spectroscopy an upper bound for the minimal length is obtained.

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I. INTRODUCTION

Recently a lot of attention has been attracted to the quantum mechanical problems linked with generalized (modified) commutation relations. Such an interest was motivated by the works on the quantum gravity and the string theory which suggested the existence of a finite lower bound to the possible resolution of length (minimal length) [1–3]. Kempf *et al.* showed that minimal length can be obtained from the deformed Heisenberg algebra [4–8]. Note that the deformed algebra leading to quantized space time was introduced by Snyder in the relativistic case [9]. In the D-dimensional case the deformed algebra proposed by Kempf reads

$$[X_i,P_j]=i\hbar[\delta_{ij}(1+\beta P^2)+\beta' P_iP_j],\ [P_i,P_j]=0,$$

$$[X_i,X_j]=i\hbar\frac{(2\beta-\beta')+(2\beta+\beta')\beta P^2}{1+\beta P^2}(P_iX_j-P_jX_i), \quad (1)$$

where β,β' are parameters of deformation. We suppose that these quantities are positive $\beta,\beta'\geqslant 0$. It can be seen that position operators do not commute so we have a noncommutative space. From the uncertainty relation it follows that minimal length is $\hbar\sqrt{\beta+\beta'}$. Note that in the special case $\beta'=2\beta$ the position operators in linear approximation over deformation parameters commute, i.e., $[X_i,X_j]=0$.

The hydrogen atom is one of the simplest quantum systems allowing highly accurate theoretical prediction and is well studied experimentally offering the most precisely measured quantities [10]. Therefore, this simple atom has a crucial role for our understanding of the key points of modern physics. Due to the singularity of the Coulomb potential at the origin this system is in particular sensitive to whether there is a fundamental minimal length. There are only a few papers on the investigation of the hydrogen atom in quantum space with minimal length [11–13].

Brau [11] considered the special case of deformation $\beta'=2\beta$ and in the linear approximation over the deformation parameters the energy spectrum of hydrogen atom was calculated. The general case of deformation $\beta'\neq 2\beta$ was studied in Ref. [12]. Using the perturbation theory the authors calculated the correction to the energy spectrum of the hydrogen atom. But in order to calculate the corrections to the *s* levels for three-dimensional space the authors were forced to use a numerical method and cutoff procedure due to the appearance once of the term $\sim 1/r^3$ in the Hamiltonian in the linear approximation over β,β'. We would like to emphasize that without the cutoff procedure this term leads to the divergence of the correction to the *s* levels.

Note also paper [14] where the comparison between the "space curvature" effects and minimal length effects for the hydrogen spectrum was made. In Ref. [15] a one-dimensional Coulomb problem was solved exactly.

In the present paper we propose the modified perturbation theory free of divergences which gives a possibility to calculate the corrections to all energy levels including *s* levels. This paper is organized as follows. In the second section we obtain corrections to the spectrum of D-dimensional Coulomb problem using the ordinal perturbation theory. In the third section we propose a modified perturbation theory and calculate corrections to the energy of the *s* levels of hydrogen in the three-dimensional case. And finally the fourth section contains the discussion.

II. PERTURBATION OF THE ENERGY SPECTRUM

In this section we consider the ordinal perturbation theory similarly as in Ref. [12] but using another representation of the deformed algebra. This algebra is more convenient for the development of the modified perturbation theory.

We studied the eigenvalue problem for the hydrogen atom in the D-dimensional case

$$\left(\frac{\mathbf{P}^2}{2m}-\frac{e^2}{R}\right)\Psi=E\Psi, \quad (2)$$

where operators of position X_i and momentum P_i satisfy the deformed commutation relation (1), $R=\sqrt{\sum_{i=1}^{D}X_i^2}$.

We use the following representation that satisfies the algebra (1) in the first order in β, β',

$$X_i=x_i+\frac{2\beta-\beta'}{4}(x_ip^2+p^2x_i),$$

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$$P_i = p_i + \frac{\beta'}{2} p_i p^2; \tag{3}$$

where $p^2 = \sum_{k=1}^{D} p_k^2$ and operators x_i, p_i obey canonical commutation relations $[x_i, p_j] = i\hbar\delta_{ij}$. For the undeformed Heisenberg algebra the position representation may be taken: $x_i = x_i$, $p_i = i\hbar\frac{\partial}{\partial x_i}$.

We write the Hamiltonian of Eq. (2) using representation (3) and taking into account only the first order terms in β, β',

$$H = \frac{p^2}{2m} + \frac{\beta' p^4}{2m} - \frac{e^2}{\sqrt{r^2 + \frac{2\beta - \beta'}{2}(r^2p^2 + p^2r^2 + \hbar^2 D)}}, \tag{4}$$

where $r = \sqrt{\sum_{i=1}^{D} x_i^2}$ and D is the dimension of space.

Expanding the inverse distance R^{-1} in the series over parameters of deformation up to the first order β, β' we have

$$H = \frac{p^2}{2m} + \frac{\beta' p^4}{2m} - e^2\left[\frac{1}{r} - \frac{2\beta - \beta'}{4}\left(\frac{1}{r}p^2 + p^2\frac{1}{r} + \frac{\hbar^2(D-1)}{r^3}\right)\right]. \tag{5}$$

This Hamiltonian contains the correction of the first order over β, β' to the undeformed hydrogen atom Hamiltonian.

Now we can calculate the corrections $\Delta E_{nl}^{(1)}$ to the spectrum using the eigenfunctions of the undeformed hydrogen atom

$$\Delta E_{nl}^{(1)} = \frac{e^2\hbar^2}{a^3 n^3}\left(\frac{(D-1)(2\beta - \beta')}{4\bar{l}(\bar{l}+1)\left(\bar{l}+\frac{1}{2}\right)} + \frac{2\beta + \beta'}{\bar{l}+\frac{1}{2}} - \frac{\beta + \beta'}{\bar{n}}\right), \tag{6}$$

where a is the Bohr radius, $\bar{n} = n + \frac{D-3}{2}$, $\bar{l} = l + \frac{D-3}{2}$, n is the principal quantum number, and l is the orbital quantum number.

Expression (6) is in agreement with the results calculated in the paper [12]. It is worth it to mention that in the special case $D = 3$ and $l = 0$ expression (6) gives the divergent contribution. It is caused by the term proportional to $1/r^3$ in Hamiltonian (5).

III. MODIFIED PERTURBATION THEORY. CORRECTIONS TO THE ENERGY OF *s*-LEVELS IN THREE-DIMENSIONAL CASE

In this section we propose a modified perturbation theory which gives a possibility to overcome the problem of divergence of the corrections to the s levels in three-dimensional case. The idea is to use a shifted expansion of inverse distance R^{-1} which does not contain divergent terms like $1/r^3$. So, we rewrite R as follows:

$$R = \sqrt{r^2 + b^2 + \alpha(r^2p^2 + p^2r^2 + \hbar^2 D - \bar{b}^2)}, \tag{7}$$

where $\alpha = (2\beta - \beta')/2$ and $b^2 = \alpha\bar{b}^2$. Next we consider the expansion over $\alpha(r^2p^2 + p^2r^2 + \hbar^2 D - \bar{b}^2)$ in the vicinity of point $r^2 + b^2$. We will choose the introduced parameter b from the condition that terms proportional to $1/r^3$ that are absent in the series.

In the first order over α we can write

$$\sqrt{r^2 + b^2 + \alpha(r^2p^2 + p^2r^2 + \hbar^2 D - \bar{b}^2)} = \sqrt{r^2 + b^2} + \alpha\hat{C}. \tag{8}$$

Squaring left and right hand side of (8) and taking into account only the term of first order over α, we obtain the following equation for operator $\hat{C}$:

$$r^2p^2 + p^2r^2 + \hbar^2 D - \bar{b}^2 = r\hat{C} + \hat{C}r. \tag{9}$$

In order to find $\hat{C}$ we write the left side of (9) in the following form:

$$r^2p^2 + p^2r^2 + \hbar^2 D - \bar{b}^2 = \frac{1}{2}\left[r\left(rp^2 + p^2r + \frac{A}{r}\right) + \left(rp^2 + p^2r + \frac{A}{r}\right)r\right], \tag{10}$$

where parameter $A = \hbar^2(D-1) - \bar{b}^2$.

It is straightforward to show using (10) that

$$\hat{C} = \frac{1}{2}\left(rp^2 + p^2r + \frac{A}{r}\right). \tag{11}$$

So we have the following expansion for the distance:

$$R = \sqrt{r^2 + b^2} + \frac{\alpha}{2}\left(rp^2 + p^2r + \frac{A}{r}\right). \tag{12}$$

It is easy to obtain the inverse distance R^{-1} using (12),

$$\frac{1}{R} = \frac{1}{\sqrt{r^2 + b^2}} - \frac{\alpha}{2\sqrt{r^2 + b^2}}\left(rp^2 + p^2r + \frac{A}{r}\right)\frac{1}{\sqrt{r^2 + b^2}} = \frac{1}{\sqrt{r^2 + b^2}} - \frac{\alpha}{2}\left(\frac{1}{r}p^2 + p^2\frac{1}{r} + \frac{A}{r^3}\right). \tag{13}$$

The contributions $1/\sqrt{r^2 + b^2}$ in the second term of expansion (13) can be replaced with $1/r$ in the linear approximation over α. We demand that our expansion does not contain the terms proportional to $1/r^3$ so we conclude that $A = 0$, i.e.,

$$b = \hbar\sqrt{\alpha(D-1)}. \tag{14}$$

It should be noted that (13) takes place under condition $b^2 > 0$, i.e., $2\beta > \beta'$.

We rewrite Hamiltonian (4) applying expansion (13) as follows:

$$H = \frac{p^2}{2m} + \frac{\beta' p^4}{2m} - e^2\left[\frac{1}{\sqrt{r^2 + b^2}} - \frac{2\beta - \beta'}{4}\left(\frac{1}{r}p^2 + p^2\frac{1}{r}\right)\right] = H_0 + V, \tag{15}$$

where H_0 is the Hamiltonian of the undeformed hydrogen atom and perturbation caused by the deformation is

$$V = \frac{\beta' p^4}{2m} - e^2\left[\frac{1}{\sqrt{r^2 + b^2}} - \frac{1}{r} - \frac{2\beta - \beta'}{4}\left(\frac{1}{r}p^2 + p^2\frac{1}{r}\right)\right]. \tag{16}$$

Now we calculate the correction to the ground state energy of the hydrogen atom caused by perturbation V. First let us consider correction which is connected with $1/\sqrt{r^2+b^2}-1/r$. We have

$$\left\langle \Psi_{1s} \left| \frac{1}{\sqrt{r^2+b^2}} \right| \Psi_{1s} \right\rangle = \frac{4}{a^3}\left\{ \frac{\pi a b}{4}\left[H_1\left(\frac{2b}{a}\right) - Y_1\left(\frac{2b}{a}\right)\right] - \frac{\pi b^2}{2}\left[H_0\left(\frac{2b}{a}\right) - Y_0\left(\frac{2b}{a}\right)\right]\right\}, \tag{17}$$

where H and Y are the Struve and the Bessel functions, respectively [19]. Then up to first order over α (or b^2)

$$\left\langle \Psi_{1s} \left| \frac{1}{\sqrt{r^2+b^2}} - \frac{1}{r} \right| \Psi_{1s} \right\rangle = \frac{2b^2}{a^3}\left(\ln\frac{b}{a} + \gamma + \frac{1}{2}\right), \tag{18}$$

where $\gamma=0.57721\ldots$ is the Euler constant.

It is easy to calculate the contributions caused by the terms $\frac{1}{r}p^2+p^2\frac{1}{r}$ and p^4. As a result the correction to the ground state energy reads

$$\begin{aligned}\Delta E_{1s}^{(1)} &= \langle \Psi_{1s}|V|\Psi_{1s}\rangle \\ &= \frac{e^2\hbar^2}{a^3}\left[3\beta + \beta' - (2\beta-\beta') \times \left(\ln\frac{\hbar^2(2\beta-\beta')}{a^2} + 2\gamma + 1\right)\right].\end{aligned} \tag{19}$$

We also calculate the correction to the $2s$ level

$$\begin{aligned}\Delta E_{2s}^{(1)} &= \langle \Psi_{2s}|V|\Psi_{2s}\rangle \\ &= \frac{e^2\hbar^2}{8a^3}\left[\frac{7\beta+3\beta'}{2} - (2\beta-\beta') \times \left(\ln\frac{\hbar^2(2\beta-\beta')}{4a^2} + 2\gamma + \frac{5}{2}\right)\right].\end{aligned} \tag{20}$$

Note that in special case $2\beta=\beta'$ these results for energy levels reproduce the results of Brau [11].

Similarly as in Ref. [12] we introduce two parameters $\xi=\Delta x_{\min}/a$ and $\eta=\beta/(\beta+\beta')$ instead of β and β', where the minimal length $\Delta x_{\min}=\hbar\sqrt{\beta+\beta'}$. We have already noticed that the calculated corrections to the s levels take place under the conditions $2\beta-\beta'\geqslant 0$. The conditions for the parameters β, β' restrict the domain of the variation for the parameter η. It is easy to verify that $\frac{1}{3}\leqslant\eta\leqslant 1$. We rewrite the correction for the $1s$ level using the parameters η and ξ:

$$\Delta E_{1s}^{(1)} = \frac{e^2}{a}\xi^2\{2\eta+1-(3\eta-1)[\ln \xi^2(3\eta-1)+2\gamma+1]\}. \tag{21}$$

The correction to the $2s$ level as the function of parameters ξ and η reads

$$\Delta E_{2s}^{(1)} = \frac{e^2}{8a}\xi^2\left[\frac{1}{2}(4\eta+3) - (3\eta-1) \times \left(\ln\frac{\xi^2(3\eta-1)}{4} + 2\gamma + \frac{5}{2}\right)\right]. \tag{22}$$

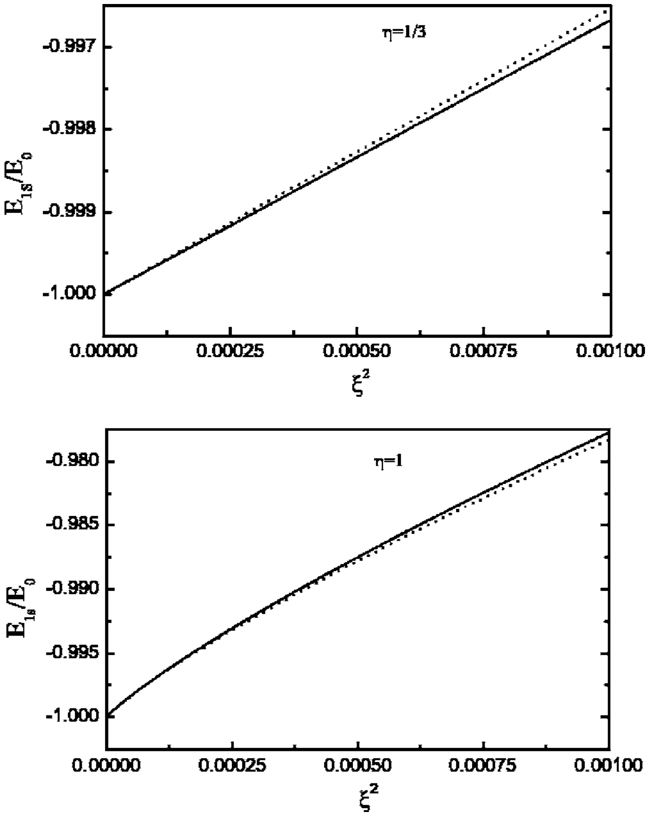

FIG. 1. Comparison of different results for the energy of $1s$ states with $\eta=1/3$ (upper graph) and $\eta=1$ (lower graph). Solid lines represent our results and dotted lines correspond to the results of Benczik (see Ref. [12]).

The energy of the $1s$ and $2s$ level as a function of parameter ξ for the fixed value η is presented in Figs. 1 and 2, respectively. The unit of energy in these figures is the absolute value of the ground state energy of the undeformed hydrogen atom $E_0=e^2/2a$.

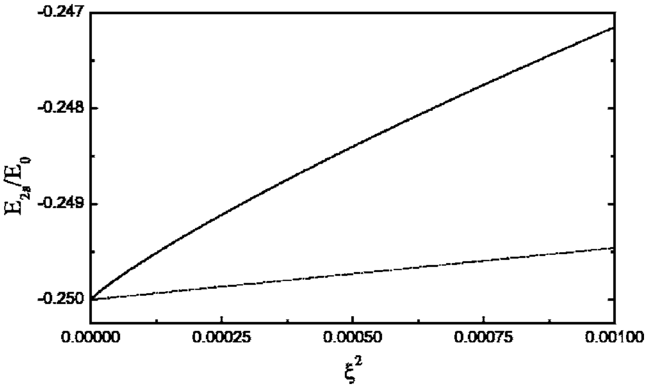

FIG. 2. Energy of the $2s$ level. The solid line shows the result when $\eta=1$ and the dash-dotted line corresponds to the parameter $\eta=1/3$.

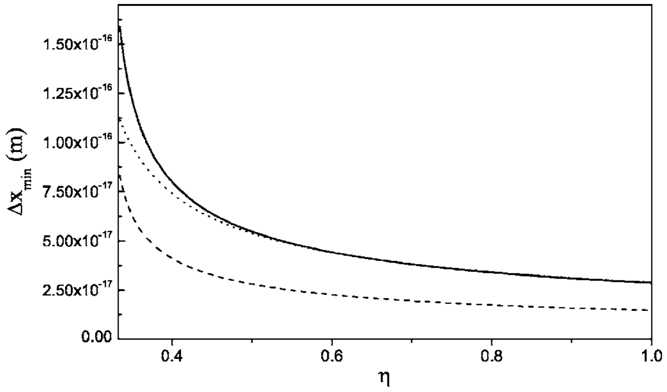

FIG. 3. Constraints on the minimal length as a function of dimensionless parameter η. The solid line represents the results obtained here and the dotted line shows the results of Ref. [12], where the experimental and theoretical data for Lamb shift from Refs. [16,17] were used. The dashed line represents the results of the present work obtained using experimental and theoretical data for the Lamb shift from Ref. [18].

Finally, we can consider constraints on the minimal length. As it was noted [12], the best estimation of the minimal length can be obtained by including the contributions of the minimal length effects in the Lamb shift. In Ref. [12] it was pointed out that the experimental Lamb shift for the 1s level of the hydrogen atom $L_{1s}^{\text{exprt}}=8172.837(22)$ MHz [16] is larger than the best theoretically obtained $L_{1s}^{\text{theor}}=8172.731(40)$ MHz [17]. To compare our results with Ref. [12] we calculate the minimal length using the same data. Assuming similarly as in Ref. [12] that the discrepancy between the experimental and theoretical values $L_{1s}^{\text{exprt}}-L_{1s}^{\text{theor}}$ is entirely attributed to the minimal length correction $\Delta E_{1s}^{(1)}$ that we find upper bound for the minimal length.

So having relation (21) we can estimate the minimal length as a function of parameter η. The limit values of the minimal length are $\Delta x_{\min}=1.64\times10^{-16}$ m for $\eta=1/3$ and $\Delta x_{\min}=2.86\times10^{-17}$ m for $\eta=1$. Let us compare our results with the estimation obtained in Ref. [12] where it was shown that for $\eta=\frac{1}{3}$ the minimal length $(\Delta x_{\min})^{-1}=1.75$ GeV or $\Delta x_{\min}=1.13\times10^{-16}$ m and for $\eta=1$ the minimal length $(\Delta x_{\min})^{-1}=6.87$ GeV or equivalently $\Delta x_{\min}=2.87\times10^{-17}$ m, where 1 m$=(\hbar c/e)$ (eV)$^{-1}$. The comparison of our results for the minimal length with results obtained in Ref. [12] for all values of parameter η is shown in Fig. 3.

We also estimate the upper bound for the minimal length using more recent experimental and theoretical results for the Lamb shift from Ref. [18], namely, $L_{1s}^{\text{exprt}}=8172.840(22)$ MHz and $L_{1s}^{\text{theor}}=8172.811(32)(2)$ MHz. The result is shown in Fig. 3 (dashed line). As one can see, the minimal length calculated with more recent data is less than the one calculated with the previous data [16,17] but both quantities are of the same order.

As one can see our results for the energy levels as well as for the estimation of the minimal length are in good agreement with the results obtained in the paper [12], especially for the $\eta=1$. The discrepancy between these two estimations are caused by the use of different methods of computation. Namely, using the modified perturbation theory we have the analytical expressions for the s energy levels. The authors of Ref. [12] in order to calculate a correction to the s levels in a three-dimensional case used a cutoff procedure and numerical calculation. More detailed comparison is made in Sec. IV.

IV. DISCUSSION

We studied the hydrogen atom in the space with the deformed Heisenberg algebra leading to nonzero minimal length. The ordinary perturbation theory proposed in Ref. [12] leads to the term proportional to $1/r^3$ in the perturbation operator. This term gives a divergent contribution to energy of the s levels in the three-dimensional case. Therefore, the authors of Ref. [12] were forced to use the cutoff of the expectation value integral $\langle 1/r^3\rangle$ at some point. In order to find the free parameter that appeared as a result of this procedure, the author used the numerical calculation.

We construct a modified perturbation theory free of divergences where instead of b^2/r^3 we have $(1/r-1/\sqrt{r^2+b^2})$ with $b=\hbar\sqrt{(D-1)(2\beta-\beta')/2}$. It gives us the possibility to calculate the analytically corrections to the energy levels caused by deformation including the s levels in the three-dimensional case. Comparing our results with experimental data from precision hydrogen spectroscopy we find that the upper bound for the minimal length is of the order of 10^{-16} and 10^{-17} m. Our results for the energy levels as well as the result for the estimation of the minimal length are in a good agreement with the results obtained in the paper [12] with the help of the numerical calculation and cutoff procedure. Note also that for the case $\eta=1/3$ $(2\beta=\beta')$ our results for the energy levels reproduce the result of Brau [11], as it must be.

Finally, we would like to draw attention to the problem which is ignored in present paper as well as in the papers of other authors. Considering the hydrogen atom in deformed space we suppose that the Coulomb potential is the same as in nondeformed space. In fact, in deformed space the Coulomb potential as a potential of a point charge might be corrected. This will lead to additional corrections to the energy spectrum of the hydrogen atom. It is an interesting problem which is worth separate investigations.

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