Generation of cluster states in ion-trap systems

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(Received 13 February 2006; published 2 June 2006)

We propose two schemes for the generation of four-qubit cluster states in ion-trap systems. The first scheme is based on resonant sideband excitation, while the second scheme does not use the vibrational mode as the memory. The schemes can be realized with presently available ion-trap techniques.

DOI: 10.1103/PhysRevA.73.065802 PACS number(s): 42.50.Dv

Entanglement is one of the most striking feature of quantum mechanics. Entangled states of three or more particles not only provide possibilities to test quantum mechanics against local hidden theory without using inequality [1], but also have practical applications in high-precision frequency measurement [2] and quantum-information processing [3]. It has been shown that there are two inequivalent classes of tripartite entanglement states, the Greenberger-Horne-Zeilinger (GHZ) class [1] and the *W* class [4], under stochastic local operation and classical communication. Recently, Briegel *et al.* have introduced another class of multiqubit entangled states, i.e., the so-called cluster states [5]. Cluster states have many interesting features. They have a high persistence of entanglement and can be regarded as an entanglement source for the GHZ states, but are more immune to decoherence than GHZ states [6]. It has been shown that a new inequality is maximally violated by the four-particle cluster state but not by the four-particle GHZ states, and the cluster states can also be used to test nonlocality without inequalities [7]. More importantly, it has been shown that the cluster states constitute a universal resource for so-called one-way quantum computation proceeding only by local measurements and feedforward of their outcomes [8].

Walther *et al.* have experimentally generated four-photon cluster states and demonstrated the feasibility of the one-way quantum computation [9]. The experimental demonstration of the cluster-state violation of Bell's inequality has also been reported [10]. On the other hand, recent advance in ion traps has opened the prospects for quantum-entanglement engineering and quantum-information processing. Very recently, research groups have realized a six-atom GHZ state [11] and eight-qubit *W* states [12] in such a system. However, multi-qubit cluster states have not been demonstrated in such a system yet. In this paper, we propose two schemes for the generation of cluster states in ion trap systems. The first scheme is based on resonant sideband excitation, while the second scheme does not use the vibrational mode as the memory. The schemes are realizable with presently available experimental techniques.

We here assume that the ions have two excited metastable states $|e\rangle$ and $|e'\rangle$ and one ground state $|g\rangle$. The quantum bit (qubit) is carried by states $|e\rangle$ and $|g\rangle$. The first ion is initially prepared in the state $|e\rangle$ and the center-of-mass vibrational mode is initially prepared in the vacuum state $|0\rangle$. We drive the first ion with a laser tuned to the first lower vibrational sideband with respect to the transition $|g\rangle\rightarrow|e\rangle$. Assume the laser is off resonant with the transition $|g\rangle\rightarrow|e'\rangle$ and thus the state $|e'\rangle$ is not affected during the interaction. In the Lamb-Dicke limit and the weak-excitation regime where the Rabi frequency is much smaller than vibrational frequency, the interaction Hamiltonian is [13]

$$H = i\frac{\eta}{2}\Omega e^{-i\phi}a|e_1\rangle\langle g_1| + \text{H.c.}, \tag{1}$$

where $a^\dagger$ and a are the creation and annihilation operators of the center-of-mass vibrational mode of the trapped ions, η is the Lamb-Dicke parameter, and Ω and ϕ are the Rabi frequency and phase of this laser field. After an interaction time τ_1 the state of the system combined by this ion and the center-of-mass mode is

$$\cos\left(\frac{\eta}{2}\Omega\tau_1\right)|e_1\rangle|0\rangle - e^{i\phi}\sin\left(\frac{\eta}{2}\Omega\tau_1\right)|g_1\rangle|1\rangle. \tag{2}$$

With the choice $\phi=\pi$ and $\eta\Omega\tau_1=\pi/2$ we obtain

$$\frac{1}{\sqrt{2}}(|e_1\rangle|0\rangle + |g_1\rangle|1\rangle). \tag{3}$$

We now follow the ideas introduced in Ref. [13] to achieve a phase gate. We drive the third ion, initially in the state $(1/\sqrt{2})(|g_3\rangle-|e_3\rangle)$, with a laser tuned to the first lower vibrational sideband with respect to the transition $|g\rangle\rightarrow|e'\rangle$. The interaction Hamiltonian is given by

$$H' = i\frac{\eta}{2}\Omega' e^{-i\phi'}a|e_3'\rangle\langle g_3| + \text{H.c.} \tag{4}$$

After an interaction time τ_2 the evolution of the state $|1\rangle|g_3\rangle$ is

$$|1\rangle|g_3\rangle \rightarrow \cos\left(\frac{\eta}{2}\Omega'\tau_2\right)|1\rangle|g_3\rangle + e^{-i\phi'}\sin\left(\frac{\eta}{2}\Omega'\tau_2\right)|0\rangle|e_3\rangle. \tag{5}$$

On the other hand, $|0\rangle|e_2\rangle$, $|0\rangle|g_2\rangle$, $|1\rangle|e_2\rangle$ do not undergo any transition during the interaction. With the choice $\eta\Omega'\tau_2=2\pi$ we obtain $|1\rangle|g_3\rangle\rightarrow-|1\rangle|g_3\rangle$. This corresponds to the phase gate between the third ion and the vibrational mode, which was first proposed by Cirac and Zoller [13]. Then we have

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$$\frac{1}{2}[|e_1\rangle(|e_3\rangle - |g_3\rangle)|0\rangle - |g_1\rangle(|e_3\rangle + |g_3\rangle)|1\rangle]. \tag{6}$$

We now map the state of the vibrational mode to the second ion by driving this ion with a laser tuned to the first lower vibrational sideband with respect to the transition $|g\rangle \rightarrow |e\rangle$, leading to

$$\frac{1}{2}[|e_1\rangle|g_2\rangle(|g_3\rangle - |e_3\rangle) - |g_1\rangle|e_2\rangle(|e_3\rangle + |g_3\rangle)]. \tag{7}$$

Performing the transformation

$$|e_1\rangle \rightarrow \frac{1}{\sqrt{2}}(|e_1\rangle + |g_1\rangle),$$

$$|g_1\rangle \rightarrow \frac{1}{\sqrt{2}}(|g_1\rangle - |e_1\rangle), \tag{8}$$

we obtain

$$\frac{1}{2\sqrt{2}}[(|g_1\rangle + |e_1\rangle)|g_2\rangle(|g_3\rangle - |e_3\rangle) - (|g_1\rangle - |e_1\rangle)|e_2\rangle(|e_3\rangle + |g_3\rangle)]. \tag{9}$$

The fourth ion is initially in the state $(1/\sqrt{2})(|e_4\rangle + |g_4\rangle)$. We map its state to the vibrational mode, resulting in

$$\frac{1}{4}[(|g_1\rangle + |e_1\rangle)|g_2\rangle(|g_3\rangle - |e_3\rangle) - (|g_1\rangle - |e_1\rangle)|e_2\rangle(|e_3\rangle + |g_3\rangle)](|0\rangle + |1\rangle). \tag{10}$$

We now perform the phase gate operation between the third ion and the vibrational mode, leading to

$$\begin{aligned}\frac{1}{4}\{&[(|g_1\rangle + |e_1\rangle)|g_2\rangle(|g_3\rangle - |e_3\rangle)\\ &- (|g_1\rangle - |e_1\rangle)|e_2\rangle(|e_3\rangle + |g_3\rangle)]|0\rangle\\ &+ [(|g_1\rangle + |e_1\rangle)|g_2\rangle(-|g_3\rangle - |e_3\rangle)\\ &- (|g_1\rangle - |e_1\rangle)|e_2\rangle(|e_3\rangle - |g_3\rangle)]|1\rangle\}.\end{aligned} \tag{11}$$

Mapping the state of the vibrational mode back to the fourth ion, we have

$$\begin{aligned}\frac{1}{4}\{&[(|g_1\rangle + |e_1\rangle)|g_2\rangle(|g_3\rangle - |e_3\rangle)\\ &- (|g_1\rangle - |e_1\rangle)|e_2\rangle(|e_3\rangle + |g_3\rangle)]|g_4\rangle\\ &+ [(|g_1\rangle + |e_1\rangle)|g_2\rangle(-|g_3\rangle - |e_3\rangle)\\ &- (|g_1\rangle - |e_1\rangle)|e_2\rangle(|e_3\rangle - |g_3\rangle)]|e_4\rangle\}.\end{aligned} \tag{12}$$

We can rewrite Eq. (12) as

$$\frac{1}{4}(|g_1\rangle\sigma_z^2 + |e_1\rangle)(|g_2\rangle\sigma_z^3 + |e_2\rangle)(|g_3\rangle\sigma_z^4 + |e_3\rangle)(|g_4\rangle + |e_4\rangle), \tag{13}$$

where

$$\sigma_z^j = |g_j\rangle\langle g_j| - |e_j\rangle\langle e_j|. \tag{14}$$

The state of Eq. (13) is just a four-qubit cluster state.

We note that the cluster states can also be generated without using the vibrational mode as the memory. We consider four identical three-level ions, having two ground states $|e\rangle$ and $|g\rangle$ and an excited state $|r\rangle$, confined in a linear trap. We now follow the ideas introduced in Ref. [14] to achieve a geometric phase gate. The transition $|e\rangle \rightarrow |r\rangle$ of the first three ions is driven by two classical laser fields with detunings Δ and $\Delta + \omega_0 - \delta$, with ω_0 being the frequency of the center-of-mass vibrational mode. Assume that $\Delta \gg \Omega_j$, ω_0, with Ω_j $(j=1,2)$ being the Rabi frequency of the jth laser. The relative detuning of the two lasers is close to ω_0, i.e., $\omega_0 \gg \delta$. In this case we can neglect other vibrational modes. In the rotating-wave approximation, the Hamiltonian for this system is given by (assuming $\hbar = 1$)

$$\begin{aligned}H = \omega_0 a^\dagger a + \sum_{j=1}^{3}\Bigg(&\Omega e^{-i[(\omega_0-\delta)t - \eta(a+a^\dagger)]}\\ &+ \Omega e^{i[(\omega_0-\delta)t - \eta(a+a^\dagger)]} + \frac{\Omega_1^2 + \Omega_2^2}{2\Delta}\Bigg)S_{z,j},\end{aligned} \tag{15}$$

where

$$S_{z,j} = \frac{1}{2}(|r_j\rangle\langle r_j| - |e_j\rangle\langle e_j|), \tag{16}$$

$$\Omega = \frac{\Omega_1\Omega_2}{2\Delta}. \tag{17}$$

If the ions initially have no probabilities of being populated in the state $|r\rangle$, they will remain in the ground state during the interaction. In this case the Hamiltonian reduces to

$$\begin{aligned}H = \omega_0 a^\dagger a - \sum_{j=1}^{3}\Bigg(&\frac{\Omega}{2}e^{-i[(\omega_0-\delta)t - \eta(a+a^\dagger)]}\\ &+ \frac{\Omega}{2}e^{i[(\omega_0-\delta)t - \eta(a+a^\dagger)]} + \frac{\Omega_1^2 + \Omega_2^2}{4\Delta}\Bigg)|e_j\rangle\langle e_j|.\end{aligned} \tag{18}$$

Consider the behavior of the trapped ions in the Lamb-Dicke regime, where $\eta\sqrt{\bar{n}+1} \ll 1$, with $\bar{n}$ being the mean phonon number of the center-of-mass mode. Assume that $\eta\Omega \ll \omega_0$ and thus we can neglect the terms oscillating at frequencies of the order of ω_0. In the interaction picture the interaction Hamiltonian is given by

$$\begin{aligned}H_i = -\sum_{j=1}^{3}\Bigg(&\frac{\Omega_1^2 + \Omega_2^2}{4\Delta} + \frac{\Omega}{2}[2\cos(-\delta t + \omega_0 t)\\ &+ i\eta(a^\dagger e^{i\delta t} - a e^{-i\delta t})]\Bigg)|e_j\rangle\langle e_j|.\end{aligned} \tag{19}$$

Define the symmetrical state $|k\rangle$ with k atoms being in the state $|e\rangle$, i.e., the well-known Dicke state [15]. During the infinitesimal interval $[t, t+dt]$ the interaction induces the evolution

$$|k\rangle|\phi_k(t)\rangle \rightarrow e^{-iH_i dt}|k\rangle|\phi_k(t)\rangle$$
$$= e^{ik[(\Omega_1^2+\Omega_2^2)/4\Delta+\Omega\cos(-\delta t+\omega_0 t)]dt}D(d\alpha_k)|k\rangle|\phi_k(t)\rangle, \tag{20}$$

where

$$d\alpha_k = -k\eta\frac{\Omega}{2}e^{i\delta t}dt. \tag{21}$$

$|\phi_k(t)\rangle$ denotes the vibrational state correlated with the qubit state $|k\rangle$ at the time t. Assume that the vibrational mode is initially in the state $|\phi(0)\rangle$. Then we obtain the evolution

$$|k\rangle|\phi(0)\rangle \rightarrow e^{ik[(\Omega_1^2+\Omega_2^2)/4\Delta]t+[\Omega/(-\delta+\omega_0)]\sin(-\delta t+\omega_0 t)}$$
$$\times e^{i\phi_k}D(\alpha_k)|k\rangle|\phi(0)\rangle, \tag{22}$$

where

$$\alpha_k = -k\int_0^t \eta\frac{\Omega}{2}e^{i\delta t'}dt' = -k\eta\frac{\Omega}{2i\delta}(e^{i\delta t}-1), \tag{23}$$

$$\phi_k = \mathrm{Im}\int_\gamma \alpha_k'^* d\alpha_k' = -\mathrm{Im}\int_0^t\left(k\eta\frac{\Omega}{2}\right)^2\frac{1}{i\delta}(1-e^{i\delta t'})dt'$$
$$= -\left(k\eta\frac{\Omega}{2}\right)^2\mathrm{Im}\left\{\frac{1}{i\delta}\left[t-\frac{1}{i\delta}(e^{i\delta t}-1)\right]\right\}$$
$$= \left(k\eta\frac{\Omega}{2}\right)^2\left[\frac{t}{\delta}-\frac{1}{\delta^2}\sin(\delta t)\right]. \tag{24}$$

With the choice

$$\delta t = 2\pi, \tag{25}$$

we obtain

$$|k\rangle|\phi(0)\rangle \rightarrow e^{ik\theta}e^{ik^2\phi}|k\rangle|\phi(0)\rangle, \tag{26}$$

where

$$\theta = \frac{\Omega_1^2+\Omega_2^2}{4\Delta}t + \frac{\Omega}{2(-\delta+\omega_0)}\sin(\omega_0 t), \tag{27}$$

$$\phi = \left(\eta\frac{\Omega}{2}\right)^2\frac{t}{\delta}. \tag{28}$$

In this case, the vibrational state is displaced along a circular path, returning to its original point in the pase space, acquiring a geometric phase conditional on the electronic states [14].

We now assume that each atom is initially in the state $(|e_j\rangle+|g_j\rangle)/\sqrt{2}$. Then the state for the first three ions can be written as a Bloch state [16],

$$\frac{1}{\sqrt{8}}\sum_{k=0}^{3}\binom{k}{3}^{1/2}|k\rangle. \tag{29}$$

Using Eqs. (26) and (29), we obtain the state of the qubit system after an interaction time t,

$$\frac{1}{\sqrt{8}}\sum_{k=0}^{3}e^{ik\theta}e^{i(k\eta\Omega/2)^2t/\delta}\binom{k}{3}^{1/2}|k\rangle. \tag{30}$$

Setting

$$\eta^2\frac{\Omega^2}{4\delta}t = \pi/2, \tag{31}$$

we have

$$\frac{1}{2\sqrt{2}}\sum_{k=0}^{3}[e^{-i\pi/4}e^{ik\theta}+e^{i\pi/4}(-)^k e^{ik\theta}]\binom{k}{3}^{1/2}|k\rangle$$
$$= \frac{e^{-i\pi/4}}{4}\left(\prod_{j=1}^{3}(|g_j\rangle+e^{i\theta}|e_j\rangle)+i\prod_{j=1}^{3}(|g_j\rangle-e^{i\theta}|e_j\rangle)\right). \tag{32}$$

We can satisfy Eqs. (25) and (31) by choosing $\delta=\eta\Omega$ and $t=2\pi/(\eta\Omega)$. The transformations $|e_1\rangle\rightarrow e^{-i\theta}|e_1\rangle$, $(|g_2\rangle+e^{i\theta}|e_2\rangle)/\sqrt{2}\rightarrow|g_2\rangle$, and $(|g_2\rangle-e^{i\theta}|e_2\rangle)/\sqrt{2}\rightarrow i|e_2\rangle$ lead to

$$\frac{e^{-i\pi/4}}{2\sqrt{2}}[(|g_1\rangle+|e_1\rangle)|g_2\rangle(|g_3\rangle+e^{i\theta}|e_3\rangle)$$
$$-(|g_1\rangle-|e_1\rangle)|e_2\rangle(|g_3\rangle-e^{i\theta}|e_3\rangle)]. \tag{33}$$

We then drive the third and fourth ions with the abovementioned lasers. After an interaction time $t=2\pi/\delta$ we obtain

$$\frac{e^{-i\pi/4}}{4}\{[(|g_1\rangle+|e_1\rangle)|g_2\rangle(|g_3\rangle+e^{2i\theta}e^{i\phi}|e_3\rangle)$$
$$-(|g_1\rangle-|e_1\rangle)|e_2\rangle(|g_3\rangle-e^{2i\theta}e^{i\phi}|e_3\rangle)]|g_4\rangle$$
$$+[(|g_1\rangle+|e_1\rangle)|g_2\rangle(e^{i\theta}e^{i\phi}|g_3\rangle+e^{3i\theta}e^{4i\phi}|e_3\rangle)$$
$$-(|g_1\rangle-|e_1\rangle)|e_2\rangle(e^{i\theta}e^{i\phi}|g_3\rangle-e^{3i\theta}e^{4i\phi}|e_3\rangle)]|e_4\rangle\}. \tag{34}$$

We again set $\eta^2(\Omega^2/4\delta)t=\pi/2$ and perform the transformations $|e_3\rangle\rightarrow -e^{-2i\theta}e^{-i\phi}|e_3\rangle$ and $|e_4\rangle\rightarrow -e^{-i\theta}e^{-i\phi}|e_4\rangle$. Then we have

$$\frac{e^{-i\pi/4}}{4}\{[(|g_1\rangle+|e_1\rangle)|g_2\rangle(|g_3\rangle-|e_3\rangle)$$
$$-(|g_1\rangle-|e_1\rangle)|e_2\rangle(|g_3\rangle+|e_3\rangle)]|g_4\rangle$$
$$+[(|g_1\rangle+|e_1\rangle)|g_2\rangle(-|g_3\rangle-|e_3\rangle)$$
$$-(|g_1\rangle-|e_1\rangle)|e_2\rangle(-|g_3\rangle+|e_3\rangle)]|e_4\rangle\}. \tag{35}$$

We can rewrite this state as

$$\frac{e^{-i\pi/4}}{4}[(|g_1\rangle\sigma_z^2+|e_1\rangle)(|g_2\rangle\sigma_z^3+|e_2\rangle)$$
$$\times(|g_3\rangle\sigma_z^4+|e_3\rangle)(|g_4\rangle+|e_4\rangle)], \tag{36}$$

where σ_z^j is given by Eq. (14). In this way we also obtain a four-particle cluster state.

It is necessary to estimate the fidelity of the operations. The error mainly results from two- or three-qubit operations. The first scheme consists of six sideband excitations which couple the internal and external degrees of freedom and trivial single-qubit rotations. The fidelity of each sideband excitation is about 0.93 [17,18]. In this case, the fidelity of the whole procedure is about 0.65. The second scheme involves a three-ion coupling and a two-ion coupling. The respective fidelities are about 0.89 and 0.95 [14,19] and thus the fidelity of the whole procedure is about 0.85.

In conclusion, we have described two schemes for the generation of four-qubit cluster states with trapped ions. The required experimental techniques of the schemes are within the scope of what can be obtained in the ion-trap setup. The experimental implementation of the schemes is useful for the test of fundamental aspects of quantum physics and quantum-information processing.

This work was supported by Fok Ying Tung Education Foundation under Grant No. 81008, the National Fundamental Research Program under Grant No. CB309300, the National Natural Science Foundation of China under Grant Nos. 60008003 and 10225421, and Funds from Fuzhou University.

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