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Purity and squeezing exchange between coupled bosonic modes

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We study the dynamics of squeezing and purity exchange between two coupled bosonic modes in the model of the quantum parametric converter, comparing it with the dynamics of entanglement.

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In the recent paper [1] Boukobza and Tannor called attention to an interesting feature of coupled quantum systems. Considering the Jaynes-Cummings model, they showed that if the quantum state of the whole system is *mixed*, then the rates of the change of entropies of the field and atomic modes can have opposite signs, in contrast to the case of a *pure* state of the whole system, where the entropies of both modes *coincide*, being equal to zero initially. In particular, two modes can *exchange* their entropies. Actually, such a behavior of partial entropies of the field and atomic subsystems was noticed earlier in [2]. Moreover, the possibility of the *purity exchange* between two resonantly coupled field modes in a cavity with oscillating boundary was discovered in [3]. Consequently, this phenomenon is not specific to the Jaynes-Cummings system. The aim of our paper is to analyze the effect of entropy (purity) exchange for coupled *bosonic* modes, comparing it with the exchange of other properties, in particular, the degree of the *squeezing* of each mode. Earlier we studied the squeezing exchange between resonantly coupled bosonic modes in [4], but only in the particular case of initial pure states of both modes.

Two bosonic modes can be described with the aid of lowering and/or raising operators $\hat{a}_k$ and $\hat{a}_k^\dagger$, satisfying the standard commutation relations $[\hat{a}_k,\hat{a}_j^\dagger]=\delta_{kj}$ (hereafter $k,j=1,2$), or by means of the Hermitian dimensionless quadrature components operators, defined according to the decomposition $\hat{a}_k=(\omega_k\hat{x}_k+i\hat{p}_k)/\sqrt{2\omega_k}$. We define the symmetrical real quadrature covariances as $q_{\alpha\beta}\equiv\frac{1}{2}\langle\hat{q}_\alpha\hat{q}_\beta+\hat{q}_\beta\hat{q}_\alpha\rangle$, where q_α are components of the four-dimensional vector $\mathbf{q}=(\hat{x}_1,\hat{p}_1,\hat{x}_2,\hat{p}_2)$. We assume for simplicity that all mean values of quadrature operators are equal to zero (otherwise it is sufficient to replace the operators $\hat{a}_k$ by $\hat{a}_k-\langle\hat{a}_k\rangle$). It is convenient to gather the covariances in the symmetrical 4×4 covariance matrix $\mathcal{Q}$, splitting it in 2×2 blocks as follows:

$$\mathcal{Q}=\|q_{\alpha\beta}\|=\left\|\begin{matrix}\mathcal{Q}_{11} & \mathcal{Q}_{12}\\ \mathcal{Q}_{21} & \mathcal{Q}_{22}\end{matrix}\right\|,\quad \mathcal{Q}_{jk}=\tilde{\mathcal{Q}}_{kj},$$

where the tilde over the matrix means matrix transposition.

For the *Gaussian* states, the von Neumann's entropy $\mathcal{G}=-\mathrm{Tr}(\hat{\rho}\ln\hat{\rho})$ and the *quantum purity* $\mu=\mathrm{Tr}(\hat{\rho}^2)$ of the whole system, described by the statistical operator $\hat{\rho}$, and its subsystems, described by the reduced statistical operators of the form $\hat{\rho}_1=\mathrm{Tr}_2(\hat{\rho})$, can be expressed in terms of the determinants $D_0=\det\mathcal{Q}$ and $D_{ij}=\det\mathcal{Q}_{ij}$ [5,6]. In particular, $\mu=(16D_0)^{-1/2}$ and $\mu_k=(4D_{kk})^{-1/2}$. Therefore, the quantities D_0 and D_{kk} can also serve as simple entropylike characteristics (equivalent to $\mathcal{G}$ and μ for Gaussian states and independent for non-Gaussian ones). They can be interpreted as the effective volumes in the phase space [6–8]. One can verify the formula

$$D_{kk}=\left\langle\hat{a}_k^\dagger\hat{a}_k+\tfrac{1}{2}\right\rangle^2-|\langle\hat{a}_k^2\rangle|^2. \tag{1}$$

The *degree of squeezing* in each mode is characterized in the best and simplest way by the *invariant squeezing coefficient* [9] (or "principal squeezing" coefficient [10])

$$\mathcal{S}_k=1+2\langle\hat{a}_k^\dagger\hat{a}_k\rangle-2|\langle\hat{a}_k^2\rangle|, \tag{2}$$

which is nothing but twice the minimal value of the variances of quadrature components of the family of operators $\hat{a}e^{i\gamma}$, when parameter γ varies in the interval $0\le\gamma<2\pi$ (so that $\mathcal{S}_k=1$ for coherent states and $\mathcal{S}_k<1$ if the state is squeezed).

We suppose that there are no correlations between the modes in the initial states. Then one can always make independent rotations in the phase planes of the modes (or multiply the operators $\hat{a}_k$ by some phase factors), which eliminate the covariances between the "coordinate" and "momentum" quadratures. Thus we shall use the following parametrization of the initial second order moments:

$$\langle\hat{a}_k^\dagger\hat{a}_k\rangle(0)=\vartheta_k\cosh(2r_k)-\tfrac{1}{2},\quad \langle\hat{a}_1^\dagger\hat{a}_2\rangle(0)=0, \tag{3}$$

$$\langle\hat{a}_k^2\rangle(0)=\vartheta_k\sinh(2r_k),\quad \langle\hat{a}_1\hat{a}_2\rangle(0)=0. \tag{4}$$

The parameters ϑ_k determine the initial purities of each subsystem, $D_{kk}(0)=\vartheta_k^2$, while the parameters r_k give the initial degrees of squeezing,

$$S_k(0)=2\vartheta_k\exp(-2r_k). \tag{5}$$

We suppose that $r_k\ge0$, assuming that initially squeezed quadratures are the coordinate ones. The inequality $\vartheta_k\ge1/2$ must hold due to the Schrödinger-Robertson uncertainty relations [11].

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We consider the evolution of the coefficients D_{kk} and S_k in the case of parametric converter, described by the Hamiltonian (we assume $\hbar=1$)

$$\hat{H}_c = \omega_1 \hat{a}_1^\dagger \hat{a}_1 + \omega_2 \hat{a}_2^\dagger \hat{a}_2 + \kappa \hat{a}_1^\dagger \hat{a}_2 e^{i\eta t} + \kappa^* \hat{a}_2^\dagger \hat{a}_1 e^{-i\eta t}, \quad (6)$$

where we set $\eta=\omega_2-\omega_1$, confining ourselves to the simplest case of *exact* resonance. In this case the Heisenberg equations of motion have exact solutions [12]

$$\hat{a}_1(t) = e^{-i\omega_1 t}[\hat{a}_1(0)\cos\tau - i\gamma\hat{a}_2(0)\sin\tau],$$

$$\hat{a}_2(t) = e^{-i\omega_2 t}[\hat{a}_2(0)\cos\tau - i\gamma^*\hat{a}_1(0)\sin\tau],$$

where $\tau\equiv|\kappa|t$ and $\gamma=\kappa/|\kappa|$. Thus we have

$$D_{11} = \vartheta_1^2 c^4 + \vartheta_2^2 s^4 + 2\chi^+ \vartheta_1 \vartheta_2 s^2 c^2, \quad (7)$$

$$\dot{D}_{11}(\tau) = 4cs[s^2\vartheta_2(\vartheta_2-\chi^+\vartheta_1) - c^2\vartheta_1(\vartheta_1-\chi^+\vartheta_2)], \quad (8)$$

$$\begin{aligned}\mathcal{S}_1(\tau) = 2\{&\vartheta_1\cosh(2r_1)c^2 + \vartheta_2\cosh(2r_2)s^2 \\ &- [\vartheta_1^2\sinh^2(2r_1)c^4 + \vartheta_2^2\sinh^2(2r_2)s^4 \\ &- 2\sigma\vartheta_1\vartheta_2\sinh(2r_1)\sinh(2r_2)c^2s^2]^{1/2}\},\end{aligned} \quad (9)$$

where $c\equiv\cos(\tau)$, $s\equiv\sin(\tau)$, $\sigma\equiv\mathrm{Re}(\gamma^2)$, and

$$\chi^\pm = \cosh(2r_1)\cosh(2r_2) \pm \sigma\sinh(2r_1)\sinh(2r_2). \quad (10)$$

For the second mode one should change the indices $1\leftrightarrow 2$. Note that $\chi^\pm \geq 1$ for any values of r_k and σ, because $\cosh[2(r_1-r_2)]\leq\chi^\pm\leq\cosh[2(r_1+r_2)]$.

We see from Eqs. (7) and (9) that the modes periodically *exchange* their phase volumes and degrees of squeezing. However, this exchange is not always "straightforward" (monotonous), and namely this fact seems to us interesting enough to justify the publication of our observations. Let us suppose for definiteness that $\vartheta_1\geq\vartheta_2$. Then Eq. (8) (with the substitution $1\leftrightarrow 2$) shows that initially $\dot{D}_{22}>0$, which seems to be quite natural. But when $\tau\to\pi/2$ (approaching the moment of total exchange of properties of the modes), the sign of this derivative is the same as the sign of combination $\vartheta_1-\chi^+\vartheta_2$. Consequently, a monotonous increase of the phase volume D_{22} during the first half of a period is observed, provided that ϑ_1 is not simply bigger than ϑ_2, but a stronger inequality $\vartheta_1>\chi^+\vartheta_2$ must be satisfied. Under the same condition, the derivative $\dot{D}_{11}(\tau)$ is negative for $0<\tau<\pi/2$. If $\chi^+\vartheta_2>\vartheta_1>\vartheta_2$ (this can happen if at least one of coefficients r_1 and r_2 is different from zero), then the behavior of the functions $D_{kk}(\tau)$ is nonmonotonous, and their derivatives change the sign at the moments of time $\tau_{1,2}^D$ determined by the equations

$$\tan^2(\tau_1^D) = \frac{\vartheta_1(\vartheta_1-\chi^+\vartheta_2)}{\vartheta_2(\vartheta_2-\chi^+\vartheta_1)} = \cot^2(\tau_2^D). \quad (11)$$

One can verify that $\tau_1^D\leq\pi/4\leq\tau_2^D$. The derivatives $\dot{D}_{11}(\tau)$ and $\dot{D}_{22}(\tau)$ have opposite signs in the interval $\tau_1^D\leq\tau\leq\tau_2^D$. The length of this interval diminishes with decrease of the difference $\vartheta_2-\vartheta_1$ (for the fixed value of χ^+), going to zero if $\vartheta_2=\vartheta_1$. In this special case $D_{11}(\tau)\equiv D_{22}(\tau)\geq D_{11}(0)$, so that the entropies of both subsystems always coincide, independently of the values of other parameters; moreover, the phase volume of each mode increases monotonously, until reaching the maximal value $\vartheta_1^2(\chi^++1)/2$ at $\tau=\pi/4$, and then returns monotonously to the initial value ϑ_1^2. The three different regimes of the evolution of phase volumes are illustrated in Fig. 1.

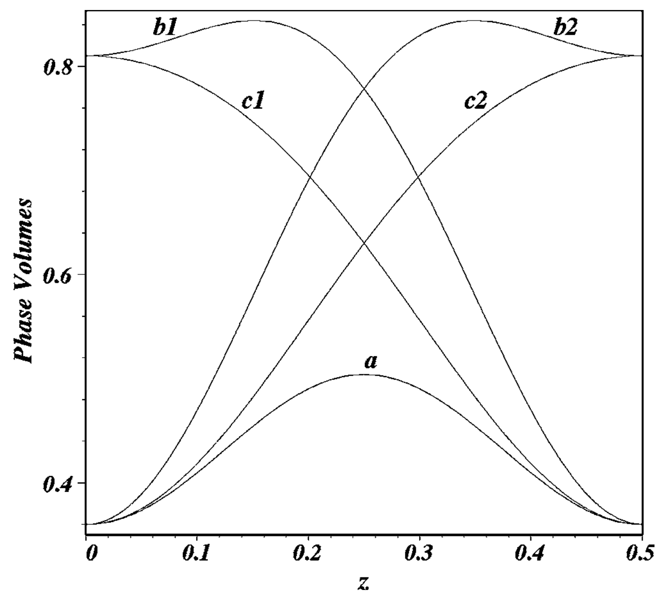

FIG. 1. The coefficients D_{kk} versus $z=\tau/\pi$. The single line (a) corresponds to the case $\vartheta_1=\vartheta_2=0.6$ and $\chi^+=1.8$. The curves ($b1$) and ($b2$) give the coefficients D_{11} and D_{22} (respectively) for $\vartheta_1=0.9$, $\vartheta_2=0.6$, and $\chi^+=1.8$. The curves ($c1$) and ($c2$) show the case $\vartheta_1=0.9$, $\vartheta_2=0.6$, and $\chi^+=1.25$.

Now let us study the behavior of the squeezing coefficient, confining ourselves for simplicity to the extreme cases $\sigma=\pm1$, when the square root in (9) can be calculated exactly. If $\sigma=-1$, then

$$\mathcal{S}_1^- = 2\vartheta_1\exp(-2r_1)c^2 + 2\vartheta_2\exp(-2r_2)s^2, \quad (12)$$

and the degree of squeezing changes monotonously. The situation is quite different in the case $\sigma=1$, when

$$\mathcal{S}_1^+ = \begin{cases} 2\vartheta_1\exp(-2r_1)c^2 + 2\vartheta_2\exp(2r_2)s^2, & \tau<\tau_1^S \\ 2\vartheta_1\exp(2r_1)c^2 + 2\vartheta_2\exp(-2r_2)s^2, & \tau>\tau_1^S \end{cases}$$

so that the time derivatives of the functions $\mathcal{S}_{1,2}^+(\tau)$ has identical finite negative jumps

$$\Delta\dot{\mathcal{S}}_{1,2}^+ = -8[\vartheta_1\sinh(2r_1) + \vartheta_2\sinh(2r_2)]cs\big|_{\tau=\tau_{1,2}^S}$$

at the instants of time $\tau_{1,2}^S$ determined by the equation

$$\tan^2(\tau_1^S) = \frac{\vartheta_1\sinh(2r_1)}{\vartheta_2\sinh(2r_2)} = \cot^2(\tau_2^S). \quad (13)$$

Note that the instants $\tau_{1,2}^S$ have no relation to the instants $\tau_{1,2}^D$ defined by Eq. (11), moreover, τ_1^S can be greater than τ_2^S. Since the derivatives $\dot{\mathcal{S}}_{1,2}^+(\tau)$ do not change the sign in the time intervals before and after $\tau_{1,2}^S$, one can easily see that there are only two possibilities for the functions $\mathcal{S}_1^+(\tau)$ and

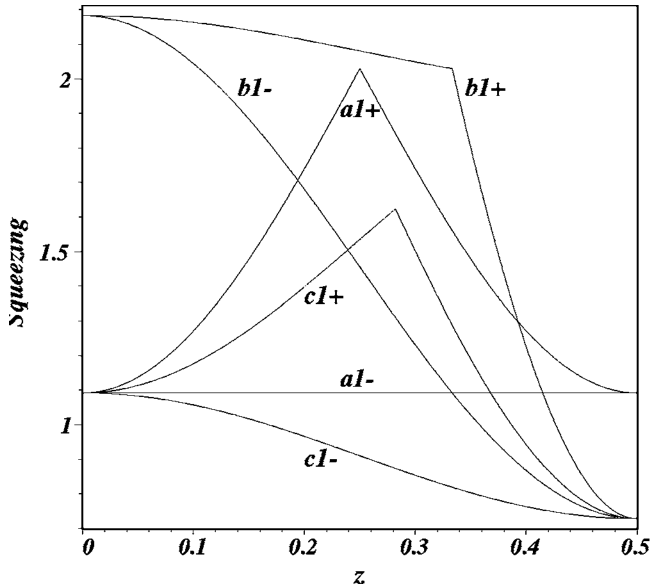

FIG. 2. The squeezing coefficient S_1 versus $z=\tau/\pi$ for identical squeezing parameters $r_1=r_2=0.25$ and different sets of the initial purity coefficients (the signs + and − correspond to $\sigma=\pm1$): (a) $\vartheta_1=\vartheta_2=0.9$; (b) $\vartheta_1=1.8$ and $\vartheta_2=0.6$; (c) $\vartheta_1=0.9$ and $\vartheta_2=0.6$. The curves corresponding to the second mode can be obtained by the reflection of this figure with respect to the vertical line z=0.25.

$S_2^+(\tau)$: (i) one of them monotonously increases while another monotonously decreases; (ii) each function increases until τ achieves the value τ_1^S or τ_2^S and decreases after that instant. In both cases, the functions $S_1^+(\tau)$ and $S_2^+(\tau)$ have identical negative jumps of their first derivatives at the instants τ_1^S and τ_2^S, respectively. The situation when both the functions first decrease and then increase is impossible. This means that the squeezing coefficient of each mode never can become smaller than the smallest of the two initial values given by Eq. (5). At the moments $\tau_{1,2}^S$ we have

$$S_1^+(\tau_1) = S_2^+(\tau_2) = \frac{2\vartheta_1\vartheta_2\sinh(2r_1+2r_2)}{\vartheta_1\sinh(2r_1)+\vartheta_2\sinh(2r_2)}.$$

Comparing this value with Eq. (5), one can see that case (ii) is possible if two inequalities hold simultaneously:

$$\vartheta_1 e^{-2r_1} < \vartheta_2 e^{2r_2}, \quad \vartheta_2 e^{-2r_2} < \vartheta_1 e^{2r_1}. \tag{14}$$

This can happen if $\vartheta_1=\vartheta_2$ or if each mode was initially in the true squeezed state (when the left-hand sides of both the inequalities are smaller than $\frac{1}{2}$). If one of the inequalities in (14) is not satisfied, then we have case (i). It can be realized if some mode was initially in a highly mixed and non-squeezed state. Examples illustrating the behavior of functions $S_1(\tau)$ are shown in Fig. 2.

The coefficients D_{kk} and S_k characterize statistical properties of each mode alone. On the other hand, the interaction between the modes gives rise to their statistical dependence. It seems natural to suppose that the peculiarities in the behavior of functions $D_{kk}(\tau)$ and $S_k(\tau)$, demonstrated in Figs. 1 and 2, could be explained by this interaction. For the *Gaussian* states, all information on the statistical correlations between the modes is contained in the off-diagonal block $\mathcal{Q}_{12}$ of the total covariance matrix. There are two invariants of this 2×2 block: its trace and determinant [13],

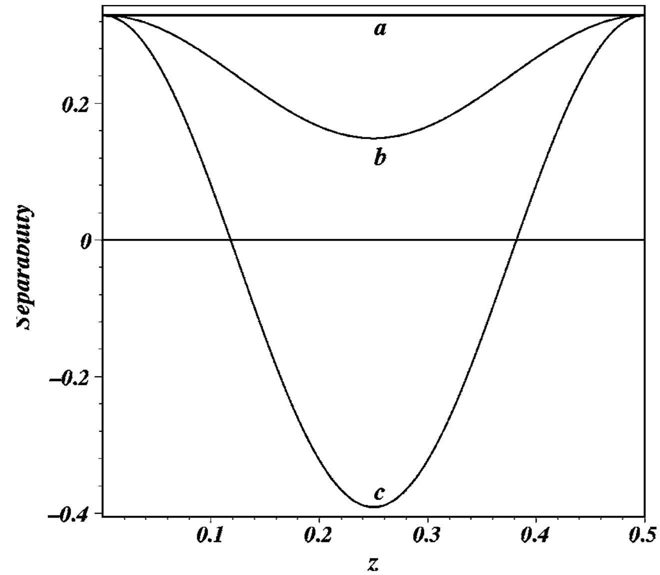

FIG. 3. The separability coefficient $\mathcal{Z}$ versus $z=\tau/\pi$ for $\vartheta_1=1.8$ and $\vartheta_2=0.6$ (so that $\chi_e\simeq2.28$). The curve (*a*) corresponds to the case $\chi^+=1.25$ (when $\chi_q\simeq-0.41$). The curves (*b*) and (*c*) correspond to $\chi_q>0$, with $\chi^+=2$ and $\chi^+=3$, respectively.

$$\mathcal{T}_{12} \equiv \tfrac{1}{2}\,\mathrm{Tr}(\mathcal{Q}_{12}\mathcal{Q}_{21}) = |\langle\hat a_1\hat a_2^\dagger\rangle|^2 + |\langle\hat a_1\hat a_2\rangle|^2, \tag{15}$$

$$D_{12} \equiv \det\mathcal{Q}_{12} = |\langle\hat a_1\hat a_2^\dagger\rangle|^2 - |\langle\hat a_1\hat a_2\rangle|^2. \tag{16}$$

For the Gaussian states with zero mean values $\langle\hat a_k\rangle$ the coefficient $\mathcal{T}_{12}$ gives a measure of correlations between fluctuations of the *numbers of photons* in each mode [14]:

$$\mathcal{T}_{12} = \langle\hat{\mathcal{N}}_1\hat{\mathcal{N}}_2\rangle - \langle\hat{\mathcal{N}}_1\rangle\langle\hat{\mathcal{N}}_2\rangle, \quad \hat{\mathcal{N}}_k \equiv \hat a_k^\dagger\hat a_k. \tag{17}$$

In the case concerned we have

$$\mathcal{T}_{12} = s^2c^2[\vartheta_1^2\cosh(4r_1) + \vartheta_2^2\cosh(4r_2) - 2\chi^-\vartheta_1\vartheta_2]. \tag{18}$$

Since the right-hand side of Eq. (18) is non-negative for any values of parameters, the dynamics of function $\mathcal{T}_{12}(\tau)$ shows nothing interesting: this periodical function starts from zero, increases monotonously as $\sin^2(2\tau)$ until the moment $\tau=\pi/4$ and then returns to the initial value. No correlation with the dynamics of functions $D_{kk}(\tau)$ and $S_k(\tau)$ is observed.

The evolution of the determinant of the cross-covariance matrix is quite similar to Eq. (18),

$$D_{12}(\tau) = s^2c^2[\vartheta_1^2 + \vartheta_2^2 - 2\vartheta_1\vartheta_2\chi^+]. \tag{19}$$

However, the sign of the right-hand side of (19) depends on the concrete values of parameters. This observation is important in view of the problem of *separability* of the density matrix of the total system of two coupled modes. Indeed, the known Simon's separability criterion [15] for Gaussian states can be represented in the form of a simple inequality $\mathcal{Z}\ge0$, where [14,16]

$$\mathcal{Z} = D_0 + \frac{1}{16} - \frac{1}{4}(D_{11} + D_{22}) - \frac{1}{2}|D_{12}|. \quad (20)$$

Moreover, if the system's dynamics is governed by a Hamiltonian which is a *quadratic form* of lowering and/or raising or quadrature operators with arbitrary time-dependent coefficients, and if the density operator of the whole system was factorized at the initial moment $t=0$ [i.e., $\mathcal{Q}_{12}(0)=0$], then [14]

$$\mathcal{Z}(t) = \mathcal{Z}(0) + \frac{1}{2}[D_{12}(t) - |D_{12}(t)|], \quad (21)$$

due to the existence of two *general symplectic invariants* [8] $\mathcal{I}_2 \equiv D_{11}(t)+D_{22}(t)+2D_{12}(t)$ and $\mathcal{I}_0 \equiv D_0(t)$. In the case concerned we have $\mathcal{Z}(0)=(\vartheta_1^2-1/4)(\vartheta_2^2-1/4)$. If

$$\chi_q \equiv \chi^+ - (\vartheta_1^2 + \vartheta_2^2)/(2\vartheta_1\vartheta_2) \le 0, \quad (22)$$

then D_{12} is always non-negative, and the joint quantum state of two modes remains separable forever. The condition (22) resembles the condition of monotonous entropy exchange, $\chi^+ \le \vartheta_1/\vartheta_2$, but the coefficient is different. Consequently, again there is no strict correlation between the behavior of entropies of each subsystem and the separability of the total system.

If condition (22) is violated, then function $\mathcal{Z}(\tau)$ decreases for $0<\tau<\pi/4$, but the state can become *entangled* ($\mathcal{Z}<0$) during some time interval if only

$$\chi^+ > \chi_e \equiv 2\vartheta_1\vartheta_2 + (8\vartheta_1\vartheta_2)^{-1}. \quad (23)$$

The instant τ_e of the "phase transition" from a separable to an entangled state is determined by the equation

$$\tan^2(2\tau_e) = \frac{2(\vartheta_1^2 - 1/4)(\vartheta_2^2 - 1/4)}{\vartheta_1\vartheta_2(\chi^+ - \chi_e)}, \quad (24)$$

but this instant coincides neither with $\tau_{1,2}^S$ nor with $\tau_{1,2}^D$, as can be seen by comparing Figs. 1, 2, and 3.

Concluding, we have studied the influence of initial parameters on the dynamics of the entropy (purity) and squeezing exchange between two coupled bosonic modes in the model of the quantum parametric converter. This dynamics appears nontrivial, because it can be nonmonotonous. At the same time, the dynamics of the degree of separability (entanglement) of the joint mixed quantum state turned out to be much more simple, and we did not discover any strict correlation between the dynamics of the squeezing and purity in each mode and the dynamics of entanglement.

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