States of the optimum Fisher measure of information for quantum interferometry

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Fisher's measure of information is compared to the usual measure of sensitivity of the SU(2) and SU(1,1) interferometers. The states of optimum Fisher information measure and prescribed mean photon-number difference are defined, and the properties of these states are studied in the SU(2) interferometer.

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I. INTRODUCTION

The optical interferometry is known to provide high-precision measurements of various physical quantities. The enhancement of precision can be connected with the use of strong input fields, but this could damage the device. Therefore, in the framework of quantum optics, the optimization of the input state has been considered with the subsidiary condition that the mean total number of photons at the input is prescribed. The phase-shift measurement scheme was devised by Caves [1,2]. It is assumed that photon numbers are measured at the outputs, and from these numbers, the phase shift is to be inferred. This is an indirect measurement of the phase shift. In the course of time, the assumption of the coherent state input has been replaced by that of the squeezed light and special quantum state input.

The SU(2) group representation of the Mach-Zehnder interferometer has been shown to be advantageous [3]. A phase sensitivity measure has been introduced to assess the precision of measurements. In the connection with the group theoretical approach, the constraint of the prescribed mean photon number [4] has been strengthened to the assumption that the total photon number is prescribed. Use of the two-mode four-wave mixer has been proposed leading to an SU(1,1) interferometer, an efficient device that can be analyzed in terms of the SU(1,1) group [3]. It has been natural to assume that the photon-number difference is known. A formal property of the phase sensitivity has drawn attention to a relevant uncertainty relation and to the corresponding minimum-uncertainty states [5], which are also called intelligent states [6,7].

Most of the papers have been devoted to the Mach-Zehnder interferometer. Holland and Burnett [8] have raised great interest. In that paper, Fock state inputs of the same photon number have been assumed. Although these states are not of "quality" with respect to the phase sensitivity measure known by then (they can obey a prescribed total photon number if it is even), the analysis has shown that the Heisenberg limit is reached for them. It has been obvious that the phase shift is measured up to the sign, but such a loss of certainty has not been emphasized anywhere because the analysis was excellent. In [9], the first beam splitter in the Mach-Zehnder interferometer has been treated as the device that produces phase-difference squeezing from amplitude squeezing, cf. [8].

The Fock-state inputs with arbitrary photon numbers have been investigated in [10]. The authors have considered various possibilities of the measurements at the outputs. Even though some schemes have been measurements of the photon-number difference up to the sign, it has emerged that new measures of the phase sensitivity can be useful.

Maximally entangled states of a system of N two-level hot trapped ions have been realized experimentally and have an optical analogue. Maximally entangled states of a two-mode field have application in interferometry and lithography [11,12]. High photon numbers are generally not available. De Martini *et al.* [13] have reported on optical (in fact, on optical stochastic) interferometry with single-photon Fock states and with a couple of photons. In [14], three types of input state have been discussed for the Heisenberg-limited measurement protocols. Campos *et al.* [15] assume that the parity operator for a mode associated with one of the output beams is measured. Studying optical multimode interferometers as in [16] may shed light on the intricate relationship between repeated measurements with two-mode interferometers and single measurements with multimode interferometers. Gerry *et al.* [17] have commented on [8] after a lapse of time (cf., however, [15,18]). Progress in optical quantum computing stimulates the study of the generation or preparation of quantum correlations for the Heisenberg-limited interferometry [19].

A recent review on the states for quantum interferometry is [20]. Mostly theoretic investigations of the quantum phase have looked for the application to the quantum interferometry [21]. It is not excluded that only a small complication of the interferometer will lead to a great advancement in the employment of the indirectly measured output quantum phase difference. It is natural to utilize the Cramér-Rao lower bound [4] as an alternative of the usual sensitivity measure.

In [22], we concentrated on the Fisher measure of information, which is an important constituent of the lower bound. The problem of finding the special states that make the Fisher measure maximum was formulated. Mathematically, it resembles [23,24], the authors of which, however, looked for a minimum because they study entanglement and extreme spin squeezing. Recently, the Cramér-Rao sensitivity limits have been considered with respect to the Poisson deconvolution problem [25].

In Sec. II, we introduce not only the SU(2) interferometers, but also the SU(1,1) and M(2) interferometers and the conventional phase sensitivity for them. In Sec. III, we will present the Cramer-Rao inequality and appropriate lower bound to be compared to the usual sensitivity measure. The Fisher measure of information is expressed for the SU(2) and SU(1,1) generalized coherent states and a coherent state. In Sec. IV, a special form of the Fisher information will be derived at zero phase shift, i.e., under the same assumption which is used for the usual phase sensitivity. A constraint optimalization problem is formulated that leads to an eigenvalue problem. This problem is solved on a computer for special values of parameters. In Sec. V, a sample optimum state will be visualized using quasidistributions. With respect to the intricacy of the proposed optimum state, the problem of attachment of the lower bound has not been touched. In particular, the maximum likelihood estimation has not been formulated based on the proper state. In [26], the Fock state inputs are assumed, the Fisher information of the same form as in [22] is derived, and the question of the lower bound is pursued and the usual regularity assumptions are reformulated. The main result of this paper is in the study of special states which render the Fisher measure of information maximum.

II. INTERFEROMETERS, SENSITIVITY MEASURES, AND COHERENT STATES

Yurke *et al.* [3] were first to emphasize the importance of the SU(2) group representation of the Mach-Zehnder interferometer as well as the role of the Heisenberg and Schrödinger pictures for this description. The Heisenberg picture resembles the classical description in that it relates the output annihilation operators $\hat{a}_{j\text{out}}$, $j=1,2$, to the input ones $\hat{a}_{j\text{in}}$, $j=1,2$, leaving the state of physical system unchanged. The equivalent Schrödinger picture consists in a transformation of the input state to the output state, whereas the operators that are to be averaged do not evolve.

A. SU(2) interferometers

The beam splitters and the phase-shift imparted by the measured medium as well as the detectors can be described using the following operators:

$$\hat{J}_1 = \tfrac{1}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1),$$

$$\hat{J}_2 = \tfrac{1}{2i}(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1),$$

$$\hat{J}_3 = \tfrac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2), \tag{2.1}$$

$$\hat{N} = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2. \tag{2.2}$$

In the Mach-Zehnder interferometer the output state $|\text{out}\rangle$ is related to the input state $|\text{in}\rangle$ as follows:

$$|\text{out}\rangle = \hat{U}_2 \hat{U}(\phi) \hat{U}_1 |\text{in}\rangle, \tag{2.3}$$

with

$$\hat{U}_1 = \exp\left(-i\frac{\pi}{2}\hat{J}_1\right),$$

$$\hat{U}(\phi) = \exp(-i\phi \hat{a}_1^\dagger \hat{a}_1),$$

$$\hat{U}_2 = \hat{U}_1^\dagger, \tag{2.4}$$

where the unitary operators $\hat{U}_j$, $j=1,2$, describe operation of beam splitters and the unitary operator $\hat{U}(\phi)$ describes the phase shift ϕ imparted in one arm of the interferometer. Using the relation $\hat{a}_1^\dagger \hat{a}_1 = \frac{1}{2}\hat{N} + \hat{J}_3$ and exploiting the group formalism, we can rewrite (2.3) in the form

$$|\text{out}\rangle = \exp\left(-i\frac{\phi}{2}\hat{N}\right)\exp(-i\phi\hat{J}_2)|\text{in}\rangle. \tag{2.5}$$

At the output, the operator $\hat{J}_3$ is measured. Naturally, it means that the photon-number difference is measured. The assumption that the total photon number is known is then equivalent to the knowledge of photon number at both output ports. The phase sensitivity or the minimum detectable phase shift (the uncertainty of the phase measurement) depends on the unknown phase shift (to be measured) and, on the assumption of small phase shifts, which are most interesting, it can be approximated by the value for $\phi=0$. It is then determined by

$$(\delta\phi)^2 = \frac{\langle(\Delta\hat{J}_3)^2\rangle}{\langle\hat{J}_1\rangle^2}, \quad \langle\hat{J}_1\rangle \neq 0, \tag{2.6}$$

where the expectation values are computed for the input state. Assuming that the interferometer is fed with the Glauber coherent state $|\alpha_1\rangle_1|\alpha_2\rangle_2$, it can be found that the optimal choice among the coherent states with fixed $|\alpha_j|$, $j=1,2$, is a coherent state with $\text{Im}(\alpha_1^*\alpha_2)=0$.

Among the eigenstates of the total photon-number operator belonging to the eigenvalue $2j$, the SU(2) generalized coherent states $|j,\zeta\rangle$ are the most important,

$$(\hat{N}+\hat{1})|j,\zeta\rangle = (2j+1)|j,\zeta\rangle, \tag{2.7}$$

and they are defined in any of the following ways:

$$\begin{aligned}|j,\zeta\rangle &= \exp(\xi\hat{J}_+ - \xi^*\hat{J}_-)|0\rangle_1|2j\rangle_2 \\ &= (1+|\zeta|^2)^{-j}\exp(\zeta\hat{J}_+)|0\rangle_1|2j\rangle_2 \\ &= (1+|\zeta|^2)^{-j}\sum_{n_1=0}^{2j}\binom{2j}{n_1}^{1/2}\zeta^{n_1}|n_1\rangle_1|2j-n_1\rangle_2,\end{aligned} \tag{2.8}$$

where $\zeta=\frac{\xi}{|\xi|}\tan|\xi|\neq 0$ or $\zeta=\xi=0$ and

$$\hat{J}_+ = \hat{J}_1 + \text{i}\hat{J}_2 = \hat{a}_1^\dagger\hat{a}_2,$$

$$\hat{J}_- = \hat{J}_1 - \text{i}\hat{J}_2 = \hat{a}_1\hat{a}_2^\dagger \tag{2.9}$$

are the raising and lowering operators. Slightly more refined calculations than in the case of ordinary coherent state show that the phase uncertainty is minimum for $\text{Im}\,\zeta=0$ when minimized under the constraint that $|\zeta|$ is fixed.

For any input state,

$$|\text{in}\rangle = \sum_{n_1=0}^{2j} c_{n_1}|n_1\rangle_1|2j-n_1\rangle_2 \tag{2.10}$$

from the SU(2)-irreducible invariant space of the states $|\psi\rangle$ obeying the equation

$$(\hat{N}+\hat{1})|\psi\rangle = (2j+1)|\psi\rangle, \tag{2.11}$$

we can formulate a criterion of optimality. It holds that the expectation value $\langle\psi|(\Delta\hat{J}_3)^2|\psi\rangle$ comprises only the products $|c_k||c_l|$, whereas $\langle\psi|\hat{J}_1|\psi\rangle^2$ depends only on $\text{Re}(c_k^* c_{k+1})$. It can be derived that these are terms with a plus sign each. Therefore, among the two-mode states that have the moduli $|c_k|$ fixed, the optimum ones have real c_k such that $c_1 \gtrless 0, c_2 > 0, \ldots, c_{2j} > 0$, for $2j$ even and $c_{2j} \gtrless 0$ for $2j$ odd if $c_0 > 0$ is chosen. This rule relates to a phase difference. Invoking the theory of quantum phase, we can call such single-mode states partial phase states with the preferred phase equal to either 0 (the upper relation) or π (the lower relation) [27]. Similarly, as in the above case of the input two-mode coherent state, this phase is rather the phase difference.

B. SU(1,1) interferometers

The SU(1,1) interferometer according to [6,7] can be described by the operators

$$\hat{K}_1 = \frac{1}{2}(\hat{a}_1^\dagger\hat{a}_2^\dagger + \hat{a}_1\hat{a}_2),$$

$$\hat{K}_2 = \frac{1}{2i}(\hat{a}_1^\dagger\hat{a}_2^\dagger - \hat{a}_1\hat{a}_2),$$

$$\hat{K}_3 = \frac{1}{2}(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2\hat{a}_2^\dagger), \tag{2.12}$$

$$\hat{N}_\text{d} = \hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2. \tag{2.13}$$

The passive beam splitters are replaced here by four-wave mixers. This nonlinear interferometer relates the output state to the input state as described by the relation (2.3), with the unitary operators

$$\hat{U}_1 = \exp(i\beta\hat{K}_1),$$

$$\hat{U}(\phi) = \exp(-i\phi\hat{a}_1^\dagger\hat{a}_1),$$

$$\hat{U}_2 = \hat{U}_1^\dagger, \tag{2.14}$$

where β relates to the reflectivity $\mathcal{R}$ of the four-wave mixer via $\sinh^2(\frac{\beta}{2}) = \mathcal{R}$. Using the relation $\hat{a}_1^\dagger\hat{a}_1 = \frac{1}{2}\hat{N}_\text{d} + \hat{K}_3$ and exploiting the SU(1,1) group formalism, we can express (2.14) in the form

$$|\text{out}\rangle = \exp\left(-i\frac{\phi}{2}\hat{N}_\text{d}\right)\exp\{-i\phi[(\cosh\beta)\hat{K}_3 - (\sinh\beta)\hat{K}_2]\}|\text{in}\rangle. \tag{2.15}$$

In this kind of interferometer the operator $\hat{K}_3$ is measured, i.e., the photon-number sum is measured. Again, the assumption that the photon-number difference is known means the knowledge of photon numbers on both the output ports.

The conventional phase sensitivity (the uncertainty of the phase measurement) is given by the relation

$$(\delta\phi)^2 = \frac{\langle(\Delta\hat{K}_3)^2\rangle}{(\sinh\beta)^2\langle\hat{K}_1\rangle^2}, \quad \langle\hat{K}_1\rangle \neq 0, \tag{2.16}$$

where the expectation values are taken in the input states. As this interferometer is an active device, the most common assumption is that only the vacuum fluctuations enter the input port, leading to a known result $(\delta\phi)^2_\text{vac} = (\sinh\beta)^{-2}$.

Also in this case we concentrate on the eigenstates of a group invariant, the photon-number difference operator $\hat{N}_\text{d}$, belonging to the eigenvalue $2k-1$. The appropriate SU(1,1) generalized coherent states fulfill the equation

$$\hat{N}_\text{d}|k,\zeta\rangle = (2k-1)|k,\zeta\rangle \tag{2.17}$$

and are defined in any of the following ways:

$$|k,\zeta\rangle = \exp(\xi\hat{K}_+ - \xi^*\hat{K}_-)|2k-1\rangle_1|0\rangle_2 = (1-|\zeta|^2)^k\exp(\zeta\hat{K}_+)|2k-1\rangle_1|0\rangle_2 = (1-|\zeta|^2)^k\sum_{n_2=0}^{\infty}\binom{n_2+2k-1}{n_2}^{1/2}\zeta^{n_2}|n_2+2k-1\rangle_1|n_2\rangle_2, \tag{2.18}$$

where $\zeta = \frac{\xi}{|\xi|}\tanh|\xi| \neq 0$ or $\zeta = \xi = 0$ and

$$\hat{K}_+ = \hat{K}_1 + \text{i}\hat{K}_2 = \hat{a}_1^\dagger\hat{a}_2^\dagger,$$

$$\hat{K}_- = \hat{K}_1 - \text{i}\hat{K}_2 = \hat{a}_1\hat{a}_2 \tag{2.19}$$

are the raising and lowering operators. Calculations show that the phase uncertainty is minimum for ζ real when minimized under the constraint that $|\zeta|$ is fixed.

C. M(2) interferometers

It is illustrative to see the asymptotics of the SU(2) generalized coherent states as $j\rightarrow\infty$ and when only the first factor of the number states $|n_1\rangle_1|2j-n_1\rangle_2$ is considered. From this viewpoint, the SU(2) coherent states $|j,\zeta_j\rangle$ converge to the usual coherent state $|\alpha\rangle$ when $\zeta_j\simeq\frac{\alpha}{\sqrt{2j}}$. Similarly, the asymptotics of the SU(1,1) generalized coherent states can be considered as $k\rightarrow\infty$ and when only the second factor of the number states $|n_2+2k-1\rangle_1|n_2\rangle_2$ is taken into account. Seen in this way, the SU(1,1) coherent states $|k,\zeta_k\rangle$ converge to the usual coherent state $|\alpha\rangle$ when $\zeta_k\simeq\frac{\alpha}{\sqrt{2k}}$. In this way, both the kinds of interferometers can be illustrated with an M(2) interferometer. This hypothetical device is described by the relation (2.3), where

$$\hat{U}_1=\exp\left(-i\tfrac{1}{2}\bar{P}\hat{Q}\right),$$

$$\hat{U}(\phi)=\exp(-i\phi\hat{n}),$$

$$\hat{U}_2=\hat{U}_1^{\dagger}, \tag{2.20}$$

where $\frac{1}{2}\bar{P}$ is an amount of displacement, $\hat{Q}$ is the positionlike quadrature

$$\hat{Q}=\hat{a}+\hat{a}^{\dagger}, \tag{2.21}$$

and

$$\hat{n}=\hat{a}^{\dagger}\hat{a} \tag{2.22}$$

is the photon-number operator. To relations (2.3) and (2.20) corresponds a reformulation of the output-input relation in the form

$$|\mathrm{out}\rangle=\exp\left(-i\phi\tfrac{1}{4}\bar{P}^2\right)\exp\left[-i\phi\left(\hat{n}-\tfrac{1}{2}\bar{P}\hat{P}\right)\right]|\mathrm{in}\rangle, \tag{2.23}$$

where

$$\hat{P}=-i(\hat{a}-\hat{a}^{\dagger}) \tag{2.24}$$

is the momentumlike quadrature. The photon-number operator is measured. The conventional phase sensitivity (the uncertainty of the hypothetical phase measurement) is given by the relation

$$(\delta\phi)^2=\frac{\langle(\Delta\hat{n})^2\rangle}{\left(\frac{1}{2}\bar{P}\right)^2\langle\hat{Q}\rangle^2},\quad \langle\hat{Q}\rangle\neq 0. \tag{2.25}$$

It is obvious that the coherent state input optimized under the constraint that $|\alpha|$ is fixed attains the minimum of the phase uncertainty when α is real. For the SU(1,1) and M(2) interferometers, a similar phase condition holds as for the SU(2) interferometer.

For any input state,

$$|\mathrm{in}\rangle=\sum_{n_2=0}^{\infty}c_{n_2}|n_2+2k-1\rangle_1|n_2\rangle_2 \tag{2.26}$$

from the SU(1,1)-irreducible invariant space of the states $|\psi\rangle$, obeying the equation

$$\hat{N}_{\mathrm{d}}|\psi\rangle=(2k-1)|\psi\rangle, \tag{2.27}$$

and for any single-mode input state

$$|\mathrm{in}\rangle=\sum_{n=0}^{\infty}c_n|n\rangle, \tag{2.28}$$

we can formulate criteria of optimality. As in the case of the SU(2) interferometers, the optimum states have real c_k, such that $c_1\gtrless 0, c_2>0, c_3>0,\ldots$, if the choice $c_0>0$ is made, the optimization concerning the states with $|c_k|$ fixed. The two-mode states (2.26) have the preferred phase difference equal to either 0 (the upper relation) or π (the lower relation). The single-mode states (2.28) have the preferred phase equal to 0 and π, respectively.

III. FISHER'S MEASURE OF INFORMATION

In this section, we shall remember the Cramér-Rao inequality and Fisher's measure of information in the application to a discrete random variable. In this way we apply some notions of the classical estimation theory to interferometric measurement. We are interested in the estimation of the phase shift ϕ as a particular case of the parameter Θ. We assume that a measured random variable X takes on values from a countable set M. We require that this set be independent of Θ, which is compatible with the interferometry. We describe statistical properties of this variable using the probabilities $p(x,\Theta)$, $x\in M$, which depend on Θ from an interval Ω. In the estimation theory, it is supposed that $p(x,\Theta)>0$, $x\in M$, independent of Θ, but from the view of the application it is useful to concede a more general situation. We shall study an unspecified estimator $T\equiv T(X)$ for the parameter Θ.

We introduce the sum

$$I(\Theta)=\sum_{x\in M}\frac{1}{p(x,\Theta)}\left[\frac{\partial}{\partial\Theta}p(x,\Theta)\right]^2. \tag{3.1}$$

Applying the Schwarz inequality, we derive that

$$\left[\frac{\partial}{\partial\Theta}\langle T(X)\rangle\right]^2\leqslant\langle[T(X)-\Theta]^2\rangle I(\Theta), \tag{3.2}$$

where the square deviation $\langle[T(X)-\Theta]^2\rangle$ measures the quality of the estimator and $\left[\frac{\partial}{\partial\Theta}\langle T(X)\rangle\right]^2$ can be compared to unity, which is its preferable value, and thus, it is an alternative measure of the quality of the estimator. When $0<I(\Theta)<\infty$, we can write the Cramér-Rao inequality as follows:

$$\langle[T(X)-\Theta]^2\rangle\geqslant\frac{\left[\frac{\partial}{\partial\Theta}\langle T(X)\rangle\right]^2}{I(\phi)}. \tag{3.3}$$

The function $I(\Theta)$ is called Fisher's measure of information on the parameter Θ, which is contained in the random variable X. Let us observe two difficulties that can occur when using Fisher's measure of information. Its relationship to the square deviation is clear only for the estimators whose "bias" $\langle T(X)\rangle-\Theta$ is independent of Θ. Second, in (3.3) the equality need not be reached, i.e., there need not be any efficient

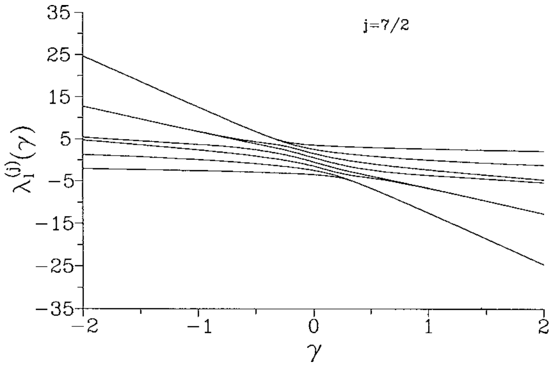

FIG. 1. The eigenvalues $\lambda_l^{(j)}(\gamma)$ vs γ for $j=\frac{7}{2}$ and $l=0,\ldots,8$ and $\gamma\in[-2,2]$. The eigenvalues are numbered by 0,…,8 from bottom up.

estimator of the parameter Θ for the probabilities $p(x,\Theta)$, $x\in M$, $\Theta\in\Omega$.

Fisher's measure of information [28,29] is a good alternative measure of sensitivity since it directly measures the rate of change of a probability distribution. Mathematically, this measure of information is introduced as a constituent of the Cramér-Rao lower bound for the variance (sometimes mean-square deviation) of any estimator of the probability distribution. Since the number of photons N_1 on the first output port is a discrete random variable, we assume a discrete random variable for all three kinds of interferometers. In the SU(2) interferometer, this variable is the number of photons on the first output port and it takes on the values $0,\ldots,2j$, whereas in the SU(1,1) interferometer it is the photon number on the second output port and it takes on the values $0,1,2,\ldots,\infty$. Fisher's measure of information on the parameter ϕ contained in the photon number N_1 is

$$I(\phi)=\sum_{n=0}^{2j}\frac{[p'(n,\phi)]^2}{p(n,\phi)}, \tag{3.4}$$

where $p(n,\phi)\neq 0$ is the distribution for the photon number at the first output port and

$$p'(n,\phi)\equiv\frac{\partial}{\partial\phi}p(n,\phi). \tag{3.5}$$

The division by zero can occur in (3.4) in inadvertent substitution of some distribution. This is connected to the regularity conditions (see the Appendix). Expanding the output state in the same basis as the input state, we introduce the coefficients

$$c(n,\phi)={}_1\langle n|_2\langle 2j-n|\mathrm{out}\rangle. \tag{3.6}$$

As it holds that

$$p(n,\phi)=|c(n,\phi)|^2, \tag{3.7}$$

we can rewrite the formula (3.4) in the form

$$I(\phi)=\sum_{n=0}^{2j}\left[\frac{c^*(n,\phi)}{|c(n,\phi)|}c'(n,\phi)+\frac{c(n,\phi)}{|c(n,\phi)|}c'^*(n,\phi)\right]^2. \tag{3.8}$$

According to an above remark, we have optimal input states with real coefficients. From the fact that the operator $(-i\hat{J}_2)$ has real matrix elements, we have, up to a common phase factor, real coefficients $c(n,\phi)$.

Among the states that have the $|c_k|$ fixed, the optimum states have real c_k such that $c_0>0,c_2<0,c_4>0,\ldots,c_{2j}>0$ for $2j\equiv 0 \pmod 4$, $c_{2j}<0$ for $2j\equiv 2 \pmod 4$, $c_{2j-1}>0$ for $2j\equiv 1 \pmod 4$, $c_{2j-1}<0$ for $2j\equiv 3 \pmod 4$ if $c_0>0$ is chosen, whereas $c_1\gtrless 0$, $c_3\gtrless 0$, $c_5\gtrless 0$, etc. This rule is different from that for the minimum detectable phase shift (for the usual measure of sensitivity). According to this remark, we have optimal input states with real coefficients.

The SU(2) and SU(1,1) generalized coherent states as well as the usual coherent state have an interesting property when used in the SU(2), SU(1,1), and M(2) interferometers, respectively. It holds that

$$(\delta\phi)^2=\frac{1}{I(0)}, \tag{3.9}$$

where

$$I(0)=2j\left[\mathrm{Re}\left(\frac{\xi}{|\xi|}\right)\right]^2, \tag{3.10}$$

$$I(0)=2k\sinh^2\beta\left[\mathrm{Re}\left(\frac{\xi}{|\xi|}\right)\right]^2, \tag{3.11}$$

$$I(0)=\bar{P}^2\left[\mathrm{Re}\left(\frac{\alpha}{|\alpha|}\right)\right]^2, \tag{3.12}$$

respectively. It is obvious that in the case of the SU(2) interferometer, the coherent state is not optimum when the phase factors of c_k vary and only the moduli $|c_k|$ are fixed.

IV. STATES OF THE OPTIMUM FISHER'S MEASURE OF INFORMATION

In this section, we restrict ourselves to the Mach-Zehnder [SU(2)] interferometer. As we have mentioned above, it is reasonable to optimize the sensitivity of measurement under the constraint that the mean photon numbers at both the input ports are known (or prescribed). Making use of the optimality of the input states with real coefficients, we may write the relation (3.8) in the form

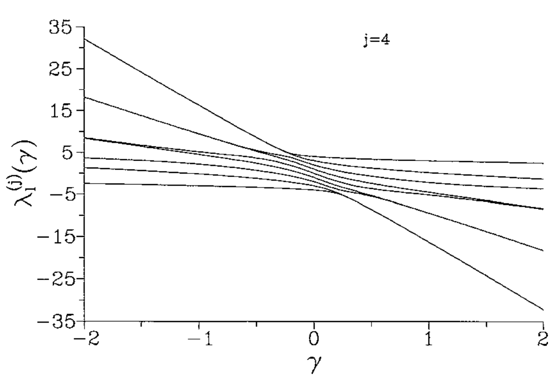

FIG. 2. The same as in Fig. 1, but for $j=4$ and $l=0,\ldots,7$.

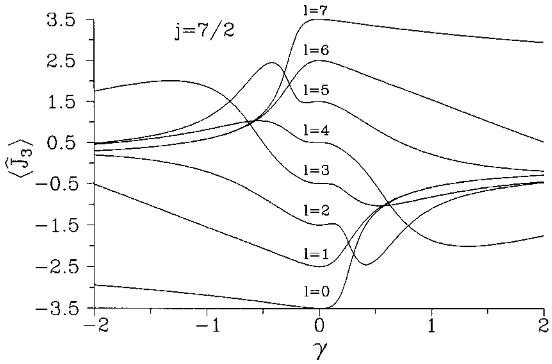

FIG. 3. The dependence of $\langle \hat{J}_3 \rangle$ on $\gamma \in [-2,2]$ for $j=\frac{7}{2}$.

$$I(\phi) = 4\sum_{n=0}^{2j} [c'(n,\phi)]^2. \tag{4.1}$$

Employing further the assumption of a compensation measurement [3], we obtain that

$$I(0) = 4\langle \hat{J}_2^2 \rangle. \tag{4.2}$$

Assuming that the input states are related to a representation of the group SU(2) and that the constraint can be formulated in terms of $\langle \hat{J}_3 \rangle$, we arrive at the eigenvalue problem

$$(\hat{J}_3 - \gamma \hat{J}_2^2)|\psi\rangle = \lambda |\psi\rangle \tag{4.3}$$

by a standard variational argument. The real parameter γ is to be determined as a function of the quantum expectation $\langle \hat{J}_3 \rangle$. The general truth of the variational calculus reads that Eq. (4.3) is only necessary condition for the eigenket $|\psi\rangle$ to provide the maximum of the Fisher measure. To have the sufficient condition at least for a local maximum, we shall follow the exposition by Magnus and Neudecker [30]. It is known that it is demanded to verify the validity of the inequality

$$\langle \chi | \left(\hat{J}_2^2 - \frac{1}{\gamma} \hat{J}_3 \right) | \chi \rangle < 0 \tag{4.4}$$

for every $|\chi\rangle \neq 0$ such that

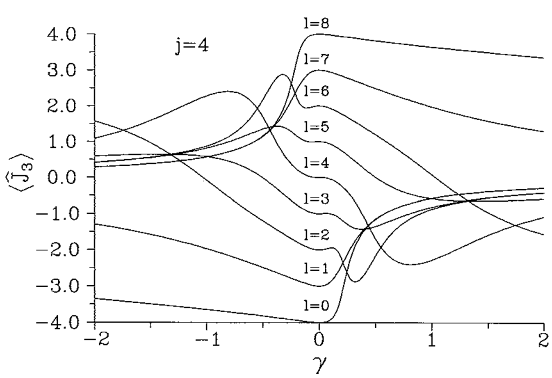

FIG. 4. The same as in Fig. 3, but for $j=4$.

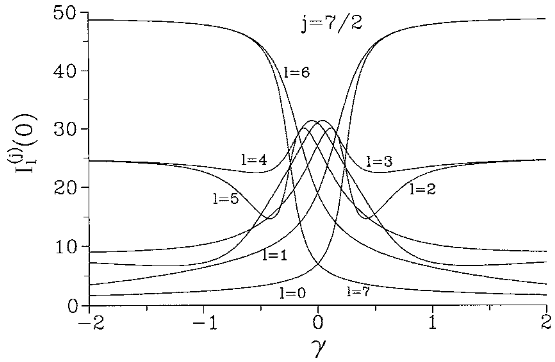

FIG. 5. Fisher's measure of information $I_l^{(j)}(\phi)$ for $\phi=0$, $j=\frac{7}{2}$, $l=0,\ldots,7$, and $\gamma \in [-2,2]$.

$$\langle \psi | \chi \rangle = 0, \quad \langle \psi | \hat{J}_3 | \chi \rangle = 0. \tag{4.5}$$

It is convenient to introduce the concept of generalized eigenvalues μ_k and eigenkets $|\chi_k\rangle$ of the operator $\left(\hat{J}_2^2 - \frac{1}{\gamma}\hat{J}_3\right)$ relative to the constraints (4.5). The condition (4.4) is then equivalent to $\max_k \mu_k < 0$.

In the eigenvalue problem (4.3), it is interesting to see the dependences of the eigenvalues on the parameter γ in Fig. 1. For $\gamma=0$, these eigenvalues coincide with those of the operator $\hat{J}_3, -j, -j+1, \ldots, j$. For $\gamma \to \pm\infty$, they exhibit the asymptotics of straight lines with the slopes given by the eigenvalues of the operator $\hat{J}_2^2$. This means that the asymptotes have only $\frac{(2j+1)}{2} = 4$ directions, but the identification or separation of the eigenvalues as $\gamma \to \pm\infty$ exhibits a more complicated behavior.

In Fig. 2, we can observe the directions of the asymptotes, whose number is $j+1$. The considered dependence of $\langle \hat{J}_3 \rangle$ on γ is monotonous in the interval under consideration for $l=0$ and $l=1$ (the negative solution decreases and the positive one increases) as follows from Fig. 3. In this interval, this dependence is monotonous also for $l=2j-1=6$ and $l=2j=7$ (the negative solution increases and the positive one decreases). For $j=4$, Fig. 4, we observe the same behavior, essentially.

In Figs. 5 and 6, we see that the Fisher measure of infor-

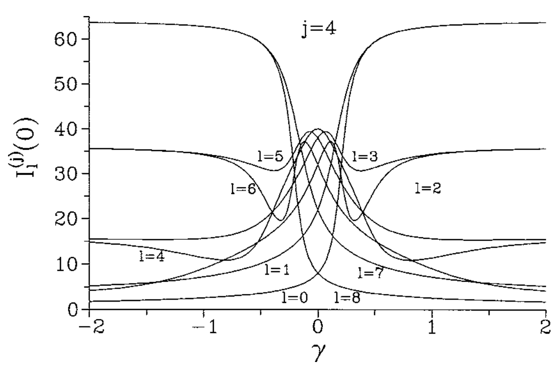

FIG. 6. Fisher's measure of information $I_l^{(j)}(\phi)$ for $\phi=0$, $j=4$, $l=0,\ldots,8$, and $\gamma \in [-2,2]$.

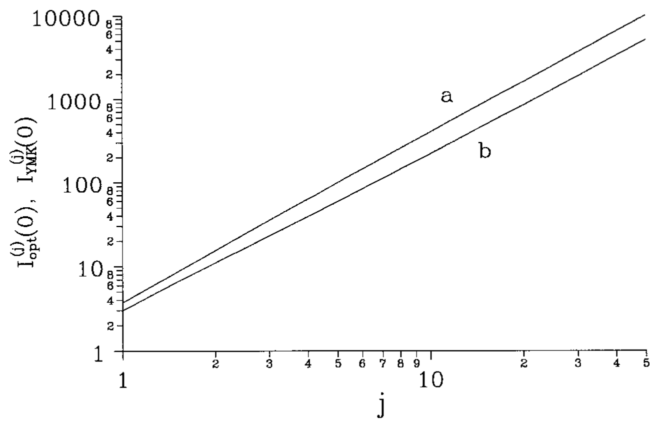

FIG. 7. Log-log plot of Fisher's measure of information $I_{\text{opt}}^{(j)}(0)$ (curve a) and $I_{\text{YMK}}^{(j)}(0)$ (curve b).

mation $I_l^{(j)}(0,\gamma)$ increases in the interval under consideration for $l=0$ and $l=1$. In this interval, this measure decreases for $l=2j-1$ and $l=2j$. From this it follows that the dependence of the Fisher measure of information $I_l^{(j)}(0)$ on $\langle \hat{J}_3 \rangle$ decreases when $\gamma_{\text{crit}}<0$ (the subscript "crit" stands for critical) and increases when $\gamma_{\text{crit}}>0$ for $l=0,1,2j-1,2j$. Taking into account that the maximum is attained for $\gamma_{\text{crit}}>0$ and $l=0$ and for $\gamma_{\text{crit}}<0$ and $l=2j$, we observe that when negative values of $\langle \hat{J}_3 \rangle$ increase to zero, the Fisher measure of information increases, and when $\langle \hat{J}_3 \rangle$ increases from zero to positive values, the Fisher measure of information decreases.

In Fig. 7, we can see the values of Fisher's measure of information for the Yurke-McCall-Klauder quality state and the optimum state of the same mean photon-number difference, $\langle \hat{J}_3 \rangle = \frac{1}{2}$. In the log-log plot, the latter dependence has a slightly convex graph and is described asymptotically by $4\alpha_{\text{YMK}} j^2$. In the log-log plot, the former dependence has a slightly concave graph and behaves asymptotically as $4\alpha_{\text{opt}} j^2$, with $1=\alpha_{\text{opt}}>\alpha_{\text{YMK}}=\frac{1}{2}$. Figure 8 helps define the optimum input state of the Mach-Zehnder interferometer relative to the constraint of $\langle \hat{J}_3 \rangle = \frac{1}{2}$. Furthermore, the optimum input states have $l=2j$. All the γ values are negative and quickly rise from a value -3.4 to the limit value -1.

In Fig. 9, the maximum generalized eigenvalues $\mu_{\text{opt,max}}^{(j)}$ for the optimum input state are plotted showing that the generalized eigenvalues are negative for $j=2,\frac{5}{2},\ldots,25$. Therefore, the condition (4.4) is fulfilled for these values of j. To characterize all the generalized eigenvalues for the optimum input states, the minimum generalized eigenvalues $\mu_{\text{opt,min}}^{(j)}$ are plotted in Fig. 10. They decrease asymptotically as fast as $-j\left(\frac{4}{3}+j\right)$.

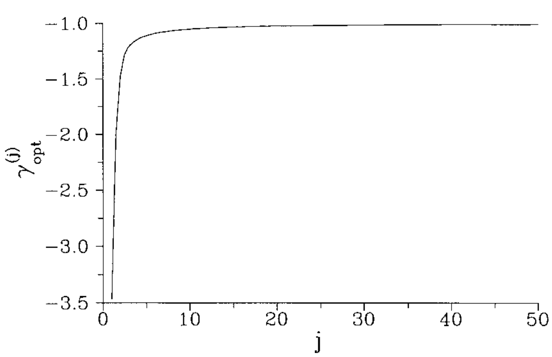

FIG. 8. The parameter $\gamma_{\text{opt}}^{(j)}$ vs j.

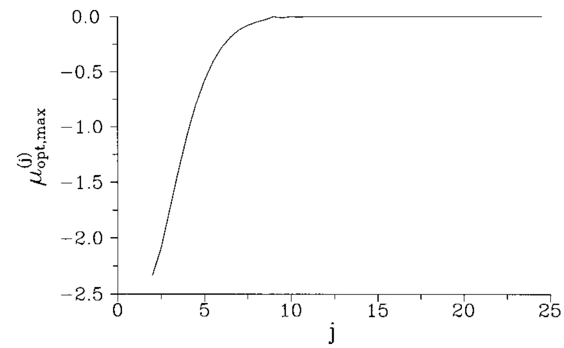

FIG. 9. The extreme generalized eigenvalues $\mu_{\text{opt,max}}^{(j)}$ for the optimum input state vs j, $j=2,\frac{5}{2},\ldots,25$.

V. QUASIDISTRIBUTIONS ON THE SPHERE

Any statistical operator $\hat{\rho}$, which describes the state of a two-mode field with the definite photon-number sum, has the property (2.7) and its $\Phi_{\mathcal{A}}$ quasidistribution (Q function) can be simplified. Concretely, the usual quasidistribution

$$\Phi_{\mathcal{A}}(\alpha_1,\alpha_2) = \frac{1}{\pi^2}\,{}_1\langle \alpha_1|_2\langle \alpha_2|\hat{\rho}^{(\mathcal{N})}|\alpha_1\rangle_1|\alpha_2\rangle_2, \tag{5.1}$$

where $|\alpha_1\rangle_1|\alpha_2\rangle_2$ is the usual coherent state, with $\mathcal{A}$ and $\mathcal{N}$ standing for the antinormal and normal orderings of the annihilation and creation operators, respectively, can be related to the generalized $\Phi^{\text{SU}(2)}(j,\zeta)$ quasidistribution

$$\Phi^{\text{SU}(2)}(j,\zeta) = \frac{4}{(1+|\zeta|^2)^2} Q_{\text{SU}(2)}(j,\mathbf{n}^{\text{SU}(2)}(\zeta)), \tag{5.2}$$

with the quasidistribution

$$Q_{\text{SU}(2)}(j,\mathbf{n}^{\text{SU}(2)}(\zeta)) = \frac{2j+1}{4\pi}\langle j,\zeta|\hat{\rho}|j,\zeta\rangle. \tag{5.3}$$

Here, $\mathbf{n}^{\text{SU}(2)}(\zeta)$ is the position vector of a point parametrized by $\zeta=\frac{\alpha_1}{\alpha_2}$,

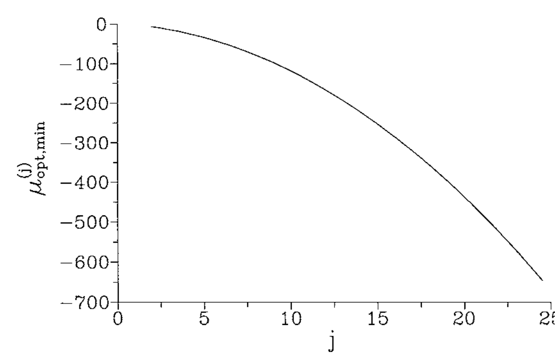

FIG. 10. The extreme generalized eigenvalues $\mu_{\text{opt,min}}^{(j)}$ for the optimum input state vs j, $j=2,\frac{5}{2},\ldots,25$.

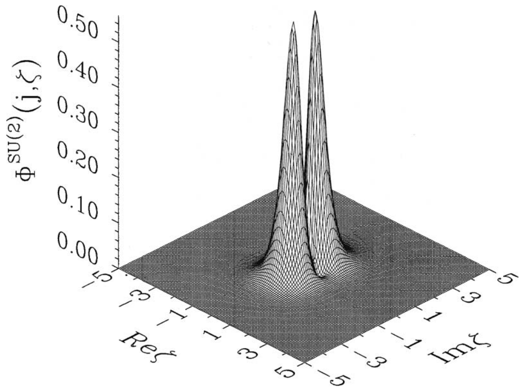

FIG. 11. The quasidistribution $\Phi^{SU(2)}(j,\zeta)$ for $j=\frac{7}{2}$ and the optimum state relative to the condition $\langle \hat{J}_3\rangle=\frac{1}{2}$.

$$\mathbf{n}^{SU(2)}(\zeta)=\frac{1}{1+|\zeta|^2}(-2\,\mathrm{Re}\,\zeta,2\,\mathrm{Im}\,\zeta,1-|\zeta|^2). \tag{5.4}$$

To see this, we introduce the quasidistribution

$$\Phi'(W,\varphi,\zeta)=\frac{W}{2(1+|\zeta|^2)}\Phi_{\mathcal{A}}\left(\frac{|\zeta|\sqrt{W}}{\sqrt{1+|\zeta|^2}}\sqrt{e^{i\varphi}\frac{\zeta}{|\zeta|}},\frac{\sqrt{W}}{\sqrt{1+|\zeta|^2}}\sqrt{e^{i\varphi}\frac{\zeta^*}{|\zeta|}}\right), \tag{5.5}$$

where the ambiguity of the complex square roots $e^{i\varphi_j}$, $j=1,2$, obeys the equations

$$e^{i\varphi_1}e^{i\varphi_2}=e^{i\varphi},\quad e^{i\varphi_1}(e^{i\varphi_2})^{-1}=\frac{\zeta}{|\zeta|}, \tag{5.6}$$

viz., it is twofold, not fourfold. The quasidistribution $\Phi'(W,\varphi,\zeta)$ can be decomposed

$$\Phi'(W,\varphi,\zeta)=P_{\mathcal{A}}(W)P(\varphi)\Phi^{SU(2)}(j,\zeta), \tag{5.7}$$

where

$$P_{\mathcal{A}}(W)=\frac{W^{2j+1}}{(2j+1)!}e^{-W}, \tag{5.8}$$

$$P(\varphi)=\frac{1}{2\pi}. \tag{5.9}$$

The meaning of the equality (5.5) is the introduction of a quasidistribution of the integrated intensity W, the phase sum φ, and the SU(2) complex amplitude. The situation with the quasidistributions in the case of the SU(2) group is not as simple as in the ordinary case, where the action of the group M(2) does not affect the visual impression essentially, but in the case under consideration the action of the group O(3) changes the shape of a quasidistribution.

To illustrate the states providing the optimum Fisher measure under the constraint $\langle \hat{J}_3\rangle=\frac{1}{2}$, we choose $j=\frac{7}{2}$ and $j=4$, which results in $l=2j$. As to the $\Phi^{SU(2)}(j,\zeta)$ quasidistribution, we can see in Figs. 11 and 12 the superposition state shape, the superposition state being

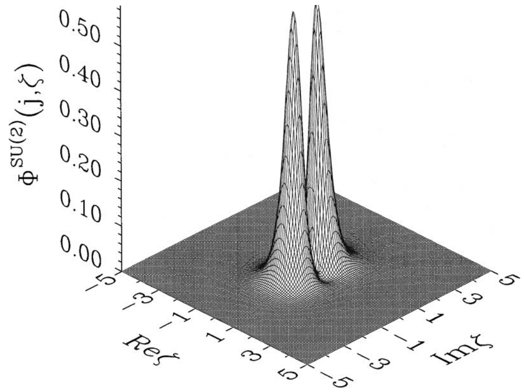

FIG. 12. The same as in Fig. 11, but for $j=4$.

$$\frac{1}{\sqrt{2}}\left(\left|j,i\sqrt{\frac{2j+1}{2j-1}}\right\rangle+\left|j,-i\sqrt{\frac{2j+1}{2j-1}}\right\rangle\right). \tag{5.10}$$

VI. CONCLUSION

In this paper we have studied the Fisher measure of information as a good alternative sensitivity measure of a Mach-Zehnder interferometer. For a description of this interferometer, we have used an SU(2) group representation and considered also an SU(1,1) interferometer. We have introduced a notion of a (fictive) M(2) interferometer to illustrate both the SU(2) and SU(1,1) interferometers when in an appropriate limit.

We have derived that the minimum detectable phase shift (the conventional phase sensitivity) is equal to the reciprocal value of the Fisher information (the Cramér-Rao lower bound) determined in zero phase shift for the interferometers under consideration if input states of such a device are SU(2) and SU(1,1) generalized coherent states and a usual coherent state.

We have tried to define states of optimum Fisher measure of information for zero phase shift. Unlike the conventional phase sensitivity, which has emerged for zero phase shift without difficulties, the Fisher measure under zero phase shift does not seem to be a firm basis for optimization. It may be connected also with the fact that the conventional phase sensitivity, which is independent of the phase shift in some cases, is an operational characteristic of an existing estimator of the phase shift for the zero phase shift, whereas the Fisher information measure can exist even when the phase estimator does not exist, whose variance is equal to the reciprocal value of the Fisher information.

We have found out that on the basis of the alternative measure, a state can still be defined that enables one to measure phase shift up to the sign. Although this limitation suggests the famous situation, which occurs for two input Fock states of the same photon numbers, we have not tried to assess the state proposed by us as an input state.

ACKNOWLEDGMENTS

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APPENDIX: REGULARITY CONDITIONS

Let n stand for a number of trials. Let $\mathcal{B}^n$ denote a σ-algebra of Borel subsets of R^n. Let T mean the transpose of a matrix. Let Θ be a one-dimensional parameter and let Ω denote the parametric space, $\Omega \subset \mathrm{R}$. Let us assume that a random vector $\mathbf{X}=(X_1,\ldots,X_n)^T$ has a density $f(\mathbf{x},\Theta)$ with respect to a σ-finite measure μ on $\mathcal{B}^n$.

Let us recall that, for a definition of the integral $\int_B f(\mathbf{x})d\mu(\mathbf{x})$, it is also required that $B \in \mathcal{B}^n$ and a not very stringent condition be satisfied that $f(\mathbf{x})$ is defined for almost all $\mathbf{x} \in \mathrm{R}^n$ (with respect to μ). We will say that a system of densities $\{f(\mathbf{x},\Theta),\Theta \in \Omega\}$ is regular if the following conditions are satisfied: (i) Ω is a nonempty open set; (ii) The set $M=\{x: f(\mathbf{x},\Theta)>0\}$ does not depend on Θ; (iii) For almost all $\mathbf{x} \in M$ (with respect to μ) a finite partial derivative $f'(\mathbf{x},\Theta)=\frac{\partial f(\mathbf{x},\Theta)}{\partial \Theta}$ exists; (iv) For all $\Theta \in \Omega$, $\int_M f'(\mathbf{x},\Theta)d\mu(\mathbf{x})=0$ holds; and (v) The integral

$$I(\Theta) = \int_M \left[\frac{f'(\mathbf{x},\Theta)}{f(\mathbf{x},\Theta)}\right]^2 f(\mathbf{x},\Theta)d\mu(\mathbf{x}) \qquad \text{(A1)}$$

is finite and positive.

This definition includes the case of a discrete random vector in which μ is a counting measure. The counting relates to elements of the set Z^n (formed by n tuples of integers), and a characteristic property of the counting measure is the identity

$$\int_B f(\mathbf{x})d\mu(\mathbf{x}) = \sum_{\mathbf{x}\in B} f(x), \quad \text{any} \quad B \subset Z^n, \qquad \text{(A2)}$$

which holds as soon as one of the sides of the relation is meaningful. The density $f(\mathbf{x},\Theta)$ is defined only in Z^n. Traditionally, we then write $p(\mathbf{x},\Theta)$ instead of $f(\mathbf{x},\Theta)$. The condition (i) remains unchanged and the other conditions become: (ii′) The set $M=\{x: p(\mathbf{x},\Theta)>0\}$ does not depend on Θ. Obviously $M \subset Z^n$; (iii′) For almost all $\mathbf{x} \in M$ (with respect to μ) a finite partial derivative $p'(\mathbf{x},\Theta)=\frac{\partial p(\mathbf{x},\Theta)}{\partial \Theta}$ exists; (iv′) For all $\Theta \in \Omega$, $\Sigma_{\mathbf{x}\in M} p'(\mathbf{x},\Theta)=0$ holds. This equality is evident if $|M|<\infty$; and (v′) The sum

$$I(\Theta) = \sum_{\mathbf{x}\in M}\left[\frac{p'(\mathbf{x},\Theta)}{p(\mathbf{x},\Theta)}\right]^2 p(\mathbf{x},\Theta) \qquad \text{(A3)}$$

is finite and positive.

In this application, the number of trials is $n=1$ and there is a difficulty with satisfying the condition (ii′). It is natural to consider a set Ω such that $\phi=0 \in \Omega$. For some input states $M(\phi)$, $\phi=0$, then consists of even integers only, but $M(\phi)$, $\phi \neq 0$, is equal to Z_+. For simplicity we consider the M(2) interferometer; the SU(2) and SU(1,1) interferometers are described with appropriate changes. For the SU(2) interferometer $|M|<\infty$, so the condition (iv′) is satisfied and in the condition (iii′) "almost all" is equivalent to "all." Thus, either we choose Ω such that $0 \bar{\in} \Omega$, or we rely on the regularization, which occurs on using the rule $p(\mathbf{x},\Theta)=|c(\mathbf{x},\Theta)|^2$.

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