Three-dimensional cavity cooling and trapping in an optical lattice

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(Received 23 March 2006; published 30 June 2006)

A robust scheme for trapping and cooling atoms is described. It combines a deep dipole-trap which localizes the atom in the center of a cavity with a laser directly exciting the atom. In that way one obtains three-dimensional cooling while the atom is dipole-trapped. In particular, we identify a cooling force along the large spatial modulations of the trap. A feature of this setup, with respect to a dipole trap alone, is that all cooling forces keep a constant amplitude if the trap depth is increased simultaneously with the intensity of the probe laser. No strong coupling is required, which makes such a technique experimentally attractive. Several analytical expressions for the cooling forces and heating rates are derived and interpreted by analogy to ordinary laser cooling.

DOI: 10.1103/PhysRevA.73.063415 PACS number(s): 32.80.Pj

I. INTRODUCTION

Optical dipole-force traps can be used to hold and manipulate atoms. They allow for a deep confinement of atoms but provide no significant cooling [1]. Often, lifetimes of particles in optical traps are limited by intensity fluctuations of the trapping beam rather than by background gas collisions. In order to lower the temperature and improve the storage times of particles in optical traps, one can apply extra cooling lasers. Here we show that the addition of only one cooling laser and an optical cavity leads to cooling forces in all directions. Those forces can be made independent of the trap depth by increasing the intensity of the cooling laser accordingly. This makes such a scheme particularly attractive as it combines deep confinement of particles and a persistent cooling mechanism. This configuration and related others techniques [2–11] (see also references therein) are of particular interest for laser cooling and trapping of particles.

The simplest cooling mechanism works as follows. The setup is depicted in Fig. 1, where two mirrors surround a trapped atom that is directly excited by a laser. Although the atomic resonances could be detuned from the pump laser, if a cavity frequency ω_{cav} is close to that of the laser ω_L, the atom will act as a mediator that efficiently transfers the radiation field from the laser into the cavity. This gives rise to a two-dimensional force acting along the laser and cavity axis [2,3]. This force cools the particle's motion if the cavity accepts photons with a frequency that is higher than the ones absorbed from the laser. We show here, the trap also induces a cavity-dependent cooling force, mainly oriented along the most tightly confining direction (i.e., the "trap axis"). In sharp contrast to a dipole trap alone, or a dipole trap with a cooling laser [12], such a cooling force can be made insensitive to the trap depth. In the simplest picture, the trap spatially modulates the internal energy levels of the atom (ac Stark shifts). An atom moving in the trap will hence see its internal energy levels modified. This leads to a spatial variation of its dipole moment, resulting in a variation of the radiation field being scattered into the cavity. This change in the cavity field is sensed by the atom and gives rise to a third cooling force, now directed along the trap axis.

Thus, with a three-dimensional arrangement of one pump laser, a cavity and a trapping potential, one obtains cooling in three dimensions. For a deep enough trap, these forces are mainly determined by the cavity characteristics: The cooling forces show a constant magnitude if one simultaneously increases the trap depth and the power of the pump laser such that the excitation probability is kept constant. This idea has been applied to explain the very long trapping times reported in a recent experiment [13]. The success of that experiment essentially relies on the presence of the trap. The cooling force along the standing wave dipole-trap axis has a major contribution because the atom was loaded into the cavity along that axis, and hence had the largest velocity component. The reported storage times are much longer than those obtained in a one-dimensional geometry, where the cavity is excited and where a dipole trap is created by the intracavity field [4,5,14–16]. In the experiment [13] an atom entering

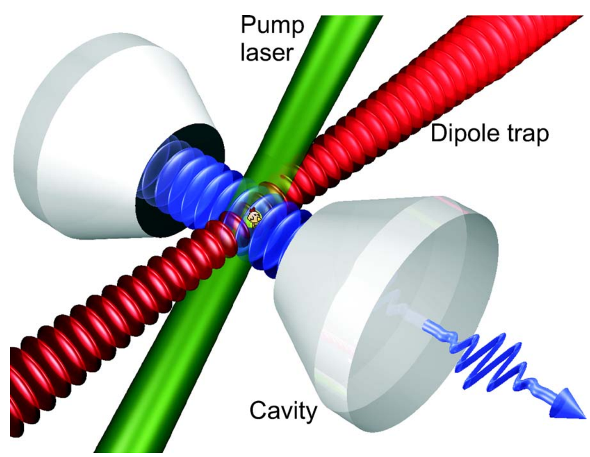

FIG. 1. (Color online) Sketch for the trapping of atoms at the center of a cavity with a near-resonant pump laser. The atom excited by the pump scatters photons into the cavity and experiences cooling forces in three dimensions due to the trap and the cavity. Cooling along the trap is cavity-induced and largely dependent on the cavity characteristics.

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the cavity is caught and cooled by scattering photons at a low but constant rate into the cavity, indicating a strong localization.

This paper provides and extends the theory highlighted in [13]. The extension takes into account all the higher orders in the coupling g between the atom and the cavity field. Our theoretical approach follows from a combination of [4,5,12], where a trap is introduced, and [7–9] where an extension of Refs. [2,3] to all orders in the coupling is treated (but at low atomic velocity). A setup close to ours has been discussed recently for the study of the quantum motion [11]. The differences are, however, (i) that we consider a more general trap (this distinction is at the origin of the above-mentioned trap-induced cooling force), and (ii) that we treat the motion classically. The difficulty so far encountered in most works is that the cooling forces and their fluctuations exhibit nontrivial structures, making a quantitative understanding difficult. A general and simple solution for the diffusion (due to light-force fluctuations) exists [17]. Here we provide a detailed discussion specific to our scheme. Several approximations for the cooling forces are given with different pictures. In particular, we give a straightforward equation for the cooling force in the regime studied in [8] and show that the resonance reported there has a natural correspondence to the case where the cavity is excited. For those reasons the paper draws a constant parallel to free-space laser cooling theory, starting from the simplest limit [2,3,13] to extended ones as we include higher orders in the coupling between atom and cavity. The paper is thus organized as follows: After introducing the model in Sec. II, we focus on the low saturation limit in Sec. III and write the needed Maxwell-Bloch equations. In Sec. IV we discuss in detail the steady-state quantities for an atom at rest, photon number, excitation probability, and forces. Those quantities are mathematically manipulated and written in different ways, which makes Sec. IV important for the understanding of the cooling forces. Those cooling forces are then addressed in Sec. V, starting from the simplest approximations to the exact (low saturation) solutions. The momentum diffusion and attainable temperature are then discussed.

Generalized cooperativity parameter

The cooperativity parameter $C=g^2/2\kappa\gamma$ has an important role in cavity QED. Here g is the coupling between atom and cavity, $1/2\gamma$ the lifetime of the atom's upper state, and $1/2\kappa$ that of a cavity photon. That parameter indicates how much the atom reacts to the mode and vice versa. Since C is independent of the frequency of any laser, it shows obvious limitations when the system is explored as a function of the atomic detuning Δ_a and cavity detuning Δ_c (those detunings are properly defined in the next section). It is more convenient to introduce the dimensionless parameter ν:

$$\nu = \frac{g^2(\boldsymbol{r})}{[\Delta_a(\boldsymbol{r}) - i\gamma](\Delta_c - i\kappa)}, \tag{1}$$

which is a generalization of C. As we shall see, the advantage is not only that the equations become transparent, but they also lead to a straightforward understanding of the presence of the cavity. Although ν is a complex number, its estimation is easy. For instance, for large detunings $\Delta_a \gg \gamma$ one has $|\nu| \approx g^2/\Delta_a\kappa$ evaluated for $\Delta_c=0$. Therefore one can neglect ν for $\Delta_a \gg g^2/\kappa$. As shown below, such a limit $|\nu| \to 0$ of "low (generalized) cooperativity" corresponds to the theories in [2,3,13]. The other limit we are concerned with is $\nu \lesssim 1$. Ignoring (γ,κ), the limit $\nu \to 1$ is reached for $\Delta_a\Delta_c \to g^2$. But that condition corresponds to driving the lowest manifold of the (atom-cavity) dressed states into resonance, which means that higher orders in the coupling g are involved (actually one can show that $\nu=1$ strictly implies $\Delta_a\Delta_c=g^2$ and $\gamma=\kappa=0$). Those two limits lead to expressions for the cooling forces that have a common form in all directions. The parameter ν is discussed in more detail in Ref. [6] where interpretations and general properties are described.

II. TRAPPED ATOM COUPLED TO A SINGLE CAVITY MODE AND A NEAR RESONANT LASER

We begin with the Jaynes-Cummings Hamiltonian for a two-state atom of momentum $\boldsymbol{p}$ and mass m, with lowering operator σ, coupled to a single cavity mode with creation operator $a^\dagger$, and we add a trap and a laser which directly excites the atom:

$$\begin{aligned} H/\hbar = {} & \frac{\boldsymbol{p}^2}{2m\hbar} + \Delta_a(\boldsymbol{r})\sigma^\dagger\sigma - \eta(\boldsymbol{r})\sigma^\dagger - \eta^*(\boldsymbol{r})\sigma + \Delta_c a^\dagger a + g(\boldsymbol{r}) \\ & \times (a\sigma^\dagger + a^\dagger\sigma) + U(\boldsymbol{r}). \end{aligned} \tag{2}$$

Here, $2g$ is the atom-cavity vacuum Rabi frequency. We have assumed the rotating wave approximation and written the Hamiltonian in the interaction picture with respect to the laser frequency ω_L. The atom is coherently excited by a laser (running wave, standing wave, or other) where $2|\eta(\boldsymbol{r})|$ is the corresponding Rabi frequency. While we also treat the standing-wave case, we focus more on a running wave laser representing photons propagating with momenta $\hbar\boldsymbol{k}_L$ [$\eta(\boldsymbol{r}) = \eta_0 e^{i\boldsymbol{k}_L\cdot\boldsymbol{r}}$]. The optical lattice is generated by a far-detuned laser standing wave. Such a trap is assumed unperturbed by the presence of both the atom and the near resonant laser. Its effect is to modulate in space the upper state energy by the amount $\hbar V_e(\boldsymbol{r})$ and the ground state by $\hbar V_g(\boldsymbol{r})$ [4,5,12]. Relative to the ground state, such an ac Stark shift leads to a shift $\hbar\Delta_S=\hbar[V_e(\boldsymbol{r})-V_g(\boldsymbol{r})]$ and to a conservative potential $\hbar U(\boldsymbol{r}) = \hbar V_g(\boldsymbol{r})$. If the detuning between the (unperturbed) atom and the laser is $\omega_{eg}-\omega_L$, the effect of the trap is contained in the effective total detuning $\Delta_a(\boldsymbol{r})=\omega_{eg}-\omega_L+\Delta_S(\boldsymbol{r})$. The laser light is near resonant to a cavity mode of frequency ω_{cav}, but can be detuned from it by the amount $\Delta_c=\omega_{cav}-\omega_L$. The master equation describing the evolution of the reduced density matrix ρ for the atom-cavity-mode system accounts for the loss mechanisms (atom and cavity decay):

$$\dot\rho = \mathcal{L}\rho = -i[H,\rho]/\hbar + \kappa L_a\rho + \gamma L_\sigma\rho, \tag{3}$$

with

$$L_a\rho = 2a\rho a^\dagger - \rho a^\dagger a - a^\dagger a\rho,$$

$$L_\sigma\rho = 2\int d^2\hat{\boldsymbol{k}}N(\hat{\boldsymbol{k}})e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\sigma\rho\sigma^\dagger e^{+i\boldsymbol{k}\cdot\boldsymbol{r}} - \rho\sigma^\dagger\sigma - \sigma^\dagger\sigma\rho,$$

where $\hat{\boldsymbol{k}}=\boldsymbol{k}/k$ ($k=\omega_{eg}/c$, recall that ω_{eg} is an atomic transition frequency) is the direction of spontaneously emitted photons, with the (dipole) angular distribution $N(\hat{\boldsymbol{k}})$ [$\int d^2\hat{\boldsymbol{k}}N(\hat{\boldsymbol{k}})=1$]. The dissipative term $\gamma L_\sigma\rho$ describes the coupling of the atom to a continuum of initially empty modes other than the cavity one. The (spontaneous) emission into the vacuum modes results in a force, which vanishes on average due to equal distribution of spontaneously emitted photons in opposite directions $\int d^2\hat{\boldsymbol{k}}\boldsymbol{k}N(\hat{\boldsymbol{k}})=0$. The fluctuations of that force contribute, however, to the diffusion through the so-called spontaneous emission term [18]. In the following we evaluate the force $\mathcal{F}=-\boldsymbol{\nabla}H$ and the diffusion in the low saturation regime by studying the Heisenberg equations of motion.

III. MAXWELL-BLOCH EQUATIONS AT LOW SATURATION

We focus on the low saturation limit, more precisely we identify the internal state of the atom with a harmonic oscillator $[\sigma,\sigma^\dagger]\to\mathbb{1}$. Most of the experiments and theories have dealt with that limit (otherwise see [6,19,20]). In that approximation the atom spends most of its time in the ground state such that one can assume equally separated fictitious higher states other than the ground and excited states. Notice that even at low saturation, such an approximation of an atom treated as a dipole oscillator might be insufficient. For example, in [13] a Sisyphus mechanism has been identified, and this is due to saturation effects. A general method to obtain such kind of a saturation-induced force is presented in [6]. By writing the time evolution of any operator O, $\langle\dot{O}\rangle_\rho=\mathrm{tr}(O\dot{\rho})$, the first two Maxwell-Bloch equations read:

$$\frac{d}{dt}\langle\sigma\rangle_\rho = -i\widetilde{\omega}_a\left\{\langle\sigma\rangle_\rho + \frac{g}{\widetilde{\omega}_a}\langle a\rangle_\rho - \frac{\eta}{\widetilde{\omega}_a}\right\}, \tag{4a}$$

$$\frac{d}{dt}\langle a\rangle_\rho = -i\widetilde{\omega}_c\left\{\langle a\rangle_\rho + \frac{g}{\widetilde{\omega}_c}\langle\sigma\rangle_\rho\right\}. \tag{4b}$$

The approximation of a harmonic oscillator is here visible only through the equation of evolution of $\langle\sigma\rangle_\rho$, where we have substituted $g\langle a(\sigma^\dagger\sigma-\sigma\sigma^\dagger)\rangle_\rho\to-g\langle a\rangle_\rho$. The complex detunings

$$\widetilde{\omega}_a = \Delta_a(\boldsymbol{r}) - i\gamma, \quad \widetilde{\omega}_c = \Delta_c - i\kappa, \tag{5}$$

lead to simplifications of the equations and will be used for intermediate calculations.

Below we first assume the atom to be at rest and extract the needed information about the light coming out through the cavity mirrors, the emission rate into free space, and on the forces. We then assume that the atom moves slowly enough $k_L v\ll(\gamma,\kappa)$ such that it is possible to expand the force up to first order in the velocity $\boldsymbol{v}$, to obtain the friction force.

IV. STEADY STATE FOR AN ATOM AT REST

A. Excitation probability and cavity-photon number

For an atom at rest, the expectation values are labeled without the index "ρ." One can understand the effects of the cavity by the following arguments. It is first assumed that the cavity is absent $g=0$, and that there is no trap [$\Delta_S=0, U(\boldsymbol{r})=0$]. The atom absorbs photons from the laser and emits them in all directions. This scattering process leads to a stationary excitation probability $P_e=\langle\sigma^\dagger\sigma\rangle$ of the upper state $|e\rangle$. The dynamics are given by the familiar Bloch equations, which in the harmonic limit reduce to Eq. (4a), and hence give the steady-state optical coherence $\langle\sigma\rangle^{free}=\eta/\widetilde{\omega}_a$ and the occupation probability $P_e^{free}\approx|\langle\sigma\rangle^{free}|^2$. The superscript "free" refers to the absence of a cavity (free space). If now the trap is switched on and if the atom sits at the bottom of the well, the excitation probability decreases due to an increase of the (effective) Stark-shifted detuning $|\Delta_a(\boldsymbol{r})|\gg\gamma$. The excitation probability has, however, the same mathematical form as before, but accounts now for the position dependence in $\Delta_a(\boldsymbol{r})$:

$$P_e^{free} = \frac{|\eta(\boldsymbol{r})|^2}{\Delta_a^2(\boldsymbol{r})+\gamma^2}. \tag{6}$$

As a function of Δ_a, this is the familiar Lorentz absorption curve for an atom in free space [21].

By adding the cavity $g\neq0$, the atom coherently emits and absorbs cavity photons before either the atom irreversibly emits the radiation into free space or a cavity photon escapes from the mirrors. The excitation probability $P_e\neq P_e^{free}$ is now changed due to the presence of a field inside the cavity $\langle a\rangle\neq0$. From Eq. (4a), in steady state the optical coherence is in fact given by the total effective field $\eta-g\langle a\rangle$ the atom interacts with: $\langle\sigma\rangle=(\eta-g\langle a\rangle)/(\Delta_a-i\gamma)$. The cavity field which is created by the presence of the atom is in turn given by Eq. (4b) $\langle a\rangle=-g\langle\sigma\rangle/(\Delta_c-i\kappa)$. Eliminating the field amplitude $\langle a\rangle$ in the equation giving $\langle\sigma\rangle$, leads to the simple equation:

$$\langle\sigma\rangle = \eta/\widetilde{\omega}_a + \nu\langle\sigma\rangle. \tag{7}$$

Since $\eta/\widetilde{\omega}_a=\langle\sigma\rangle^{free}$, we conclude that the presence of the cavity is entirely contained in the parameter ν, Eq. (1). Made explicit, the atomic expectation values read:

$$\langle\sigma\rangle = \frac{\eta(\boldsymbol{r})}{\Delta_a(\boldsymbol{r})-i\gamma}\frac{1}{1-\nu(\boldsymbol{r})}, \tag{8a}$$

$$P_e = \frac{|\eta(\boldsymbol{r})|^2}{\Delta_a^2(\boldsymbol{r})+\gamma^2}\frac{1}{|1-\nu(\boldsymbol{r})|^2}. \tag{8b}$$

The cavity-field amplitude $\langle a\rangle=-g\langle\sigma\rangle/\widetilde{\omega}_c$ and cavity photon number $N_{cav}=\langle a^\dagger a\rangle\approx|\langle a\rangle|^2$ then read with the use of Eqs. (8):

$$\langle a \rangle = -\frac{g}{(\Delta_c - i\kappa)} \frac{\eta(\boldsymbol{r})}{[\Delta_a(\boldsymbol{r}) - i\gamma]} \frac{1}{1-\nu(\boldsymbol{r})}, \tag{9a}$$

$$N_{cav} = \frac{g^2(\boldsymbol{r})}{\Delta_c^2 + \kappa^2} P_e. \tag{9b}$$

In the limit of large detuning Δ_a, $\nu \propto 1/\Delta_a \ll 1$, thus $P_e \approx P_e^{free}$. This means that the emission rate of photons from the cavity $\propto 2\kappa N_{cav}$ is a Lorentz curve as a function of the cavity detuning Δ_c (9b) [the atomic excitation (8b) is unaffected by the cavity as it reduces to Eq. (6)]. The approximation $\nu \to 0$ is met if either Δ_a is large, the cavity is far detuned with respect to the laser, or dissipation is large $(\gamma, \kappa) \gg g$. Of course, this is relative to the strength of the coupling g between atom and cavity. For example, for strong coupling and a cavity on resonance with the laser $\Delta_c = 0$, the trap should be sufficiently deep $|\Delta_a| \gg g^2/\kappa$ in order to neglect the back action (7) of the cavity-mode on the atom, and vice versa [2,3,13].

In order to get other physical pictures as well as transparent equations, one introduces the effective detuning $\Delta_{a,\mathrm{eff}} = \mathrm{Re}[\widetilde{\omega}_a(1-\nu)]$ and decay rate $\Gamma_{\mathrm{eff}} = -\mathrm{Im}[\widetilde{\omega}_a(1-\nu)]$:

$$\Delta_{a,\mathrm{eff}}(\boldsymbol{r}) = \Delta_a(\boldsymbol{r}) - \frac{g^2(\boldsymbol{r})}{\Delta_c^2 + \kappa^2}\Delta_c, \tag{10a}$$

$$\Gamma_{\mathrm{eff}}(\boldsymbol{r}) = \gamma + \frac{g^2(\boldsymbol{r})}{\Delta_c^2 + \kappa^2}\kappa. \tag{10b}$$

In that picture the excitation probability becomes $P_e = |\eta|^2/(\Delta_{a,\mathrm{eff}}^2 + \Gamma_{\mathrm{eff}}^2)$, which when compared to Eq. (6) indicates that the presence of the cavity induces (i) a light shift, i.e., one substitutes $\Delta_a \to \Delta_{a,\mathrm{eff}}$, and (ii) for a broadening of the atom's line width $2\gamma \to 2\Gamma_{\mathrm{eff}}$. In particular, for $\Delta_c = 0$, $\Gamma_{\mathrm{eff}} = \gamma(1+2C)$ is expressed in terms of the Purcell factor $1 + g^2/\kappa\gamma$. Similarly the effective cavity detuning $\Delta_{c,\mathrm{eff}} = \mathrm{Re}[\widetilde{\omega}_c(1-\nu)]$ and decay rate $K_{\mathrm{eff}} = -\mathrm{Im}[\widetilde{\omega}_c(1-\nu)]$ are introduced:

$$\Delta_{c,\mathrm{eff}}(\boldsymbol{r}) = \Delta_c - \frac{g^2(\boldsymbol{r})}{\Delta_a(\boldsymbol{r})^2 + \gamma^2}\Delta_a(\boldsymbol{r}), \tag{11a}$$

$$K_{\mathrm{eff}}(\boldsymbol{r}) = \kappa + \frac{g^2(\boldsymbol{r})}{\Delta_a(\boldsymbol{r})^2 + \gamma^2}\gamma. \tag{11b}$$

Notice that, with these new variables, the cavity photon number $N_{cav} = g^2 P_e^{free}/(\Delta_{c,\mathrm{eff}}^2 + K_{\mathrm{eff}}^2)$ refers to the free space excitation probability P_e^{free} instead of P_e. In that picture the atom acts like a refractive index which shifts the cavity resonance $\Delta_c \to \Delta_{c,\mathrm{eff}}$, and broadens the Lorentz cavity profile $\kappa \to K_{\mathrm{eff}}$. For $\nu \to 0$, $(\Delta_{c,\mathrm{eff}}, K_{\mathrm{eff}})$ reduce to (Δ_c, κ). The cavity photon number is plotted in Fig. 2. One sees that the main difference between having cooperative effects or not (ν large or small) is that already for an atom at rest the scattering rate into the cavity is no longer maximum for a cavity on resonance $\Delta_c = 0$.

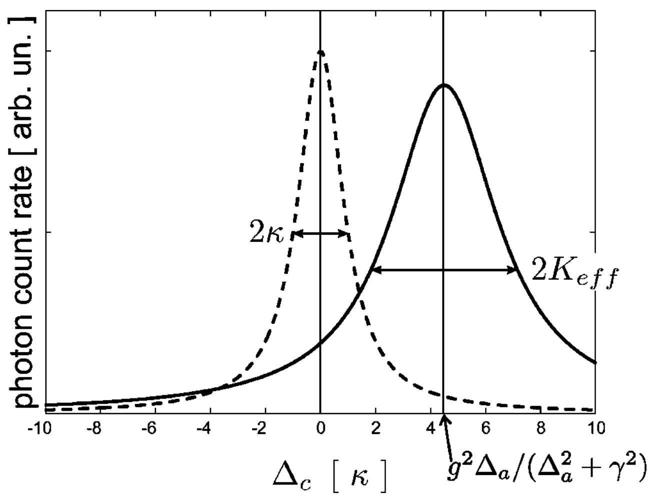

FIG. 2. Cavity-photon count rate ($\propto 2\kappa N_{cav}$) at the output of one mirror as a function of the cavity detuning Δ_c. The vertical scaling is arbitrary $[N_{cav} \propto |\eta(\boldsymbol{r})|^2]$. The count rate (solid) is centered around $g^2\Delta_a/(\Delta_a^2 + \gamma^2)$, which corresponds to $\Delta_{c,\mathrm{eff}} = 0$, and is here shifted by several cavity line widths κ. Cooling occurs on the right side of that curve $\Delta_{c,\mathrm{eff}} > 0$, and is optimum for $\Delta_{c,\mathrm{eff}} \approx K_{\mathrm{eff}}$. The dashed curve [2,3,13] reflects the limit $\nu \to 0$, reached for $\Delta_a \gg g^2/\kappa$, representing a Lorentz curve as if an empty cavity is excited through one of the cavity mirrors.

B. Dipole-trap force, cavity force, and laser-pump force

The force acting on the atom has three components: $\mathcal{F} = -\nabla H = \mathcal{F}_{trap} + \mathcal{F}_{pump} + \mathcal{F}_{cav}$,

$$\mathcal{F}_{trap}/\hbar = -[\nabla\Delta_a(\boldsymbol{r})]\sigma^\dagger\sigma - \nabla U(\boldsymbol{r}),$$

$$\mathcal{F}_{pump}/\hbar = +[\nabla\eta(\boldsymbol{r})]\sigma^\dagger + [\nabla\eta^*(\boldsymbol{r})]\sigma,$$

$$\mathcal{F}_{cav}/\hbar = -[\nabla g(\boldsymbol{r})](a\sigma^\dagger + a^\dagger\sigma). \tag{12}$$

The naming of those forces should not bring to the conclusion that they, respectively, act only along the trap, the pump, and the cavity axis. The labeling only refers to the gradients of the spatial modulations of the ac Stark shift, the coupling η, and the cavity coupling g. However, those forces are dominant along the respective axis; we shall refer to them in that sense. The mean force $\boldsymbol{F} = \langle\mathcal{F}\rangle$ [Eq. (12)] for an atom at rest can be obtained from the steady-state expectation values (8) and (9). The mean force along the trap reads

$$\boldsymbol{F}_{trap} = -(\hbar\nabla\Delta_a)P_e - \hbar\nabla U, \tag{13a}$$

and is at low excitation $P_e \ll 1$ largely independent from the presence of the cavity $\boldsymbol{F}_{trap} \approx -\hbar\nabla U(\boldsymbol{r})$, which confines the atom.

The force along the near-resonant running-wave laser is analogous to the familiar radiation pressure [1,21]. We here write it in two ways, by using Eqs. (8a), (10), and (8b):

$$\boldsymbol{F}_{pump} = \hbar\boldsymbol{k}_L 2\Gamma_{\mathrm{eff}}P_e = \hbar\boldsymbol{k}_L(2\gamma P_e + 2\kappa N_{cav}). \tag{13b}$$

The first equation shows that the radiation pressure is proportional to the enhanced decay rate Γ_{eff} [Eq. (10b)]. The picture is that of an effective atom experiencing only "spontaneous emission," occurring at a rate $2\Gamma_{\mathrm{eff}}P_e$, as in free space [1]. The second equation shows that the radiation pressure is proportional to the total photon loss rate $2\gamma P_e + 2\kappa N_{cav}$. The difference between that expression and the free space one $\hbar \boldsymbol{k}_L 2\gamma P_e^{free}$ is the presence of the cavity-photon loss rate $2\kappa N_{cav}$, and the fact that P_e is evaluated taking the cavity into account. Actually one can say that the atom experiences a radiation pressure proportional to the number of photons that are absorbed per unit time by the atom-cavity system. That static force tends to expel the atom out of the trap. It can be circumvented either by increasing the trap depth, such that the atom oscillates in the trap, or it can be balanced by a counterpropagating pump beam. For a laser standing-wave $\eta(\boldsymbol{r}) = \eta_0 \cos(\boldsymbol{k}_L \cdot \boldsymbol{r})$, or for any real function $\eta(\boldsymbol{r})$, one has $\boldsymbol{F}_{pump} = 2\hbar\, \eta(\boldsymbol{\nabla}\eta)\Delta_{a,\mathrm{eff}}/(\Delta_{a,\mathrm{eff}}^2 + \Gamma_{\mathrm{eff}}^2)$, which again presents a similarity with the dipole force in free space upon the substitution $(\Delta_{a,\mathrm{eff}}, \Gamma_{\mathrm{eff}}) \to (\Delta_a, \gamma)$. To avoid additional complexity due to the structure of a standing-wave pump laser, it is reasonable to use orthogonal polarizations for the two counterpropagating beams. In Ref. [13] the laser is retroreflected with orthogonal linear-polarizations, i.e., in a lin $\perp$ lin configuration. With the present model it is possible to understand quantitatively the basic features of the cooling process in the initial stage, until a temperature of $k_B T \approx \hbar\kappa/2$ is reached.

There is also a simple analogy with respect to the dipole force in free space for the force along the cavity axis:

$$\boldsymbol{F}_{cav} = 2(\hbar \boldsymbol{\nabla} g) g \frac{\Delta_c}{\Delta_c^2 + \kappa^2} P_e. \tag{13c}$$

That force can be derived from a potential. For $\nu \to 0$, it can be understood as follows. From Eq. (10a) the atom in the cavity experiences a cavity-induced Stark shift $\Delta_S^{cav}(\boldsymbol{r}) = -g^2(\boldsymbol{r})\Delta_c/(\Delta_c^2 + \kappa^2)$, which acts as a trap along the cavity axis $\boldsymbol{F}_{cav} \approx -\hbar \boldsymbol{\nabla}(\Delta_S^{cav} P_e^{free})$ ($P_e \approx P_e^{free} = \mathrm{const}$). The cavity thus provides a trap for $\Delta_c > 0$, $U_{cav} = \Delta_S^{cav} P_e^{free}$, where the cooling also operates. That trap is deepest for $\Delta_c = \kappa$ where it reaches the value $U_{cav} = -g^2/2\kappa P_e^{free}$. Such a cavity-induced light shift can take values in the range of megahertz, e.g., in current cavities [19,22] $U_{cav}/2\pi = (426, 91) \times P_e$ MHz.

V. COOLING FORCES

For the cooling forces we use the label "v." The friction force is obtained by expanding the force up to first order in the velocity. If the atom moves slowly, the internal variables $\langle\sigma\rangle_\rho, \langle a\rangle_\rho$ can be expanded $\langle\sigma\rangle_\rho = \langle\sigma\rangle + \sigma_v, \langle a\rangle_\rho = \langle a\rangle + a_v$, where (σ_v, a_v) are the displacements with respect to the values for an atom at rest. The equation for the friction tensor is long and derived in Appendix A. Here we start from the low cooperativity limit $\nu \to 0$ before proceeding to a higher cooperativity, and focus on the diagonal elements of the friction tensor. We then discuss the exact expressions of the forces along the pump and trap axis and extract a picture in terms of the dressed states. We then end with the case $\Delta_c = 0$ where all forces present a compact form. The reader interested on the exact expressions of those cooling forces could therefore jump directly to Secs. V E and V F.

A. Preliminary discussion

In free space an atom "sees" counterpropagating photons with a higher frequency, i.e., they are blue to the atom. Otherwise, the photons appear red. This is of course the familiar Doppler effect which corresponds to shifting the photons frequency ω_L by the amount $\boldsymbol{k}_L \cdot \boldsymbol{v}$. In Doppler cooling the laser is tuned below the natural frequency ω_{eg} of the atom, i.e., $\Delta_a = \omega_{eg} - \omega_L > 0$. In this case the counterpropagating atom resonantly absorbs the laser photons $\Delta_a + \boldsymbol{k}_L \cdot \boldsymbol{v} \approx 0$. The Lorentz absorption curve (6) is then expanded to first order in velocity after the substitution $\Delta_a \to \Delta_a + \boldsymbol{k}_L \cdot \boldsymbol{v}$, giving a friction force scaling like the slope of the Lorentz profile, $\gamma\Delta_a/(\Delta_a^2 + \gamma^2)^2$. Now, this simple substitution $\Delta_a \to \Delta_a + \boldsymbol{k}_L \cdot \boldsymbol{v}$ is an incorrect procedure when the laser light is a standing wave [21]. This substitution makes sense only when the force is averaged over a spatial period, and for vanishing saturation. Physically, by averaging, interferences between the two running waves forming the standing wave vanish, and one can view the force by adding the independent contributions of the two running waves. The same problem is encountered if one considers the waist of the laser light. When moving into the cavity setting, the same problem arises, partly due to the standing wave structure of the cavity mode. Regardless of this problem, for a dipole oscillator, and in a standing wave, the friction force still scales like $\gamma\Delta_a/(\Delta_a^2 + \gamma^2)^2$. We interpret below the cooling forces by reference to such a common structure.

B. Cooling forces in the limit $|\Delta_a| \gg g^2/\kappa$

Although in that limit one has $P_e \approx P_e^{free}$, for later generalization the expressions below are written either in terms of P_e^{free} or P_e. The friction force along the near-resonant laser takes the simple form (neglecting the free-space term)

$$\boldsymbol{F}_{pump}^{v} = -4\hbar \boldsymbol{k}_L (\boldsymbol{k}_L \cdot \boldsymbol{v}) \frac{\kappa\Delta_c}{(\Delta_c^2 + \kappa^2)^2} g^2 P_e^{free}. \tag{14a}$$

If a laser standing wave is used, or for any real function $\eta(\boldsymbol{r})$, one would substitute in that equation $\boldsymbol{k}_L \to (\boldsymbol{\nabla}\eta)/\eta$ (notice that $P_e^{free} \propto |\eta|^2$). Friction along the near-resonant laser is caused by preferential absorption of photons traveling in the direction opposite to the atom. The Doppler effect shifts these photons towards the blue by $\boldsymbol{k}_L \cdot \boldsymbol{v}$, such that the coupled atom-cavity system resonantly absorbs counterpropagating pump photons. This expression can be retrieved with the substitution $\Delta_c \to \Delta_c + \boldsymbol{k}_L \cdot \boldsymbol{v}$ in the photon number (9b) ($\nu \to 0, P_e \approx P_e^{free}$), which gives the slope of the cavity photon loss rate $2\kappa N_{cav}$ in Eq. (13b).

A similar emission-based friction force acts along the resonator axis. Photons emitted into the direction of motion are blue detuned with respect to the laser. By recoil, these forward emissions also cool the atomic motion. If now the cavity is blue detuned, the emissions into the direction of motion are favored, and hence the atom is cooled along the cavity axis

$$\boldsymbol{F}^{v}_{cav} = -4\hbar\,\nabla g(\nabla g\cdot\boldsymbol{v})\frac{\kappa\Delta_c}{(\Delta_c^2+\kappa^2)^2}P_e. \tag{14b}$$

This simple picture ignores interferences that are at the origin of the spatial modulation of the cavity field. It holds true, however, if that equation is averaged over a spatial period of the cavity axis.

The two above forces are seen as due to Doppler effects, but they have a common origin, namely the dependency of the amplitude of the cavity field on the atomic position. As the atom moves slowly, both the amplitude $\langle a\rangle_\rho=\langle a\rangle+a_v$ of the field and the atomic coherence $\langle\sigma\rangle_\rho=\langle\sigma\rangle+\sigma_v$ vary by the small amounts a_v and σ_v, respectively, and vary faster than the atomic external degrees of freedom. Neglecting the variation of the atomic dipole to the evolution of the cavity field, the equation of evolution for $\langle a\rangle_\rho$ (4b) can be written as if the atom is at rest $\langle\dot{a}\rangle_\rho\approx-i\widetilde{\omega}_c(\langle a\rangle_\rho-\langle a\rangle)=-i\widetilde{\omega}_c a_v$. The rate of variation $\langle\dot{a}\rangle_\rho$ can also be expressed by the small displacement of the atom's center of mass $\langle\dot{a}\rangle_\rho=\boldsymbol{v}\cdot\nabla\langle a\rangle_\rho\approx\boldsymbol{v}\cdot\nabla\langle a\rangle$. The variation of the amplitude of the field thus reads $a_v=-\boldsymbol{v}\cdot\nabla\langle a\rangle/i\widetilde{\omega}_c$. Since the atom is detuned, its internal variables oscillate so rapidly such that one can assume that they adapt instantaneously to the variations of the cavity field, i.e., $\sigma_v=-(g/\widetilde{\omega}_a)a_v$ [Eq. (4a)]. By noting that $\langle a\rangle=-g\langle\sigma\rangle/\widetilde{\omega}_c$, $\langle\sigma\rangle\approx\eta/\widetilde{\omega}_a$ and for large Δ_a, $1/\widetilde{\omega}_a^2\approx1/(\Delta_a^2+\gamma^2)$, one obtains from Eq. (12) the two friction forces above, depending on which gradient is considered $\nabla\eta$ or ∇g.

For the force along the trap the excitation probability $\langle\sigma^\dagger\sigma\rangle_\rho$ is linearized $\langle\sigma\rangle^*\sigma_v$+c.c., thus $\boldsymbol{F}^{v}_{trap}=-\hbar\nabla\Delta_a(\langle\sigma\rangle^*\sigma_v+\text{c.c.})$. Isolating the variations along the trap $\nabla\langle a\rangle\rightarrow\nabla\Delta_a\,\partial\langle a\rangle/\partial\Delta_a$, one obtains the new force

$$\boldsymbol{F}^{v}_{trap} = -4\hbar\,\nabla\Delta_a(\nabla\Delta_a\cdot\boldsymbol{v})\frac{\kappa\Delta_c}{(\Delta_c^2+\kappa^2)^2}\frac{g^2P_e}{\Delta_a^2+\gamma^2}. \tag{14c}$$

Notice that it acts along the steep trap-gradients, but is otherwise governed by the cavity characteristics.

C. Orders of magnitude and comparison to Doppler cooling

All the forces above have the same order of magnitude [for the evaluation $(\nabla\Delta_a)^2\propto\Delta_a^2$] and hence lead to symmetric cooling for $\Delta_c>0$ in all three dimensions while the atom is confined in the trap. It is clear from those equations that the cooling forces can be maintained constant by increasing the trap depth proportionally to the pump power such that P_e is kept fixed. It is instructive to compare those forces with the Doppler force:

$$\boldsymbol{F}^{v,free} = -4\hbar\,\boldsymbol{k}_L(\boldsymbol{k}_L\cdot\boldsymbol{v})\frac{\gamma\Delta_a}{\Delta_a^2+\gamma^2}P_e^{free}. \tag{15}$$

In a trap and for $\Delta_c=\kappa$ the ratio between that force and the ones above is on the order of $g^2\Delta_a/(4\kappa^2\gamma)=C\Delta_a/2\kappa$. This factor is already 10 for reasonable parameters $(\Delta_a,\kappa)/2\pi=(100,5)$ MHz, $C=1$. This ratio becomes large in the strong coupling regime. It makes explicit the fact that the free-space cooling forces vanish for large detuning, while the cavity forces are constant for a given excitation probability P_e. Eventually, we compare the maximum free-space Doppler cooling force, achieved for $\Delta_a=\gamma/\sqrt{3}$, with that maximally obtained in a cavity, for $\Delta_c=\kappa/\sqrt{3}$. The ratio between the two is now about g^2/κ^2, and could exceed unity by a large amount.

D. Cooling forces for higher orders in *g*

We now consider the limits of strong coupling $g>(\kappa,\gamma,|\Delta_c|)$ and small cavity detuning. Although not restricted to, the approximations below are suited for regimes $\Delta_a\gg g$. The equations above do not take into account the complete structure of the atom-cavity system. Following what is illustrated in Fig. 2, the presence of the atom can significantly shift the cavity Lorentz curve and induce a broadening, $(\Delta_c,\kappa)\rightarrow(\Delta_{c,\text{eff}},K_{\text{eff}})$ [Eqs. (11a) and (11b)]. In this limit one can simplify the equations from Appendix A to obtain a new set of forces, now scaling with $(\Delta_{c,\text{eff}},K_{\text{eff}})$:

$$\boldsymbol{F}^{v}_{pump} = -4\hbar\,\boldsymbol{k}_L(\boldsymbol{k}_L\cdot\boldsymbol{v})\frac{K_{\text{eff}}\Delta_{c,\text{eff}}}{(\Delta_{c,\text{eff}}^2+K_{\text{eff}}^2)^2}g^2P_e^{free},$$

$$\boldsymbol{F}^{v}_{cav} = -4\hbar\,\nabla g(\nabla g\cdot\boldsymbol{v})\frac{K_{\text{eff}}\Delta_{c,\text{eff}}}{(\Delta_{c,\text{eff}}^2+K_{\text{eff}}^2)^2}P_e|1+\nu|^2,$$

$$\boldsymbol{F}^{v}_{trap} = -4\hbar\,\nabla\Delta_a(\nabla\Delta_a\cdot\boldsymbol{v})\frac{K_{\text{eff}}\Delta_{c,\text{eff}}}{(\Delta_{c,\text{eff}}^2+K_{\text{eff}}^2)^2}\frac{g^2P_e}{\Delta_a^2+\gamma^2}. \tag{16}$$

All these forces generalize the ones above, and apart from the factor $|1+\nu|^2$, which reflects the fact that the force $\boldsymbol{F}^{v}_{cav}$ depends on the mode and dipole operators, the forces present the same structure as those before. In particular, cooling appears now for $\Delta_{c,\text{eff}}>0$, i.e., when the cavity detuning Δ_c exceeds the frequency shift induced by the atom Eq. (11a).

Notice that the condition $\Delta_{c,\text{eff}}=0$ matches the normal-mode resonance $\Delta_a\Delta_c=g^2$ for large detuning $\Delta_a\gg\gamma$. Actually $\boldsymbol{F}^{v}_{pump}$ can be obtained if one uses the Doppler-shift argument in absorption as above $\Delta_c\rightarrow\Delta_c+\boldsymbol{k}_L\cdot\boldsymbol{v}$, and similarly for Δ_a. Here it is the Doppler shift affecting the cooperativity parameter ν which gives $\boldsymbol{F}^{v}_{pump}$ just above. This force can be understood by saying that the atom-cavity system resonantly absorbs laser photons as soon as it is excited near the normal-mode resonances. When the cavity detuning is varied, those forces are maximum for $\Delta_{c,\text{eff}}=K_{\text{eff}}/\sqrt{3}$ (we ignore the factor $|1+\nu|^2$). The minimum temperature is obtained for $\Delta_{c,\text{eff}}=K_{\text{eff}}$ (see below). Assuming $K_{\text{eff}}\approx\kappa$ [Eq. (11b)], one obtains the extended condition $\Delta_c\approx\kappa+g^2/\Delta_a$, which is a generalization of the condition $\Delta_c\approx\kappa$ [2,3,13] (see Fig. 2).

We end by noting that the factor $K_{\text{eff}}\Delta_{c,\text{eff}}/(\Delta_{c,\text{eff}}^2+K_{\text{eff}}^2)^2$ governs the cooling forces for large atomic detunings, up to saturation effects [6], also in case the cavity is pumped. A general interpretation would be that a detuned atom oscillates very fast such that it can be considered as stationary. Its effect is then to dress the cavity mode by modifying its characteristics. Since the cavity mode is a harmonic oscillator, it follows that the force has a structure which mimics an effective two-state atom at vanishing saturation. In other words, one would compare this factor with that of free space

$\gamma\Delta_a/(\Delta_a^2+\gamma^2)^2$. In [8], the friction has been analyzed numerically in the far detuned field. This is well-approximated by the analytical form $\boldsymbol{F}^v_{pump}$ just above. Thus the "polariton resonance" as called in [8] actually gives a friction with a structure that is traced back to the atom-cavity system where the atom is far detuned, rather than a particular cavity setup.

E. General expression for the friction force along the pump and trap axis

The friction force along the cavity axis has been derived and discussed in detail via numerical plots in [7]. Above we provided an efficient approximation for the force along the cavity axis in a given limit. The full expression of that force and actually the entire friction tensor is written in Appendix A. Here we extract from Appendix A the full expressions for the forces along the pump laser and along the trap:

$$\boldsymbol{F}^v_{pump} = -4\hbar\,\boldsymbol{k}_L(\boldsymbol{k}_L\cdot\boldsymbol{v})|\eta|^2\left[\frac{\Gamma_{\rm eff}\Delta_{a,\rm eff}}{(\Delta^2_{a,\rm eff}+\Gamma^2_{\rm eff})^2} + \frac{1}{2g^2}\,\mathrm{Im}\left(\frac{\nu^2}{(1-\nu)^2}\right)\right], \tag{17a}$$

and

$$\boldsymbol{F}^v_{trap} = -4\hbar\,\boldsymbol{\nabla}\Delta_a(\boldsymbol{\nabla}\Delta_a\cdot\boldsymbol{v})P_e\left[\frac{\Gamma_{\rm eff}\Delta_{a,\rm eff}}{(\Delta^2_{a,\rm eff}+\Gamma^2_{\rm eff})^2} + \frac{1}{2g^2}\,\mathrm{Im}\left(\frac{\nu^2}{(1-\nu)^2}\right)\right]. \tag{17b}$$

If a laser standing wave is used, or for any real function $\eta(\boldsymbol{r})$, one would substitute $\boldsymbol{k}_L\to(\boldsymbol{\nabla}\eta)/\eta$ in $\boldsymbol{F}^v_{pump}$. One can see that, apart from a global factor, the friction forces are identical. This originates from the fact that both forces depend on the atom's characteristics, the optical coherence σ and occupation $\sigma^\dagger\sigma$, and not on the cavity mode operators. The first term in each of the expressions reduces to the free-space limit (in the harmonic description) when the cavity is absent. The important term is the second one, here expressed in terms of ν in order to show the symmetry upon the substitution $(\Delta_a,\gamma)\leftrightarrow(\Delta_c,\kappa)$. For large Δ_a one can approximate $\nu^2 = g^4/\widetilde{\omega}_a^2\widetilde{\omega}_c^2 \approx g^4/(\Delta_a^2+\gamma^2)\widetilde{\omega}_c^2$, thus one remains with the imaginary part of $\widetilde{\omega}_c^2(1-\nu)^2=(\Delta_{c,\rm eff}-iK_{\rm eff})^2$ [see Eqs. (11a) and (11b)], which immediately gives the corresponding two forces in Eqs. (16), themselves being a generalization of Eqs. (14).

Dressed state picture

This symmetry also allows us to extract a quantitative picture in terms of the first two dressed-states $|\pm\rangle$ of the atom-cavity system (actually they would be density matrices). The frequencies of the lowest two dressed states of the atom-cavity system relative to the pump frequency ω_L read:

$$\Delta_\pm = \frac{1}{2}(\Delta_a+\Delta_c) \pm \frac{1}{2}\,\mathrm{Re}\sqrt{4\mathrm{g}^2+[\Delta_{ac}-\mathrm{i}(\gamma-\kappa)]^2}, \tag{18a}$$

and the corresponding decay rates are

$$\Gamma_\pm = \frac{1}{2}(\gamma+\kappa) \mp \frac{1}{2}\,\mathrm{Im}\sqrt{4\mathrm{g}^2+[\Delta_{ac}-\mathrm{i}(\gamma-\kappa)]^2}, \tag{18b}$$

with the atom-cavity detuning $\Delta_{ac}(\boldsymbol{r})=\Delta_a-\Delta_c=\omega_{eg}-\omega_{cav}+\Delta_S(\boldsymbol{r})$. One then rewrites the symmetric contribution above $\nu^2/(1-\nu)^2=g^4/(\Delta_- - i\Gamma_-)^2(\Delta_+ - i\Gamma_+)^2$, which gives the force in terms of these decay rates and energies. Further approximations and interpretations are then possible. For example, in a regime where the dressed states are well resolved, e.g., $\Delta_+\gg\Gamma_+$, one can follow the procedure of approximations above, i.e., write $\nu^2/(1-\nu)^2\approx g^4/(\Delta_- - i\Gamma_-)^2(\Delta_+^2+\Gamma_+^2)$. By taking the imaginary part one obtains a force that now scales with $\Gamma_-\Delta_-/(\Delta_-^2+\Gamma_-^2)$. One can check that by symmetry there is a contribution $\Gamma_+\Delta_+/(\Delta_+^2+\Gamma_+^2)$ which is then small in that limit. Hence one can add the two contributions with a good approximation and obtain, for example, for the force along the laser beam:

$$\boldsymbol{F}^v_{pump} = -4\hbar\,\boldsymbol{k}_L(\boldsymbol{k}_L\cdot\boldsymbol{v})\left\{\frac{\Gamma_-\Delta_-}{\Delta_-^2+\Gamma_-^2}+\frac{\Gamma_+\Delta_+}{\Delta_+^2+\Gamma_+^2}\right\}N_{cav}. \tag{19}$$

That equation has a clear meaning when compared to the free-space limit Eq. (15). It shows that it is the atom-cavity system which mimics a single atom in interaction with a laser field, i.e., it is the composite system which now resonantly absorbs the radiation field. A similar expression can be written for the force along the trap direction, providing a deeper understanding of the cooling force in that direction. Since for large Δ_a one has $(\Delta_+,\Gamma_+)\approx(\Delta_a,\gamma)$ and $(\Delta_-,\Gamma_-)\approx(\Delta_c,\kappa)$, those expressions reduce to Eqs. (14) discussed above [2,3,13]. Actually, the forces in Eqs. (16) are close to Eq. (19). Here cooling occurs for $\Delta_->0$ (neglecting the contribution due to the state $|+\rangle$).

F. General expression for the friction in all directions for $\Delta_c=0$

The exact expressions in the case $\Delta_c=0$ (and hence $\Delta_{a,\rm eff}=\Delta_a$) can also be written compactly. First, the friction force along the pump and trap axis follow from Eqs. (17):

$$\boldsymbol{F}^v_{pump} = -4\hbar\,\boldsymbol{k}_L(\boldsymbol{k}_L\cdot\boldsymbol{v})\left(1-\frac{g^2}{\kappa^2}\right)\frac{\Gamma_{\rm eff}\Delta_{a,\rm eff}}{(\Delta^2_{a,\rm eff}+\Gamma^2_{\rm eff})}P_e, \tag{20a}$$

$$\boldsymbol{F}^v_{trap} = -4\hbar\,\boldsymbol{\nabla}\Delta_a(\boldsymbol{\nabla}\Delta_a\cdot\boldsymbol{v})\left(1-\frac{g^2}{\kappa^2}\right)\frac{\Gamma_{\rm eff}\Delta_{a,\rm eff}}{(\Delta^2_{a,\rm eff}+\Gamma^2_{\rm eff})^2}P_e, \tag{20b}$$

which clearly show cooling for $\Delta_a<0$ provided $g>\kappa$, the forces then scaling like $g^2/\kappa^2\gg1$. Similarly, from Appendix A the exact friction force along the cavity axis reads for $\Delta_c=0$

$$\boldsymbol{F}^{v}_{cav} = -4\hbar \nabla g(\nabla g \cdot \boldsymbol{v}) P_e \left[\frac{K_{\text{eff}}\Delta_{c,\text{eff}}}{(\Delta^2_{c,\text{eff}} + K^2_{\text{eff}})^2} - \frac{g^4}{\kappa^4} \frac{\Gamma_{\text{eff}}\Delta_{a,\text{eff}}}{(\Delta^2_{a,\text{eff}} + \Gamma^2_{\text{eff}})^2} \right]. \tag{20c}$$

Since for $\Delta_c=0$ one has $\Delta_{c,\text{eff}}=-g^2\Delta_a/(\Delta_a^2+\gamma^2)$ $(\Delta_{a,\text{eff}}=\Delta_a)$, the force along the cavity axis is now a cooling force for $\Delta_a<0$ for any value of the coupling g. The force along the pump is maximum for a detuning $\Delta_a=-\gamma(1+g^2/\kappa\gamma)/\sqrt{3}$, while the forces along the trap and along the cavity are maximum for $\Delta_a=-\gamma(1+g^2/\kappa\gamma)/\sqrt{5}$ (maximizing only the last term of $\boldsymbol{F}^{v}_{cav}$). Those values are quite close to each other. As shown now this allows one to obtain optimized three-dimensional cooling.

VI. DIFFUSION AND TEMPERATURE

All the forces acting on the atom fluctuate and hence lead to heating of the atomic motion. This heating is expressed in the diffusion coefficient $2\text{D}=d/dt\langle\Delta\boldsymbol{p}^2\rangle$, which characterizes the spread $\langle\Delta\boldsymbol{p}^2\rangle=\langle[\boldsymbol{p}-\langle\boldsymbol{p}\rangle]^2\rangle=\langle\boldsymbol{p}^2\rangle-\langle\boldsymbol{p}\rangle^2$ of the atomic momentum $\boldsymbol{p}$. A global understanding for the diffusion has been recently proposed, where a general and strikingly simple form has been demonstrated [17]. The diffusion tensor is provided in Appendix B. The sum of the diagonal elements of this tensor define the diffusion coefficient 2D, which reads the simple form [17]:

$$2\text{D} = (\hbar k)^2 2\gamma P_e + |\hbar \nabla\langle\sigma\rangle|^2 2\gamma + |\hbar \nabla\langle a\rangle|^2 2\kappa. \tag{21}$$

The gradients can be easily computed with the help of Eqs. (8a) and (9a). The first term is the familiar contribution arising from the random direction of the spontaneously emitted photons (with momentum norm $\hbar k\approx\hbar k_L$), occurring at a rate $2\gamma P_e$. The second term stems from the fluctuations of the atomic dipole coupled to a classical field [1,17]. Those two terms are identical to the free-space expression [1,21] except that $\langle\sigma\rangle$ and $P_e\approx|\langle\sigma\rangle|^2$ take into account the presence of the cavity [Eq. (8), $\nu\neq 0$]. The last term stems from the fluctuations of the cavity field coupled to a classical atomic dipole. This term has large contributions in experiments [16,22,23] (also see [15] for a possible origin of the observed heating), in particular for strong coupling between the atom and the cavity field. Equation (21) is invariant in the sense that it is independent of whether the atom or the cavity or both is/are probed, and independent of the structure of the laser light (trap and/or near fields) which is/are used. In particular, notice that the gradient acting on $\langle\sigma\rangle$ and/or $\langle a\rangle$, when modulus-squared, generates terms that are proportional to $(\nabla\eta)^2$, $(\nabla\Delta_a)^2$, and $(\nabla g)^2$, but also cross terms like, for example, $(\nabla\eta)(\nabla g)$. All these terms are contained in Eq. (21) in such a way that the equation is invariant.

Beside the spontaneous emission term, the diffusion along the probe beam [component $(\nabla\eta)^2$] can be interpreted as being due to the random number of photons that are absorbed by the atom-cavity system per unit time. By ignoring the spontaneous emission term, it can be written as $2\text{D}_{pump}=(\hbar k_L)^2 2\Gamma_{\text{eff}}P_e$. For our purpose, the temperature evaluation is similar in all directions. If we look at the diffusion along the cavity axis, and focus only on the last term of Eq. (21), then one obtains in the limit $\nu\rightarrow 0$, which gave us the cooling force (14b), $2\text{D}_{cav}=(\hbar\nabla g)^2 2\kappa P_e/(\Delta_c^2+\kappa^2)$. In the same way as for the friction force (14b), this diffusion coefficient can be interpreted if spatially averaged along the cavity axis. In this case D_{cav} is due to the randomness in the number of photons that leak out from the cavity. Such a randomness induces a random force and hence a diffusion which has precisely that form. The diffusion here has a Lorentz shape and hence preserves the same structure as Doppler diffusion. Thus the lowest temperature is reached for $\Delta_c\approx\kappa$ and is limited by $k_BT\approx\hbar\kappa/2\bar{u}$, where $\bar{u}$ is a number that determines the degree of localization around an antinode of the cavity mode (ideally $\bar{u}\rightarrow 1$ for an atom well-localized at an antinode $k_BT\rightarrow\hbar\kappa/2$). Basically this temperature is obtained in all directions.

In the limit of higher cooperativity (16) the friction along the laser direction is maximum for $\Delta_{c,\text{eff}}=K_{\text{eff}}/\sqrt{3}$, while the fluctuations of the cavity mode give the diffusion $|\hbar\nabla\langle a\rangle|^2\kappa=(\hbar\boldsymbol{k}_L)^2\kappa P_e^{free}g^2/(\Delta^2_{c,\text{eff}}+K^2_{\text{eff}})$. Thus the temperature is now minimized for $\Delta_{c,\text{eff}}=K_{\text{eff}}$ where it reaches $k_BT=\hbar\kappa/2$. By using Eq. (11), and for $K_{\text{eff}}\approx\kappa$ the optimum detuning between the cavity and the probe beam would be around $\Delta_c=\kappa+g^2\Delta_a/(\Delta_a^2+\gamma^2)$ (see Fig. 2). Eventually for $\Delta_c=0$ one obtains, for example, along the pump $k_BT=\hbar\kappa/2(1+\gamma\kappa/g^2)$ for $\Delta_a=-\gamma(1+g^2/\gamma\kappa)$.

VII. CONCLUSION

We have shown that a three-dimensional arrangement of one pump laser, a cavity, and a trapping laser is promising for cooling and trapping atoms. In particular, a strong cavity-induced cooling force is also acting along the large gradients of the trap. Such a force is not present in a isolated dipole trap and has a similar structure as those in the other directions, i.e., along the laser and cavity axis. All forces in all three directions can be made independent of the trap depth by increasing the intensity of the cooling laser accordingly. It follows that one obtains both a deep confinement of particles and a persistent cavity-induced cooling mechanism. The ideal temperature for a deep enough trap would be around $k_BT\approx\hbar\kappa$ in all directions and does not require strong coupling. Such a temperature follows from the fact that for large detuning Δ_a the atomic oscillator reacts poorly to the laser while the cavity mode being close to resonance governs the dynamics.

ACKNOWLEDGMENTS

This work was partially supported by the Deutsche Forschungsgemeinschaft (SFB 631 and research unit 635) and the European Union in the IST (QGATES, SCALA) and IHP (CONQUEST) programs.

APPENDIX A: FRICTION TENSOR IN THE HARMONIC DESCRIPTION

The friction force is obtained by expanding the force up to first order in the velocity. If the atom moves slowly, the internal variables $(\langle\sigma\rangle_\rho,\langle a\rangle_\rho)$ can be expanded $\langle\sigma\rangle_\rho=\langle\sigma\rangle+\sigma_v,\langle a\rangle_\rho=\langle a\rangle+a_v$, where (σ_v,a_v) are the displacements with respect to the steady state values for an atom at rest. Those values are determined by the Maxwell-Bloch equations for the atom-cavity system written at first order in the velocity. By using the hydrodynamic derivative $d/dt=\partial/\partial t+\boldsymbol{v}\cdot\boldsymbol{\nabla}$ inserted in Eq. (4) and from the stationary values for an atom at rest [Eqs. (8a) and (9a)], the coupled equations for (σ_v,a_v) read:

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}\langle\sigma\rangle=-i\widetilde{\omega}_a\left\{\sigma_v+\frac{g}{\widetilde{\omega}_a}a_v\right\},$$

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}\langle a\rangle=-i\widetilde{\omega}_c\left\{a_v+\frac{g}{\widetilde{\omega}_c}\sigma_v\right\},$$

which by inversion give

$$\sigma_v=-\frac{1}{1-\nu}\left[\frac{\boldsymbol{v}\cdot\boldsymbol{\nabla}\langle\sigma\rangle}{i\widetilde{\omega}_a}-\frac{g}{\widetilde{\omega}_a}\frac{\boldsymbol{v}\cdot\boldsymbol{\nabla}\langle a\rangle}{i\widetilde{\omega}_c}\right], \tag{A1a}$$

$$a_v=-\frac{1}{1-\nu}\left[\frac{\boldsymbol{v}\cdot\boldsymbol{\nabla}\langle a\rangle}{i\widetilde{\omega}_c}-\frac{g}{\widetilde{\omega}_c}\frac{\boldsymbol{v}\cdot\boldsymbol{\nabla}\langle\sigma\rangle}{i\widetilde{\omega}_a}\right]. \tag{A1b}$$

It is convenient to introduce the vectors $\boldsymbol{u},\boldsymbol{w}$, which include all the information on the gradients:

$$\boldsymbol{u}=\boldsymbol{\nabla}\eta-\langle\sigma\rangle\boldsymbol{\nabla}\Delta_a-\langle a\rangle\boldsymbol{\nabla}g, \tag{A2a}$$

$$\boldsymbol{w}=\langle\sigma\rangle\boldsymbol{\nabla}g. \tag{A2b}$$

This gives for the variations of the expectation values:

$$\boldsymbol{\nabla}\langle\sigma\rangle=\frac{1}{\widetilde{\Omega}_a}\left[\boldsymbol{u}+\frac{g}{\widetilde{\omega}_c}\boldsymbol{w}\right], \tag{A3a}$$

$$\boldsymbol{\nabla}\langle a\rangle=-\frac{1}{\widetilde{\Omega}_c}\left[\frac{g}{\widetilde{\omega}_a}\boldsymbol{u}+\boldsymbol{w}\right], \tag{A3b}$$

with $\widetilde{\Omega}_a=(\Delta_a-i\gamma)(1-\nu)=\Delta_{a,\mathrm{eff}}-i\Gamma_{\mathrm{eff}}$ and $\widetilde{\Omega}_c=(\Delta_c-i\kappa)(1-\nu)=\Delta_{c,\mathrm{eff}}-iK_{\mathrm{eff}}$. The velocity dependent force is obtained by linearizing (harmonic limit) and expanding Eqs. (12) up to first order in $\boldsymbol{v}$:

$$\boldsymbol{F}^v/\hbar=\boldsymbol{u}^*\sigma_v-\boldsymbol{w}^*a_v+\text{c.c.}, \tag{A4}$$

which is entirely known with the help of Eqs. (A1) and (A3). The cooling force has well-separated contributions, along the pump $\boldsymbol{\nabla}\eta$, the trap $\boldsymbol{\nabla}\Delta_a$, and the cavity axis $\boldsymbol{\nabla}g$. However, one should have in mind that the expectation values depend on all the parameters (η,Δ_a,g), and hence the friction force includes cross contributions. For example, if the atom moves along the cavity axis $\boldsymbol{v}\cdot\boldsymbol{\nabla}g\neq 0$ it experiences a friction force along the near resonant laser, to be added to $\boldsymbol{F}^v_{pump}$ [term $\propto\hbar\boldsymbol{\nabla}\eta(\boldsymbol{\nabla}g\cdot\boldsymbol{v})$]. This term is not present in Ref. [3], but it vanishes after spatial averaging. Separating the contributions due to the variation of the atomic dipole $\boldsymbol{\nabla}\langle\sigma\rangle$ from those due to the variation of the amplitude of the cavity field $\boldsymbol{\nabla}\langle a\rangle$ gives the total cooling force acting on the atom $\boldsymbol{F}^v=\boldsymbol{F}^{v,atom}+\boldsymbol{F}^{v,mode}$

$$\boldsymbol{F}^{v,atom}=-\frac{\hbar}{i\widetilde{\Omega}_a^2}\left[\boldsymbol{u}^*+\frac{g}{\widetilde{\omega}_c}\boldsymbol{w}^*\right]\boldsymbol{v}\cdot\left(\boldsymbol{u}+\frac{g}{\widetilde{\omega}_c}\boldsymbol{w}\right)+\text{c.c.}, \tag{A5}$$

$$\boldsymbol{F}^{v,mode}=-\frac{\hbar}{i\widetilde{\Omega}_c^2}\left[\frac{g}{\widetilde{\omega}_a}\boldsymbol{u}^*+\boldsymbol{w}^*\right]\boldsymbol{v}\cdot\left(\frac{g}{\widetilde{\omega}_a}\boldsymbol{u}+\boldsymbol{w}\right)+\text{c.c.} \tag{A6}$$

That friction force is valid for any structure of the probe beam, i.e., for any function $\eta(\boldsymbol{r})$. If now the atom is far detuned while the cavity mode is close to resonance with the pump laser, the contribution $\boldsymbol{F}^{v,atom}$ can be typically neglected, and hence the forces are mainly due to the variations of the cavity field $\boldsymbol{F}^v\approx\boldsymbol{F}^{v,mode}$. The approximations presented in the body of the paper are all extracted from $\boldsymbol{F}^{v,mode}$.

APPENDIX B: DIFFUSION TENSOR IN THE HARMONIC DESCRIPTION

We first define the diffusion tensor such that it is symmetric with respect to the spatial coordinates (i,j), i.e., $2\mathrm{D}_{ij}=\mathrm{Re}\, d\langle\Delta p_i\Delta p_j\rangle/dt$ where $\Delta p_i=p_i-\langle p_i\rangle$. The real part is needed because the momentum $\boldsymbol{p}$ is an operator. One can show that [18]

$$2\mathrm{D}_{ij}=(\hbar k)^2 2\gamma P_e E_{ij}+\mathrm{Re}\int_0^\infty d\tau[\langle\Delta\mathcal{F}_i(\tau)\Delta\mathcal{F}_j(0)\rangle+\langle\Delta\mathcal{F}_i(0)\Delta\mathcal{F}_j(\tau)\rangle],$$

where $\mathcal{F}_i$ is the component along the i-axis of the force operator $\mathcal{F}$ defined in Eq. (12), and where $E_{ij}=\int d^2\hat{\boldsymbol{\kappa}}\kappa_i\kappa_j N(\hat{\boldsymbol{\kappa}}\cdot\hat{\boldsymbol{d}})$, and $\hat{\boldsymbol{d}}$ is the orientation of the dipole-moment. Notice that $\Sigma_i E_{ii}=1$ [18]. In our general setup it is difficult to justify the two-level approximation without having restrictive assumptions on the nature of the polarization of the beams and the cavity mode. However, considering a π-transition and the quantization axis along the cavity axis z one has $N=[1-(\hat{\boldsymbol{\kappa}}\cdot\hat{\boldsymbol{d}})^2]3/8\pi$, thus $E_{zz}=1/5, E_{yy}=E_{xx}=2/5$, while for circularly polarized light $N=[1+(\hat{\boldsymbol{\kappa}}\cdot\hat{\boldsymbol{d}})^2]3/16\pi$, giving $E_{zz}=2/5, E_{yy}=E_{xx}=3/10$. The result showed in Ref. [17] allows a simple writing of the integral above, giving the diffusion tensor component D_{ij}:

$$2\mathrm{D}_{ij}(\boldsymbol{r})=(\hbar k)^2 2\gamma P_e E_{ij}+\mathrm{Re}[\hbar^2\partial_i\langle\sigma\rangle\partial_j\langle\sigma\rangle^*]2\gamma+\mathrm{Re}[\hbar^2\partial_i\langle a\rangle\partial_j\langle a\rangle^*]2\kappa,$$

where the partial derivatives are taken with respect to the components r_i of the atomic position $\boldsymbol{r}$, $\partial_i=\partial/\partial r_i$. The diffusion tensor enters the Fokker-Planck equation for the distribution function f as $\Sigma_{i,j}\mathrm{D}_{ij}(\boldsymbol{r})\partial^2 f/\partial p_i\partial p_j$. The diffusion coefficient studied in the text is the sum of the diagonal elements $2\mathrm{D}=\Sigma_i\mathrm{D}_{ii}$ [Eq. (21)]. The diffusion tensor (and hence 2D [Eq. (21)]) is valid for any structure of the pump beam, i.e., for any function $\eta(\boldsymbol{r})$. Moreover, the diffusion still has that solution if any of the frequencies $(\Delta_a,\eta,\Delta_c,g,\eta)$ depends on the atomic position $\boldsymbol{r}$.

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