Optimal control, geometry, and quantum computing

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(Received 17 March 2006; published 16 June 2006)

We prove upper and lower bounds relating the quantum gate complexity of a unitary operation, U, to the optimal control cost associated to the synthesis of U. These bounds apply for any optimal control problem, and can be used to show that the quantum gate complexity is essentially equivalent to the optimal control cost for a wide range of problems, including time-optimal control and finding minimal distances on certain Riemannian, sub-Riemannian, and Finslerian manifolds. These results generalize the results of [Nielsen, Dowling, Gu, and Doherty, Science **311**, 1133 (2006)], which showed that the gate complexity can be related to distances on a Riemannian manifold.

DOI: 10.1103/PhysRevA.73.062323 PACS number(s): 03.67.Lx, 02.30.Yy

I. INTRODUCTION

Quantum computers have caused great interest due to their potential use in efficiently solving problems considered intractable on conventional classical computers [1,2]. Despite this interest, there is as yet no general framework for constructing efficient quantum algorithms, nor for proving limitations on the power of quantum computers.

Recent work [3,4] has proposed a geometric approach to quantum computation, based on the observation that finding quantum circuits of the minimal size required to perform some desired computation is equivalent to a problem in Riemannian geometry. More precisely, the size of the minimum quantum circuit synthesizing a unitary U is, up to polynomial factors and some technical caveats (see Sec. IV for precise statements), equal to the distance $d(I,U)$ between the identity operation I and U, according to some Riemannian metric. This equivalence means that problems in quantum computation can be recast in terms of equivalent problems in Riemannian geometry.

The results of [3,4] establish an equivalence between the number of gates needed to synthesize U and the minimal distance according to some *specific* Riemannian metric. However, inspection of the proof in [3,4] shows that many of the properties used in the proof are rather generic, and there are certainly other Riemannian metrics with the same property. One may therefore ask what is the most general class of Riemannian metrics that can be connected to gate complexity. Even more generally, the problem of finding minimal geodesics in Riemannian geometry may be viewed as an instance of the problem of optimizing some cost function in the framework of nonlinear optimal control (see, e.g. [5]), and it is interesting to ask whether it is possible to make any general connections between optimal control and gate complexity.

The purpose of the present paper is to identify a large family of optimal control problems whose optimal cost is equivalent to the minimal gate complexity of the desired unitary operation. As special cases of our results we obtain the geometric results of [3,4], but also identify many other classes of optimization problems which can be connected to gate complexity, including problems from time-optimal control, and from Riemannian, subriemannian, and Finslerian geometry. Of course, in some (though not all) of these examples more straightforward techniques may be used to relate the optimal cost to quantum gate complexity. The benefit of the analysis in the present paper is that it provides a unified and generalized framework for deriving connections between quantum gate complexity and optimal control.

By identifying this large family of optimal control problems we identify the essential features of the geometric problem in [3,4] that are responsible for the equivalence to quantum computation. We also widen the class of problems in optimal control which may be analyzed in order to obtain insight into quantum computation. A considerable body of work has been done on optimal control in quantum physics (see references later in the paper), and we hope that the close connection between optimal quantum control and quantum gate complexity will stimulate further work on optimal quantum control.

The structure of the paper is as follows. Section II describes background material on quantum computing and optimal control theory that is useful later in the paper. Section III proves a general theorem relating the optimal cost for a control problem to quantum gate complexity. In Sec. IV we illustrate this theorem through a series of applications to example problems, including time-optimal control, and problems from Riemannian, subriemannian, and Finsler geometry. Section V concludes our paper.

II. BACKGROUND

In this section we introduce some background material on quantum computation (Section II A) and optimal control (Section II B) that will be useful later in the paper.

A. Quantum computation and gate complexity

We assume the reader is familiar with basic notions of quantum circuits (e.g., Chapter 4 of [2]). Suppose U is an n-qubit unitary operation. We define the *exact gate complexity* $G(U)$ to be the minimal number of one- and two-qubit

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quantum gates required to synthesize U *exactly*, with no ancilla qubits allowed to assist in the preparation of U. We define the *approximate gate complexity* $G(U,\epsilon)$ to be the minimal number of gates required to synthesize some n-qubit unitary operation V satisfying $\|U-V\|<\epsilon$, where $\|\cdot\|$ is the usual matrix norm. Once again, no ancilla qubits are allowed to assist in the synthesis. Note that in [3,4] the notation $m(U)$ was used for the gate complexity.

Our results connect problems in optimal control to the values of $G(U)$ and $G(U,\epsilon)$. The typical object of interest in optimal control is the optimal *cost* $C(U)$ associated to a unitary, U, according to a cost function which is defined precisely below. Our goal is to identify control problems such that $C(U)$ provides good lower bounds on the exact gate complexity $G(U)$, and good upper bounds on the approximate gate complexity $G(U,\epsilon)$. As a result, up to polynomial factors the exact synthesis of U without ancilla must take at least $C(U)$ quantum gates, and U can be synthesized to accuracy ϵ using at most $C(U)$ quantum gates.

One might naturally ask if it is possible to extend these results to prove a similar lower bound involving approximate computation, or an upper bound involving exact computation. Parameter counting can be used to show that a bound of the form $G(U)\leq \mathrm{poly}(C(U),n)$ is not possible. Whether a bound of the form $\mathrm{poly}(C(U),n,1/\epsilon)\leq G(U,\epsilon)$ is possible remains an open problem. Fortunately, lower bounds for exact computation and upper bounds for approximate computation remain of great interest.

B. Optimal control on SU(2^n)

We now sketch the basic ideas of optimal control theory, following the standard approach (e.g., [5]), but omitting mathematical details regarding smoothness and regularity conditions, as these are not important for our purposes.

Let $H_1,\ldots,H_m$ be a set of linearly independent matrices in the Lie algebra su(2^n) of traceless n-qubit Hermitian matrices.[1] Our control system is based on Schrödinger's equation:

$$\frac{dU}{dt} = -iH(t)U(t); \quad H(t)\equiv \sum_{j=1}^{m} h_j(t)H_j, \tag{1}$$

where $h(t)=(h_1(t),\ldots,h_m(t))$ is known as the *control function*, and we impose the initial condition $U(0)=I$. Defining the notation $H_h\equiv\Sigma_{j=1}^m h_j H_j$, we see that $H(t)=H_{h(t)}$. We refer to $H(t)$ as the *control Hamiltonian* corresponding to the control function $h(t)$. Note that to any control Hamiltonian $H(t)$ defined on an interval $[0,T]$ there exists a unique solution $U(t)$ to Eq. (1) defined on the same interval.

In general, the control function $h(t)$ is not allowed to take arbitrary values, but is constrained to lie in an *allowed control region* $A\subseteq R^m$. We denote the corresponding set of allowed control Hamiltonians by $\mathcal{H}_A$.

To complete the specification of the control problem we must also specify a *cost function*, which is a real-valued function $c:A\rightarrow R$ on the allowed control region. Equivalently, it may be regarded as a function $c:\mathcal{H}_A\rightarrow R$ on allowed control Hamiltonians, and it is this viewpoint we shall take most often. The cost function allows us to assign a cost to a control Hamiltonian $H(t)$ defined on an interval $[0,T]$ by $C(H(t))\equiv\int_0^T dtc(H(t))$. This allows us to define the *cost of a unitary* U by $C(U)\equiv \inf_{T,H(t)} C(H(t))$, where we take the infimum over all intervals $[0,T]$, and over all control functions $H(t)$ such that $H(t)\in\mathcal{H}_A$ for all times t, and $U(T)=U$. Note that in general there is no reason why this infimum should exist, as there may be no allowed control Hamiltonian $H(t)$ which can be used to synthesize the desired unitary U. However, if we assume that the Lie algebra generated by $H_1,\ldots,H_m$ is the full Lie algebra su(2^n), and that the allowed control region $\mathcal{H}_A$ is not trivial, we can ensure that such a control function exists, and so the infimum is defined [6]. This condition is known as the condition that the control system be *bracket generating*. Provided reasonable continuity assumptions are made about the cost function $c(\cdot)$ it can also be shown that the infimum is achieved for some control function $H(t)$.

The allowed control region $\mathcal{H}_A$ and the cost function $c(\cdot)$ jointly specify the control problems we shall be interested in. Such control problems are known as *right-invariant control problems* on the Lie group SU(2^n), and we shall denote them using the notation $(\mathcal{H}_A,c)$.

III. BOUNDS RELATING OPTIMAL CONTROL AND QUANTUM GATE COMPLEXITY

In this section we develop some general relationships between the cost function $C(U)$ of a right-invariant control system $(\mathcal{H}_A,c)$ on SU(2^n) and the exact and approximate gate complexities, $G(U)$ and $G(U,\epsilon)$. Our results generalize and extend the ideas in [3,4].

Splittings. The key tool we use to relate the cost $C(U)$ to the gate complexities $G(U)$ and $G(U,\epsilon)$ is an object we refer to as *splitting*. We define splittings in two steps. First, we identify a special set $\mathcal{H}_P\subseteq\mathcal{H}_A$ of *preferred* Hamiltonians, which we shall assume are bracket generating. Second, we identify a *projection map* $P:\mathcal{H}_A\rightarrow\mathcal{H}_P$ which takes any allowed Hamiltonian H and projects it onto a preferred Hamiltonian $H_P\equiv P(H)$. Note that this can be an arbitrary function, and need not be a projection in the linear algebraic sense. We call the pair $(\mathcal{H}_P,P)$ a *splitting* for the control problem $(\mathcal{H}_A,c)$.

The bounds relating the control cost $C(U)$ to gate complexity will depend on the particular splitting we choose. For examples of "good" choices of splitting (i.e., choices resulting in fairly tight bounds between control cost and gate complexity) see the later examples. For now we suppose that the choice of splitting has been fixed, and will show how it can be used to relate the control cost to gate complexity.

[1] Note that physicists' and mathematicians' definitions of Lie algebras differ by a factor of i, and so our definition of su(2^n) is consistent with the usual mathematical definition in terms of traceless *skew*-Hermitian matrics.

Our construction is rather abstract, and many readers may prefer to first read the statement of Theorem 1, and then to read Sec. IV, where that theorem is applied to several example control problems.

Relationship between $C(U)$ ***and*** $G(U)$. To express this relationship we need to define two quantities associated to the splitting. The first quantity is the maximal cost of applying any preferred Hamiltonian, $c_P \equiv \sup_{H\in\mathcal{H}_P} c(H)$. Note that we use the subscript P as a mnemonic to indicate that c_P is a cost associated to the set of preferred Hamiltonians. The second quantity is the maximal time T_P required to exactly generate an arbitrary one- or two-qubit unitary operation by applying time-dependent preferred Hamiltonians.

Observe that we can synthesize any one- or two-qubit quantum gate for a cost at most $c_P T_P$. Since U can be synthesized exactly using $G(U)$ one- and two-qubit gates, we deduce the desired bound relating $C(U)$ and $G(U)$:

$$C(U) \leq c_P T_P G(U). \tag{2}$$

Note that the value of $C(U)$ depends only on the control system, $(\mathcal{H}_A, c)$, not on the choice of splitting, $(\mathcal{H}_P, P)$. Thus, different choices of splitting can give rise to different bounds, and it is necessary to choose the splitting in an intelligent way to get the best possible bound. In particular, one should choose the splitting to minimize the product $c_P T_P$.

Relationship between $C(U)$ *and* $G(U,\epsilon)$. This relationship is rather more complex than that between $C(U)$ and $G(U)$, and is expressed in terms of four quantities associated to the splitting. The first quantity is a ratio defined by[2] $R \equiv \max_{H\in\mathcal{H}_A} \|H-H_P\|/c(H)$. The second quantity is the maximum matrix norm $N_P \equiv \max_{H\in\mathcal{H}_P} \|H\|$ of any preferred Hamiltonian.

The third quantity requires a more complex explanation. Suppose $\Delta > 0$ and $\delta > 0$. We define a Δ-averaged Hamiltonian to be a Hamiltonian $\bar{H}$ which can be written in the form $\bar{H} = \int_0^\Delta dt H(t)$ for some Hamiltonian control function which remains in the preferred set, $H(t) \in \mathcal{H}_P$. We define the Δ-averaged unitaries to be the set of unitary operations which can be written in the form $\exp(-i\bar{H})$ for some Δ-averaged Hamiltonian $\bar{H}$. We define $g(\Delta,\delta)$ to be the maximum number of one- and two-qubit gates required to approximate an arbitrary Δ-averaged unitary to an accuracy better than δ in matrix norm.

The fourth quantity is the minimal cost associated to any allowed Hamiltonian, $c_A \equiv \min_{H\in\mathcal{H}_A} c(H)$. This quantity arises in our proof as a way of getting a bound on the time T associated to the optimal Hamiltonian control $H(t)$. The argument is to observe that $C(U) = \int_0^T dt c(H(t)) \geq T c_A$, and so $T \leq C(U)/c_A$.

[2]Note that here and elsewhere we write max and min rather than sup and inf. Our proofs are easily modified for the case when (for example) the maximum is not defined, but this does make the discussion less transparent, and so we have avoided it.

With these quantities defined, we can relate $C(U)$ and $G(U,\epsilon)$. The first step is to take the Hamiltonian control $H(t)$ which achieves the optimal control cost $C(U)$, and to form the corresponding projected Hamiltonian $H_P(t) \equiv P(H(t))$. We suppose $H_P(t)$ generates a unitary U_P, and aim to show that U_P is a pretty good approximation to U. As in the proof of Lemma 1 in the supporting online materials for [4], we can apply the triangle inequality repeatedly to obtain:

$$\|U - U_P\| \leq \int_0^T dt \|H(t) - H_P(t)\|. \tag{3}$$

The definition of the ratio R ensures that $\|H-H_P\| \leq Rc(H)$ for all H, and thus:

$$\int_0^T dt \|H(t) - H_P(t)\| \leq R \int_0^T dt c(H) = RC(U). \tag{4}$$

Putting these inequalities together we obtain $\|U-U_P\| \leq RC(U)$. Intuitively, provided the control problem and splitting are such that R is much smaller than $1/C(U)$, we ensure that U and U_P will be quite close.

In the next step of the proof we discretize the evolution according to $H_P(t)$, and show that it can be approximated by a suitable sequence of Δ-averaged Hamiltonians. The key to doing this is the following lemma, which appeared as Lemma 2 in [4]. We have made some minor notational changes to the statement of the lemma, but the essential content of the lemma, and the proof, which is an easy application of the Dyson operator expansion, is unchanged.

Lemma 1. Let V be an n-qubit unitary generated by applying a time-dependent Hamiltonian $H_P(t) \in \mathcal{H}_P$ over a time interval $[s, s+\Delta]$. Then defining the corresponding Δ-averaged Hamiltonian $\bar{H} \equiv \int_s^{s+\Delta} dt H(t)$ we have:

$$\|V - e^{-i\bar{H}\Delta}\| \leq 2(e^{N_P\Delta} - 1 - N_P\Delta) = O(N_P^2\Delta^2), \tag{5}$$

where N_P is the maximum matrix norm of any preferred Hamiltonian, as defined earlier.

To apply this lemma, we divide the time interval $[0,T]$ up into a large number N of time intervals each of length $\Delta = T/N$. Let U_P^j be the unitary operation generated by $H_P(t)$ over the jth time interval. Let U_M^j (the unitary corresponding to the mean Hamiltonian) be the unitary operation generated by the Δ-averaged Hamiltonian over the corresponding time interval. Then the lemma implies that $\|U_P^j - U_M^j\| \leq O(N_P^2\Delta^2)$. By assumption, we can then synthesize a unitary operation U_A^j using at most $g(\Delta,\delta)$ one- and two-qubit gates, and satisfying $\|U_M^j - U_A^j\| \leq \delta$. We define U_A (the actual unitary to be synthesized by our gate sequence) to be the result of applying the unitaries U_A^j in sequence. Note that U_A can be generated using $Ng(\Delta,\delta) = Tg(\Delta,\delta)/\Delta$ one- and two-qubit quantum gates.

Repeated application of the triangle inequality, substitution of the inequalities obtained above, and using the fact that $N = T/\Delta$, yields:

$$\|U - U_A\| \tag{6}$$

$$\leq \|U - U_P\| + \|U_P - U_A\| \tag{7}$$

$$\leq RC(U) + \sum_{j=1}^{N} \|U_P^j - U_A^j\| \tag{8}$$

$$\leq RC(U) + \sum_{j=1}^{N} (\|U_P^j - U_M^j\| + \|U_M^j - U_A^j\|) \tag{9}$$

$$\leq RC(U) + O(N_P^2 T\Delta) + \frac{T}{\Delta}\delta. \tag{10}$$

Substituting the bound on T obtained earlier, $T \leq C(U)/c_A$, we deduce that we can synthesize an operation U_A satisfying

$$\|U - U_A\| \leq RC(U) + O\left(\frac{N_P^2 C(U)\Delta}{c_A}\right) + \frac{C(U)\delta}{c_A\Delta} \tag{11}$$

using $C(U)g(\Delta,\delta)/c_A\Delta$ one- and two-qubit gates.

Summing up, we have the following theorem:

Theorem 1. Consider a control problem $(\mathcal{H}_A, c)$ and a splitting $(\mathcal{H}_P, P)$ for that problem. Then we have:

(i) Let $c_P \equiv \max_{H\in\mathcal{H}_P} c(H)$ be the maximal cost of any preferred Hamiltonian, and suppose T_P is the maximal time required to generate an arbitrary one- or two-qubit unitary operation using preferred Hamiltonians. Then:

$$C(U) \leq c_P T_P G(U). \tag{12}$$

(ii) Let $R \equiv \max_{H\in\mathcal{H}_A} \|H - H_P\|/c(H)$, $N_P \equiv \max_{H\in\mathcal{H}_P} \|H\|$, $c_A \equiv \min_{H\in\mathcal{H}_A} c(H)$. Suppose that if $\bar{H}$ is a Δ-average of Hamiltonians in $\mathcal{H}_P$, i.e., can be written in the form $\bar{H} = \int_0^\Delta dt H(t)$ for some Hamiltonian control function $H(t)$ which remains in the preferred set, then the corresponding unitary $\exp(-i\bar{H})$ can be simulated to an accuracy δ using a number of gates $g(\Delta,\delta)$. Then we can synthesize an operation U_A satisfying

$$\|U - U_A\| \leq RC(U) + O\left(\frac{N_P^2 C(U)\Delta}{c_A}\right) + \frac{C(U)\delta}{c_A\Delta} \tag{13}$$

using $C(U)g(\Delta,\delta)/c_A\Delta$ one- and two-qubit gates.

We stress that this theorem does not necessarily give tight connections between optimal costs and gate complexity. Finding such connections depends on making an appropriate choice of the cost function, and of the splitting. However, the examples in the next section will show that such choices can be made for a wide variety of interesting cost functions.

IV. EXAMPLES

We will now describe a sequence of examples illustrating Theorem 1. These examples are not exhaustive, but illustrate the wide range of situations in which Theorem 1 can be used to relate problems of optimal control and quantum gate complexity.

Note that in each of the examples described in the present section, we are imagining that there is a *family* $U = U_n$ of unitary operations, one for each value of n, acting on n qubits. Correspondingly, in each of our examples we will describe an entire family of cost functions and splittings, one for each value of n. Our goal is to prove results of the form $\text{poly}(C(U),n) \leq G(U)$ and $G(U,\epsilon) \leq \text{poly}(C(U),n,1/\epsilon)$ for suitable polynomial functions.

Subriemannian metric. Suppose the allowed Hamiltonians $\mathcal{H}_A$ are of the form $H = \sum_\sigma h_\sigma \sigma$, where the sum is restricted to be over Pauli sigma matrices containing only one- and two-qubit terms, and we require that $\sum_\sigma h_\sigma^2 = 1$. We define the cost function by $c(H) \equiv \sqrt{\sum_\sigma h_\sigma^2}$ so for allowed Hamiltonians we have $c(H) = 1$. This cost function $C(U)$ is an example of the distance associated to a subriemannian metric [7], and the problem of finding $C(U)$ is that of finding the minimal length geodesics on a subriemannian manifold. We choose the splitting to be trivial, with $\mathcal{H}_P = \mathcal{H}_A$ and $P(H) = H$.

With this control problem $(\mathcal{H}_A, c)$ and splitting $(\mathcal{H}_P, P)$, we may apply part (i) of Theorem 1. In that notation, it follows immediately from the definitions that $c_P = 1$ and T_P is a constant of order one, independent of the number of qubits, n. Thus $C(U) \leq T_P G(U)$, and so, up to a constant factor, the subriemannian distance $C(U)$ provides a lower bound on the exact gate complexity $G(U)$.

To apply part (ii) of Theorem 1, note that we have $R = 0$ and $c_A = 1$, again directly from the definitions. It follows from elementary norm inequalities that $N_P = O(n)$.[3] To understand the behavior of $g(\Delta,\delta)$, suppose that $\bar{H} = \int_0^\Delta dt\, H(t)$ is a Δ-averaged Hamiltonian over Hamiltonians in $\mathcal{H}_P$. Lemma 3 in [4] implies that $\exp(-i\bar{H})$ can be simulated to an accuracy of order $O(n^4\Delta^3)$ using $O(n^2/\Delta)$ gates. Thus $g(\Delta, O(n^4\Delta^3)) \leq O(n^2/\Delta)$. We deduce that we can synthesize an operation U_A satisfying

$$\|U - U_A\| \leq O(C(U)n^2\Delta) + O(C(U)n^4\Delta^2) \tag{14}$$

using $O(C(U)n^2/\Delta^2)$ gates. It follows that by choosing Δ appropriately, we can synthesize a good approximation to U using a number of gates that scales in a fashion comparable to $C(U)$. To see this, let $\Delta = \epsilon/n^2 C(U)$. Then we see that we can synthesize an operation U_A satisfying $\|U - U_A\| \leq O(\epsilon)$ using $O(C(U)^3 n^6/\epsilon^2)$ gates. It follows that:

$$G(U,\epsilon) \leq O(C(U)^3 n^6/\epsilon^2), \tag{15}$$

which is the required result—$G(U,\epsilon)$ scales as no more than a polynomial in $C(U)$, n and $1/\epsilon$.

Time-optimal control. If $c(H) = 1$, then $C(U)$ is the minimal time taken to generate U using control Hamiltonians in the allowed control region, $\mathcal{H}_A$. This is known as the *time-optimal* control problem. A common variant of the time-optimal control problem is to constrain the set of allowed

[3] $\|H\| \leq \sum_\sigma |h_\sigma| \leq (\sqrt{3}/2)n\sqrt{\sum_\sigma h_\sigma^2} = (\sqrt{3}/2)n$, $\forall H \in \mathcal{H}_P$. The second inequality follows from $\|\vec{v}\|_1 \leq \sqrt{d}\|\vec{v}\|_2$, where d is the dimension of the real vector $\vec{v}$, $\|\vec{v}\|_1 = \sum_{i=1}^d |v_i|$ and $\|\vec{v}\|_2 = \sqrt{\sum_{i=1}^d v_i^2}$. In our case $d = 9n(n-1)/2 + 3n$, the number of one- and two-qubit terms.

controls so that $h_1(t)=1$, i.e., so that the Hamiltonian H_1 is always being applied. This is known as the time-optimal control problem with *drift*, and H_1 is known as the *drift Hamiltonian*. The time-optimal control problem in quantum physics has received considerable attention; see, e.g., [8–12] for recent work, and further references. Of particular interest in this context is work such as [9], which studies the time complexity of various quantum computing primitives, such as the quantum Fourier transform, and applies powerful tools from optimal control theory such as the Pontryagin maximum principle [13] (see, e.g., [5]) to obtain time-optimal implementations of these primitives.

The time-optimal control problem with drift takes a particularly simple and appealing form in the case where there are only two terms in the control Hamiltonian, i.e., $H=H_1+h(t)H_2$, and it is this case we shall focus on; analogous results can also be proved for other time-optimal control problems using essentially the same ideas. We will assume that the control region is such that the allowed range of values for $h(t)$ is $|h(t)|\leq 1$. *A priori* it is not obvious that it is possible to find examples of Hamiltonians H_1 and H_2 which are bracket-generating. However, it follows from results of [14,15] (c.f. [16]) that if we choose H_1 and H_2 at random, then with probability one they will be bracket-generating. Of course, this does not mean that they are universal for quantum computation in the usual sense. It may take such a H_1 and H_2 exponential time to generate standard quantum gates such as the controlled-NOT, or even single-qubit unitaries. Conversely, it may not be possible to efficiently simulate H_1 and H_2 in the standard quantum gate model of computation.

We will now provide examples of families of Hamiltonians H_1 and H_2 such that the time-optimal control cost scales as a polynomial in the quantum gate complexity. The key to this is the following theorem, which is of independent interest:

Theorem 2. There is a family of n-qubit Hamiltonians H_1 and H_2 such that: (1) any one- or two-qubit unitary gate can be synthesized exactly in a time bounded above by a value that scales as a polynomial in n; and (2) using one- and two-qubit gates we can simulate any unitary of the form $\exp(-i\Delta(H_1+\alpha H_2))$ (with $|\alpha|\leq 1$) to an accuracy δ using $g(\Delta,\delta)=O(p(n)\Delta^2/\delta)$ one- and two-qubit gates, for some polynomial $p(n)$.

Proof (outline): We choose H_1 to be a Hamiltonian acting on the first two qubits in a manner specified more precisely below. We choose H_2 so that $\exp(-iH_2)$ permutes qubits 2 through n by a cyclic displacement, i.e., the state of qubit 2 becomes the state of qubit 3, the state of qubit 3 becomes the state of qubit 4, and so on, with the state of qubit n becoming the state of qubit 2.

With these choices, conclusion (2) follows from standard quantum simulation techniques for simulating a sum of Hamiltonians, and the observation that the Hamiltonians H_1 and H_2 can both be efficiently simulated (the latter using the quantum Fourier transform [1,2]).

Conclusion (1) requires a little more effort. In particular, note that using H_1 and H_2 we can simulate the Hamiltonian $\exp(-iH_2)H_1\exp(iH_2)=\tilde{H}_1$, where the tilde denotes that $\tilde{H}_1$ is the same Hamiltonian as H_1, but now acts on qubits 1 and 3. It can now be verified numerically or by hand that for many choices of two-qubit Hamiltonian H_1, the Hamiltonians H_1 and $\tilde{H}_1$ generate the full Lie algebra on qubits one, two, and three.[4] As a result, in constant time we can generate an arbitrary unitary operation on qubits one, two, and three. Conjugating repeatedly by$\exp(-iH_2)$ we can use this to generate an arbitrary unitary on qubits 1 and j, where j is any qubit. Standard techniques then suffice to efficiently generate an arbitrary unitary on any pair of qubits. **Q.E.D.**

Suppose we consider the time-optimal control problem where H_1 and H_2 have been chosen as in Theorem 2. As in the subriemannian case we again choose the trivial splitting, $\mathcal{H}_P=\mathcal{H}_A$ and $P(H)=H$. Applying part (1) of Theorem 1, we see that $c_P=1$ and $T_p\leq q(n)$, for some polynomial $q(n)$. As a result, we have $C(U)\leq q(n)G(U)$.

Applying part (2) of Theorem 1, we have $R=0$, $N_P=O(1)$, $c_A=1$, and $g(\Delta,\delta)=O(p(n)\Delta^2/\delta)$, for some polynomial $p(n)$. As a result, we conclude that it is possible to synthesize a unitary U_A satisfying

$$\|U-U_A\|\leq O(C(U)\Delta)+O\left(\frac{C(U)\delta}{\Delta}\right) \tag{16}$$

using $O(C(U)p(n)\Delta/\delta)$ gates. Setting $\Delta=\epsilon/C(U)$ and $\delta=\epsilon^2/C(U)^2$, we see that we can synthesize a unitary U_A satisfying $\|U-U_A\|\leq O(\epsilon)$ using $O(C(U)^2p(n)/\epsilon)$ gates, and so we conclude that

$$G(U,\epsilon)\leq O(C(U)^2p(n)/\epsilon), \tag{17}$$

which is the desired polynomial scaling.

Riemannian metric. We now analyze the metric considered in [4], and show how to recover the results of [4]. This is our first example which makes use of a nontrivial splitting. Expanding the control Hamiltonian as $H=\Sigma_\sigma h_\sigma\sigma$, where the sum is over all n-qubit Pauli matrices, the cost function of [4] (which is just the norm associated to the metric) is defined by:

$$c(H)\equiv\sqrt{\sum_\sigma{}' h_\sigma^2+p^2\sum_\sigma{}'' h_\sigma}, \tag{18}$$

where the primed sum is over one- and two-qubit Pauli terms, and the double primed sum is over three- and more-qubit Pauli terms. The parameter p is a penalty whose value we set later. $\mathcal{H}_A$ is defined to contain all those Hamiltonians such that $c(H)=1$. For the splitting, we choose the set of preferred Hamiltonians $\mathcal{H}_P$ so that it contains all Hamiltonians containing just one- and two-qubit terms. The projection P takes an arbitrary Hamiltonian, and eliminates all terms in the Pauli expansion except the one- and two-qubit terms, i.e., it takes $\Sigma'_\sigma h_\sigma\sigma+\Sigma''_\sigma h_\sigma\sigma$ to $\Sigma'_\sigma h_\sigma\sigma$. In the language

[4]Examples of this phenomenon were found numerically by the present author and H. L. Haselgrove [17]. C. Hill and Haselgrove [18] have recently constructed rather more elegant examples demonstrating essentially the same phenomenon as described in this theorem, but making use of a relatively simple (and more physically plausible) two-body Hamiltonian in place of H_2, which involves complex many-body terms.

of part (1) of Theorem 1 we have $c_P=1$ and T_P is a constant, and so $C(U)\leq T_P G(U)$, i.e., the control cost is a lower bound on the gate complexity $G(U)$, to within a constant factor.

Next, we evaluate the quantities defined in part (2) of Theorem 1. To evaluate R, observe that the Hamiltonian H achieving the maximum must contain only terms which are three- or more-body. Thus:

$$R=\frac{\left\|\sum_{\sigma}{}'' h_\sigma \sigma\right\|}{p\sum_{\sigma}{}'' h_\sigma^2}\leq\frac{\sum_{\sigma}{}''|h_\sigma|}{p\sum_{\sigma}{}'' h_\sigma^2}\leq\frac{2^n}{p}, \tag{19}$$

where the first inequality follows from the triangle inequality, and the second inequality follows from the Cauchy-Schwarz inequality. For the same reasons as in the subriemannian case, $N_P=O(n)$, $c_A=1$ and $g(\Delta,O(n^4\Delta^3))\leq O(n^2/\Delta)$.

Applying part (2) of Theorem 1 we deduce that we can synthesize an operation U_A satisfying

$$\|U-U_A\|\leq\frac{2^n}{p}C(U)+O(C(U)n^2\Delta)+O(C(U)n^4\Delta^2) \tag{20}$$

using $O(C(U)n^2/\Delta^2)$ gates. Again we choose $\Delta=\epsilon/n^2C(U)$. Then we see that we can synthesize an operation U_A satisfying $\|U-U_A\|\leq 2^nC(U)/p+O(\epsilon)$ using $O(C(U)^3n^6/\epsilon^2)$ gates. Standard results on universality (see, e.g., [19] and references therein) imply that $C(U)\leq O(4^n)$ for all unitaries U, and so by choosing $p=8^n/\epsilon$ we obtain

$$G(U,\epsilon)\leq O(C(U)^3n^6/\epsilon^2), \tag{21}$$

which is the desired polynomial scaling.

Other control problems. It is not difficult to generate many other examples of optimal control problems whose cost scales in essentially the same way as the gate complexity. An example is the following cost function that was conjectured in [3] to be equivalent to the gate complexity:

$$c\left(\sum_\sigma h_\sigma\sigma\right)\equiv\sum_\sigma{}'|h_\sigma|+p\sum_\sigma{}''|h_\sigma|. \tag{22}$$

Once again, p is a penalty parameter that we shall choose to be large. We define $\mathcal{H}_A$ to consist of all Hamiltonians such that $c(H)=1$. We define the splitting as for the Riemannian metric considered above, setting $\mathcal{H}_P$ to be those Hamiltonians in $\mathcal{H}_A$ containing only one- and two-qubit terms, and the projection P to remove all three- and more qubit terms from the Pauli expansion. A similar analysis to the Riemannian case allows us to relate the cost $C(U)$ to the gate complexity. The only significant difference is in the evaluation of R, where we obtain $R\leq 1/p$, and thus it is possible in this case to choose more modest values of p and still achieve a close relationship between the scaling of the cost and of the gate complexity.

V. CONCLUSION

We have proved a general theorem relating quantum gate complexity to the optimal control cost for an arbitrary control problem. Application of the theorem depends on the use of a tool known as *splitting*, which must be chosen appropriately in order to obtain good bounds. We have illustrated this theorem with examples showing that quantum gate complexity is essentially equivalent to the optimal control cost for problems including time-optimal control and finding minimal distances on certain Riemannian, subriemannian, and Finslerian manifolds. It is possible to improve the scaling in many of these results with a more refined use of the Dyson operator expansion [20] and Suzuki-Trotter type formulas [21], and it would be interesting to determine what the optimal bounds are. It also seems likely that the results can be further generalized using tools more sophisticated than the notion of a splitting that we have introduced. However, the most important direction of future work will be to better understand the optimal cost for specific choices of the control problem, and what it implies for quantum gate complexity.

ACKNOWLEDGMENTS

Thanks to Lyle Noakes for his encouragement, and for emphasizing the importance of isolating the essential features of optimal control problems responsible for the equivalence to quantum gate complexity.

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