

# Universal destabilization and slowing of spin-transfer functions by a bath of spins

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We investigate the effect of a spin bath on the spin-transfer functions of a permanently coupled spin system. When each spin is coupled to a separate environment, the effect on the transfer functions in the first excitation sector is amazingly simple: the group velocity is slowed down by a factor of 2, and the fidelity is destabilized by a modulation of  $|\cos Gt|$ , where  $G$  is the mean square coupling to the environment.

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## I. INTRODUCTION

Recently suggested protocols [1–3] give a new perspective to the physics of strongly coupled spin systems. They demonstrate that the coherent transfer of spin flips can be used to transfer unknown quantum states and entanglement, a task of paramount importance in any quantum-information application [4]. Generally, the relevant quantities determining the performance of the mentioned protocols are the time-dependent transition amplitudes of local spin flips in a ferromagnetic ground state. We will refer to these amplitudes as “spin-transfer functions.” The same functions also occur in the charge- and energy-transfer dynamics in molecular systems [5] and in continuous-time random walks [6] to which our results equally apply.

It is both important and interesting to ask how these transfer functions change if the intended couplings between the spins are accompanied by unwanted couplings to environmental spins which do not take part in the transport. It is well known from the theory of open quantum systems [7] that this can lead to dissipation and decoherence, which also means that quantum information is lost. Here we consider a model where the system is coupled to a spin environment through an exchange interaction because the same type of coupling is also responsible for the transport of the information through the system. Moreover, this coupling offers the unique opportunity of an analytic solution of our problem without any approximations regarding the strength of system-environment coupling (in most treatments of the effect of an environment on the evolution of a quantum system, the system-environment coupling is assumed to be weak) and allows us to include inhomogeneous interactions of the bath spins with the system. For such coupling, decoherence is possible for mixed (thermal) initial bath states [8]. However if the system and bath are both initially cooled to their ground states, is there still a nontrivial effect of the environment on the spin-transfer functions? In this paper we find that there are two important effects: the spin-transfer functions are slowed and destabilized due to the environment. This has both positive and negative implications for the use of strongly coupled spin systems as quantum-communication channels.

## II. MODEL

We choose to start with a specific spin system, i.e., an open spin chain of arbitrary length  $N$ , with a Hamiltonian given by

$$H_S = -\frac{1}{2} \sum_{\ell=1}^{N-1} J_{\ell} (X_{\ell} X_{\ell+1} + Y_{\ell} Y_{\ell+1}), \quad (1)$$

where  $J_{\ell}$  are some arbitrary couplings and  $X_{\ell}$  and  $Y_{\ell}$  are the Pauli- $X$  and  $Y$  matrices for the  $\ell$ th spin. Toward the end of the paper we will, however, show that our results hold for any system where the number of excitations is conserved during dynamical evolution. In addition to the chain Hamiltonian, each spin  $\ell$  of the chain interacts with an independent bath of  $M_{\ell}$  environmental spins (see Fig. 1) via an inhomogeneous Hamiltonian,

$$H_I^{(\ell)} = -\frac{1}{2} \sum_{k=1}^{M_{\ell}} g_k^{(\ell)} (X_{\ell} X_k^{(\ell)} + Y_{\ell} Y_k^{(\ell)}). \quad (2)$$

In the above expression, the Pauli matrices  $X_{\ell}$  and  $Y_{\ell}$  act on the  $\ell$ th spin of the chain, whereas  $X_k^{(\ell)}$  and  $Y_k^{(\ell)}$  act on the  $k$ th environmental spin attached to the  $\ell$ th spin of the chain. We denote the total interaction Hamiltonian by

$$H_I \equiv \sum_{\ell=1}^N H_I^{(\ell)}. \quad (3)$$

The total Hamiltonian is given by  $H = H_S + H_I$ , where it is important to note that  $[H_S, H_I] \neq 0$ . We assume that a homogeneous magnetic field along the  $z$  axis is applied. The ground state of the system is then given by the fully polarized state  $|0, 0\rangle$ , with all chain and bath spins aligned along the  $z$  axis. The above Hamiltonian describes an extremely complex and disordered system with a Hilbert space of dimension  $2^{N+NM}$ . In the context of state transfer, however, only the dynamics of the first excitation sector is relevant. We proceed by mapping this sector to a much simpler system [9,10]. For  $\ell = 1, 2, \dots, N$  we define the states

$$|\ell, 0\rangle \equiv \sigma_{\ell}^{+} |0, 0\rangle \quad (4)$$

and

$$|0, \ell\rangle \equiv \frac{1}{G_{\ell}} \sum_{k=1}^{M_{\ell}} g_k^{(\ell)} \sigma_k^{+} |0, 0\rangle \quad (5)$$

with

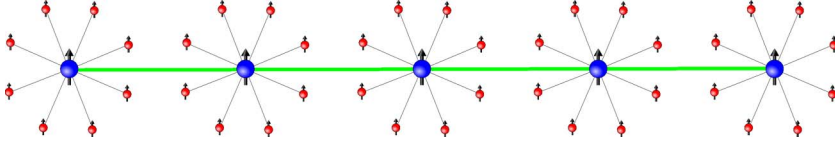


FIG. 1. (Color online) A spin chain of length  $N=5$  coupled to independent baths of spins.

$$G_\ell = \sqrt{\sum_{k=1}^{M_\ell} (g_k^{(\ell)})^2}. \quad (6)$$

It is easily verified that (setting  $J_0=J_N=0$ )

$$H_S|\ell, 0\rangle = -J_{\ell-1}|\ell-1, 0\rangle - J_\ell|\ell+1, 0\rangle,$$

$$H_S|0, \ell\rangle = 0, \quad (7)$$

and

$$H_I|\ell, 0\rangle = -G_\ell|0, \ell\rangle, \quad (8)$$

$$H_I|0, \ell\rangle = -G_\ell|\ell, 0\rangle. \quad (9)$$

Hence these states define a  $2N$ -dimensional subspace that is invariant under the action of  $H$ . This subspace is equivalent to the first excitation sector of a system of  $2N$  spin-1/2 particles, coupled as is shown in Fig. 2.

Our main assumption is that the bath couplings are *in effect* the same, i.e.,  $G_\ell=G$  for all  $\ell$ . Note, however, that the individual numbers of bath spins  $M_\ell$  and bath couplings  $g_k^{(\ell)}$  may still depend on  $\ell$  and  $k$  as long as their mean square average is the same. Also, our analytic solution given in the next section relies on this assumption, but numerics show that our main result [Eq. (25)] remains a good approximation if the  $G_\ell$  slightly vary and we take  $G \equiv \langle G_\ell \rangle$ .

### III. RESULTS

In this section, we solve the Schrödinger equation for the model outlined above and discuss the spin-transfer functions. First, let us denote the orthonormal eigenstates of  $H_S$  alone by

$$H_S|\psi_k\rangle = \epsilon_k|\psi_k\rangle \quad (k=1, 2, \dots, N) \quad (10)$$

with

$$|\psi_k\rangle = \sum_{\ell=1}^N a_{k\ell}|\ell, 0\rangle. \quad (11)$$

For what follows, it is not important whether analytic expressions for the eigensystem of  $H_S$  can be found. Our result holds even for models that are not analytically solvable, such as the randomly coupled chains considered in [2]. We now make an ansatz for the eigenstates of the full Hamiltonian, motivated by the fact that the states

$$|\phi_\ell^n\rangle \equiv \frac{1}{\sqrt{2}}[|\ell, 0\rangle + (-1)^n|0, \ell\rangle] \quad (n=1, 2) \quad (12)$$

are eigenstates of  $H_I^{(\ell)}$  with the corresponding eigenvalues  $\pm G$  [this follows directly from Eqs. (8) and (9)]. Define the vectors

$$|\Psi_k^n\rangle \equiv \sum_{\ell=1}^N a_{k\ell}|\phi_\ell^n\rangle \quad (13)$$

with  $k=1, 2, \dots, N$  and  $n=0, 1$ . The  $|\Psi_k^n\rangle$  form an orthonormal basis in which we express the matrix elements of the Hamiltonian. We can easily see that

$$H_I|\Psi_k^n\rangle = -(-1)^n G|\Psi_k^n\rangle \quad (14)$$

and

$$H_S|\Psi_k^n\rangle = \frac{\epsilon_k}{\sqrt{2}} \sum_{\ell=1}^N a_{k\ell}|\ell, 0\rangle = \frac{\epsilon_k}{2}(|\Psi_k^0\rangle + |\Psi_k^1\rangle). \quad (15)$$

Therefore the matrix elements of the full Hamiltonian  $H = H_S + H_I$  are given by

$$\langle \Psi_{k'}^n | H | \Psi_k^n \rangle = \delta_{kk'} \left( -(-1)^n G \delta_{nn'} + \frac{\epsilon_k}{2} \right). \quad (16)$$

The Hamiltonian is not diagonal in the states of Eq. (13). But  $H$  is now block diagonal, consisting of  $N$  blocks of size 2, which can be easily diagonalized analytically. The orthonormal eigenstates of the Hamiltonian are given by

$$|E_k^n\rangle = c_{kn}^{-1} \{ [(-1)^n \Delta_k - 2G] |\Psi_k^0\rangle + \epsilon_k |\Psi_k^1\rangle \} \quad (17)$$

with the eigenvalues

$$E_k^n = \frac{1}{2} [\epsilon_k + (-1)^n \Delta_k] \quad (18)$$

and the normalization

$$c_{kn} \equiv \sqrt{[(-1)^n \Delta_k - 2G]^2 + \epsilon_k^2}, \quad (19)$$

where

$$\Delta_k = \sqrt{4G^2 + \epsilon_k^2}. \quad (20)$$

Note that the ansatz of Eq. (13) that put  $H$  in block-diagonal form did not depend on the details of  $H_S$  and  $H_I^{(\ell)}$ . The methods presented here can be applied to a much larger class of systems, including the generalized spin star systems (which include an interaction within the bath) discussed in [10].

After solving the Schrödinger equation, let us now turn to quantum state transfer. The relevant quantity [1–3] is given by the transfer function

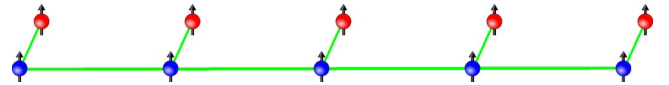


FIG. 2. (Color online) In the first excitation sector, the system can be mapped into an effective spin model where the bath spins are replaced by a single effective spin, as indicated here for  $N=5$ .

$$\begin{aligned}
 f_{N,1}(t) &\equiv \langle N,0 | \exp\{-iHt\} | 1,0 \rangle \\
 &= \sum_{k,n} \exp\{-iE_k^n t\} \langle E_k^n | 1,0 \rangle \langle N,0 | E_k^n \rangle.
 \end{aligned}$$

The modulus of  $f_{N,1}(t)$  is between 0 (no transfer) and 1 (perfect transfer) and fully determines the fidelity of state transfer. Since

$$\begin{aligned}
 \langle \ell, 0 | E_k^n \rangle &= c_{kn}^{-1} [(-1)^n \Delta_k - 2G] \langle \ell, 0 | \Psi_k^0 \rangle + \epsilon_k \langle \ell, 0 | \Psi_k^1 \rangle \\
 &= \frac{c_{kn}^{-1}}{\sqrt{2}} [(-1)^n \Delta_k - 2G + \epsilon_k] a_{k\ell},
 \end{aligned}$$

we get

$$f_{N,1}(t) = \frac{1}{2} \sum_{k,n} e^{(-it/2)[\epsilon_k + (-1)^n \Delta_k]} \frac{[(-1)^n \Delta_k - 2G + \epsilon_k]^2}{[(-1)^n \Delta_k - 2G]^2 + \epsilon_k^2} a_{k1} a_{kN}^*. \quad (21)$$

Equation (21) is the main result of this paper, fully determining the transfer of quantum information and entanglement in the presence of the environment. In the limit  $G \rightarrow 0$ , we have  $\Delta_k \approx \epsilon_k$ , and  $f_{N,1}(t)$  approaches the usual result [1–3] without an environment,

$$f_{N,1}^0(t) \equiv \sum_k \exp\{-it\epsilon_k\} a_{k1} a_{kN}^*. \quad (22)$$

In fact, a series expansion of Eq. (21) yields that the first modification of the transfer function is of the order of  $G^2$ ,

$$G^2 \sum_k a_{k1} a_{kN}^* \left[ \exp\left\{-it\epsilon_k\right\} \left( -\frac{1}{\epsilon_k} - \frac{it}{\epsilon_k} \right) + \frac{1}{\epsilon_k^2} \right]. \quad (23)$$

Hence the effect is small for very weakly coupled baths. However, as the chains get longer, the lowest-lying energy  $\epsilon_1$  usually approaches zero, so the changes become more significant (scaling as  $1/\epsilon_k$ ). For intermediate  $G$ , we evaluated Eq. (21) numerically and found that the first peak of the transfer function generally becomes slightly lower, and gets shifted to higher times (Figs. 3 and 4). A numeric search in the coupling space  $\{J_\ell, \ell = 1, \dots, N-1\}$ , however, also revealed some rare examples where an environment can also slightly improve the peak of the transfer function (Fig. 5).

In the strong-coupling regime  $G \gg \epsilon_k/2$ , we can approximate Eq. (20) by  $\Delta_k \approx 2G$ . Inserting it in Eq. (21) then gives

$$\begin{aligned}
 f_{N,1}(t) &\approx \frac{1}{2} e^{-iGt} \sum_k \exp\left\{-it\epsilon_k \frac{1}{2}\right\} a_{k1} a_{kN}^* \\
 &\quad + \frac{1}{2} e^{iGt} \sum_k \exp\left\{-it\epsilon_k \frac{1}{2}\right\} a_{k1} a_{kN}^* \\
 &= \cos(Gt) f_{N,1}^0\left(\frac{t}{2}\right). \quad (24)
 \end{aligned}$$

This surprisingly simple result consists of the normal transfer function, slowed down by a factor of  $1/2$ , and modulated by a quickly oscillating term (Figs. 3 and 4). Our derivation actually did not depend on the indices of  $f(t)$  and we get for the transfer from the  $n$ th to the  $m$ th spin of the chain that

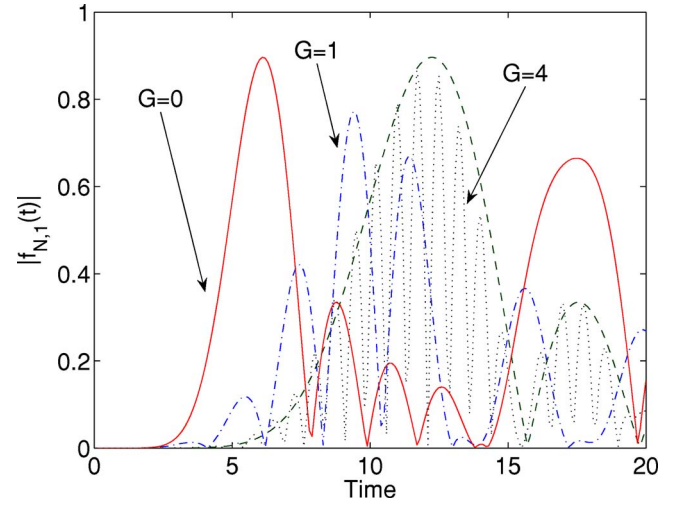


FIG. 3. (Color online) The absolute value of the transport function  $f_{N,1}(t)$  of a uniform spin chain (i.e.,  $J_\ell=1$ ) with length  $N=10$  for three different values of the bath coupling  $G$ . The dashed line is the envelope of the limiting function for  $G \gg \epsilon_k/2$  given by  $|f^0(t/2)|$ . We can see that Eq. (25) becomes a good approximation already at  $G=4$ .

$$f_{n,m}(t) \approx \cos(Gt) f_{n,m}^0\left(\frac{t}{2}\right). \quad (25)$$

It may look surprising that the matrix  $f_{n,m}$  is no longer unitary. This is because we are considering the dynamics of the chain only, which is an open quantum system [7]. A heuristic interpretation of Eq. (25) is that the excitation oscillates back and forth between the chain and the bath (hence the modulation), and spends half of the time trapped in the bath (hence the slowing). If the time of the maximum of the transfer function  $|f_{n,m}^0(t)|$  for  $G=0$  is a multiple of  $\pi/2G$ , then this maximum is also reached in the presence of the bath.

Finally, we want to stress that Eq. (25) is *universal* for any

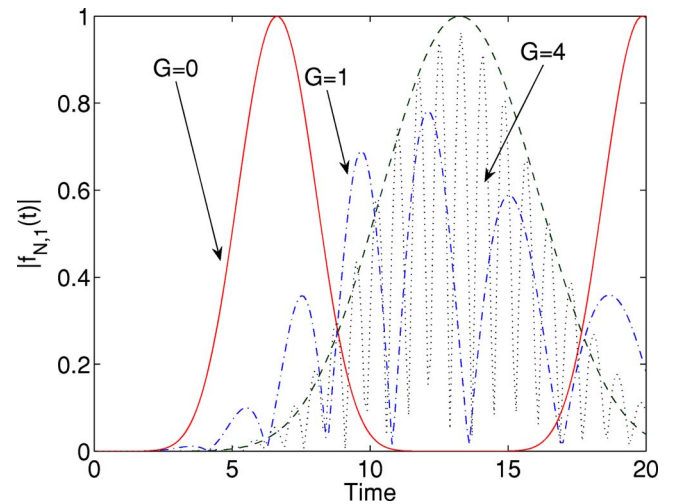


FIG. 4. (Color online) The same as Fig. 3, but now for an engineered spin chain [i.e.,  $J_\ell = \sqrt{\ell(N-\ell)}$ ] as in [3]. For comparison, we have rescaled the couplings such that  $\sum_\ell J_\ell$  is the same as in the uniform coupling case.

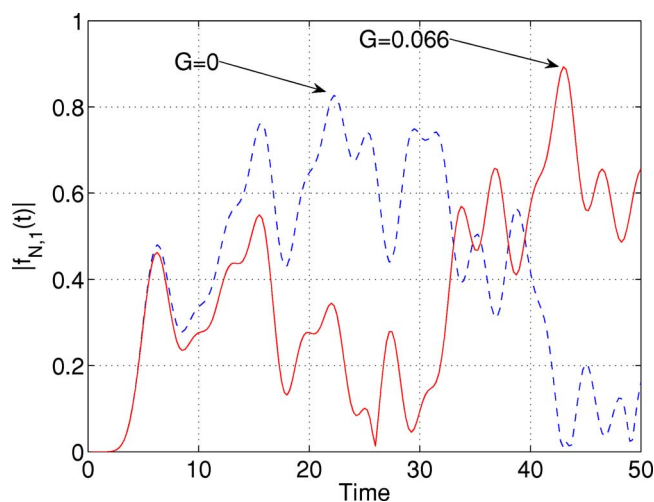


FIG. 5. (Color online) A weakly coupled bath may even improve the transfer function for some specific choices of the  $J_\ell$ . This plot shows the transfer function  $|f_{N,1}(t)|$  for  $N=10$ . The couplings  $J_\ell$  were found numerically.

spin Hamiltonian that conserves the number of excitations, i.e., with  $[H_S, \sum_\ell Z_\ell] = 0$ . Thus our restriction to chainlike topology and exchange couplings for  $H_S$  is not necessary. In fact the only difference in the whole derivation of Eq. (25) for a more general Hamiltonian is that Eq. (7) is replaced by

$$H_S |\ell, 0\rangle = \sum_{\ell'} h_{\ell'} |\ell', 0\rangle. \quad (26)$$

The Hamiltonian can still be formally diagonalized in the first excitation sector as in Eq. (11), and the states of Eq. (17)

will still diagonalize the total Hamiltonian  $H_S + H_I$ . Also, rather than considering an exchange Hamiltonian for the interaction with the bath, we could have considered a Heisenberg interaction [11], but only for the special case where all bath couplings  $g_k^{(\ell)}$  are all the same [12]. Up to some irrelevant phases, this leads to the same results as for the exchange interaction.

#### IV. CONCLUSION

We found a surprisingly simple and universal scaling law for the spin-transfer functions in the presence of spin environments. In the context of quantum state transfer [1–3] this result is double edged: on one hand, it shows that even for very strongly coupled baths quantum state transfer is possible, with the same fidelity and only reasonable slowing. On the other hand, it also shows that the fidelity as a function of time becomes destabilized with a quickly oscillating modulation factor. In practice, this factor will restrict the time scale in which one has to be able read the state from the system. This demonstrates that even though a bath coupling need not introduce decoherence or dissipation to the system, it may cause other dynamical processes such as destabilization that can be problematic for quantum-information processing.

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