Connection between quantum-mechanical and classical time evolution via a dynamical invariant

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The time evolution of a quantum system with at most quadratic Hamiltonian is described with the help of different methods, namely the time-dependent Schrödinger equation, the time propagator or Feynman kernel, and the Wigner function. It is shown that all three methods are connected via a dynamical invariant, the so-called Ermakov invariant. This invariant introduces explicitly the quantum aspect via the position uncertainty and its possible time dependence. The importance of this aspect, also for the difference between classical and quantum dynamics, and in particular the role of the initial position uncertainty is investigated.

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I. INTRODUCTION

For a complete description of a physical system, the classical point of view is generally not sufficient, but one also has to take into account the quantum aspect. This also holds for time-dependent problems where, in addition to the time evolution of position and momentum, which represent the classical particle aspect, also the change in time of the position, and momentum uncertainties, representing the wave mechanical aspect, have to be considered.

An interesting question in this context is also, what are the characteristic differences between the quantum mechanical and the classical description of a dynamical system and what do they depend upon?

Different methods exist to describe the dynamics of quantum systems, namely (i) the time-dependent Schrödinger equation (SE) [with the corresponding continuity equation (CE) for the probability density]; (ii) the description using a time propagator, also called Feynman kernel or space representation of the Green function; and (iii) the time-dependent Wigner function.

We will show that all three methods are linked via a dynamical invariant, the so-called Ermakov invariant (discovered by Ermakov in 1880 [1], obviously without any relation to the not yet existing quantum theory).

In the next section, for one dimensional and at most quadratic Hamiltonians (with possibly time-dependent frequency), we will show how this invariant emerges from the wave packet (WP) solution of the corresponding time-dependent SE and how the occurring variables are related to position, momentum, and the respective uncertainties. Also, the CE corresponding to the time-dependent SE will be briefly considered, in particular to show a characteristic difference compared to the classical situation.

In Sec. III, the time evolution of an initial system will be described using a so-called propagator or Feynman kernel. For the derivation of this kernel, Feynman solely uses the classical Lagrangian of the respective system. The Feynman kernel of the free motion and of the harmonic oscillator (HO) will be given explicitly. Since this formalism is based entirely on the classical Lagrangian, no information about the initial quantum uncertainties of position (or momentum) enter, a fact that has some consequences which will be discussed afterwards.

For the quadratic systems with Gaussian WP solutions, the time propagator can also be derived very simply by using a Gaussian ansatz where the time dependence enters via two parameters that are connected with the description in Sec. II, in particular with the Ermakov invariant. Since this invariant not only depends on the classical variables, but also on the quantum uncertainties, some information about this aspect, especially concerning the initial state, will enter this form of the propagator.

With the help of our propagator, the WP solution for the free motion and the HO, with emphasis on the behavior of the WP width (proportional to the position uncertainty), will be studied and compared with the usual treatment of these systems via Feynman kernel. Specifically, the limit of disappearing oscillator frequency, $\omega \to 0$, will be discussed. Also, the relation between the Ermakov invariant and the time-dependent parameters of our propagator will be explained.

The same form of our propagator can be obtained using the Ermakov system as a starting point to modify the Feynman procedure in a way that also the quantum uncertainty aspect enters from the beginning, in addition to the classical Lagrangian. Following the elegant method of Dhara and Lawande [2] or a similar treatment by Nassar [3], where clearly the aspect of the time dependence of the position uncertainty becomes obvious, one also arrives at our above-mentioned propagator. After the inclusion of the uncertainty aspect into the propagator formalism via the Ermakov invariant has been specified, the influence of the initial uncertainty on the time evolution of the quantum system will be discussed.

The connection between the Ermakov invariant and the time-dependent Wigner function will be shown in Sec. IV. The consequences of a time dependence of the position uncertainty for the equations of motion connected with the Wigner function will be specified, in detail.

Finally, in Sec. V, our results will be summarized, the occurrence and importance of time-dependent position uncertainties will be stressed—also for the difference between classical mechanics and quantum mechanics—the special

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role of the initial position uncertainty will be emphasized, and an outlook for further developments will be given.

II. DESCRIPTION OF QUANTUM DYNAMICS VIA TIME-DEPENDENT SCHRÖDINGER EQUATION

One way of describing the dynamics of a quantum system is the time-dependent SE. In the following, we restrict our discussion to one spatial dimension and Hamiltonians that are, at most, quadratic in their variables. We will discuss explicitly the important model problems of the free motion and the HO [the possibility of a time-dependent frequency $\omega=\omega(t)$ will also be mentioned], because in these cases, for the relevant occurring quantities, exact analytic expressions exist that allow a direct comparison of the considered methods.

We begin with the time-dependent SE for the one-dimensional HO in position space

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t)=\left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{m}{2}\omega^2x^2\right\}\Psi(x,t). \tag{1}$$

For this equation, analytic solutions in the forms of Gaussian WPs exist

$$\Psi_{WP}(x,t)=N(t)\exp\left\{i\left[y(t)\tilde{x}^2+\frac{1}{\hbar}\langle p\rangle\tilde{x}+K(t)\right]\right\}, \tag{2}$$

where $\tilde{x}=x-\langle x\rangle=x-\eta(t)$ with $\langle x\rangle=\int_{-\infty}^{+\infty}\Psi^*x\Psi dx=\eta(t)=$ classical trajectory, $\langle p\rangle=m\dot{\eta}=$ classical momentum; $N(t)$ (normalizing factor) and $K(t)$ are purely time-dependent terms whose meaning will become obvious later. The complex, possibly time-dependent, quantity $y(t)=y_R(t)+iy_I(t)$ is connected with the WP width, or the position uncertainty, via $y_I=\frac{1}{4\langle\tilde{x}^2\rangle}$ with $\langle\tilde{x}^2\rangle=\langle x^2\rangle-\langle x\rangle^2$.

Inserting WP (2) into SE (1) yields the equations of motion for $\eta(t)$ and $y(t)$. The equation of motion for the WP maximum, located at $x=\eta(t)$, is just the classical equation of motion

$$\ddot{\eta}+\omega^2\eta=0, \tag{3}$$

where overdots denote time derivatives.

The equation of motion for $y(t)$ is given by the complex Riccati equation

$$\frac{2\hbar}{m}\dot{y}+\left(\frac{2\hbar}{m}y\right)^2+\omega^2=0, \tag{4}$$

which can be separated into real and imaginary parts

$$\text{Im:}\quad \frac{2\hbar}{m}\dot{y}_I+2\left(\frac{2\hbar}{m}y_I\right)\left(\frac{2\hbar}{m}y_R\right)=0, \tag{5}$$

$$\text{Re:}\quad \frac{2\hbar}{m}\dot{y}_R+\left(\frac{2\hbar}{m}y_R\right)^2-\left(\frac{2\hbar}{m}y_I\right)^2+\omega^2=0. \tag{6}$$

The real part $y_R(t)$ can be eliminated from Eq. (6) by solving Eq. (5) for y_R and inserting the result into Eq. (6).

It is useful to introduce a new variable, $\alpha(t)$, that is connected with $y_I(t)$ via

$$\frac{2\hbar}{m}y_I=\frac{1}{\alpha^2(t)}, \tag{7}$$

where $\alpha(t)$ is directly proportional to the WP width, or position uncertainty, respectively, i.e.

$$\alpha=\sqrt{\frac{m}{2\hbar y_I}}=\sqrt{\frac{2m\langle\tilde{x}^2\rangle}{\hbar}}. \tag{8}$$

Inserting definition (7) into Eq. (5) shows that the real part of $y(t)$ just describes the relative change in time of the WP width

$$\frac{2\hbar}{m}y_R=\frac{\dot{\alpha}}{\alpha}=\frac{1}{2}\frac{\frac{d}{dt}\langle\tilde{x}^2\rangle}{\langle\tilde{x}^2\rangle}. \tag{9}$$

Together with definition (7) this finally turns Eq. (6) into

$$\ddot{\alpha}+\omega^2\alpha=\frac{1}{\alpha^3}. \tag{10}$$

It has been shown by Ermakov in 1880 [1] that the system of differential equations (3) and (10), coupled via the possibly time-dependent frequency ω, leads to a dynamical invariant, I, that has been rediscovered by several authors in the 20th century [4], also in a quantum mechanical context [5], and is also known in the literature as Ermakov-Lewis invariant. Ermakov's way of obtaining this invariant (see also Ray and Reid in [6]) used Eq. (3) to eliminate ω^2 in Eq. (10) (a method that we will encounter again later in the context of the Feynman kernel), leading to $\ddot{\alpha}-\frac{\ddot{\eta}}{\eta}\alpha=\frac{1}{\alpha^3}$. After some basic manipulations, one arrives at the expression

$$\frac{1}{2}\frac{d}{dt}(\dot{\eta}\alpha-\eta\dot{\alpha})^2=-\frac{1}{2}\frac{d}{dt}\left(\frac{\eta}{\alpha}\right)^2, \tag{11}$$

which yields the invariant

$$I=\frac{1}{2}\left[(\dot{\eta}\alpha-\eta\dot{\alpha})^2+\left(\frac{\eta}{\alpha}\right)^2\right]=\text{const}. \tag{12}$$

This invariant not only depends on the classical variables $\eta(t)$ and $\dot{\eta}(t)$, like, e.g., the classical Lagrangian, but also on the quantum uncertainty, connected with $\alpha(t)$, and its change in time, $\dot{\alpha}(t)$, which will be of importance later on.

In order to discuss the differences between the classical and the quantum dynamics of the same model system, we briefly mention a property of the density function $\rho(x,t)=\Psi^*(x,t)\Psi(x,t)$, corresponding to the solution of the SE (1). This probability density fulfills the CE

$$\frac{\partial}{\partial t}\rho+\frac{\partial}{\partial x}(\rho v)=\frac{\partial}{\partial x}\rho+v\frac{\partial}{\partial x}\rho+\rho\frac{\partial}{\partial x}v=0, \tag{13}$$

where the velocity field v in the probability current $j=\rho v$ is defined as

$$v = \frac{\hbar}{2mi}\left(\frac{\frac{\partial}{\partial x}\Psi}{\Psi} - \frac{\frac{\partial}{\partial x}\Psi^*}{\Psi^*} \right) = \frac{\hbar}{2mi}\frac{\partial}{\partial x}\ln\frac{\Psi}{\Psi^*}, \tag{14}$$

i.e., it depends on the phase of Ψ. In the case of our WP solutions, one obtains for the quantum system

$$\frac{\partial}{\partial x}v = \frac{\dot{\alpha}}{\alpha} = \frac{1}{2}\frac{\frac{d}{dt}\langle\tilde{x}^2\rangle}{\langle\tilde{x}^2\rangle}. \tag{15}$$

This quantity only vanishes for WPs with constant width, but for $\dot{\alpha}\neq 0$ it is different from zero. For the classical case, a continuity equation equivalent to Eq. (13) exists, but there for the distribution of a virtual ensemble over the phase space, described by a density function ρ_{cl} of the generalized coordinates and momenta and of time. From Hamilton's equations of motion it follows that

$$\nabla v = 0, \tag{16}$$

(but now in phase space with ∇ representing derivatives with respect to position and momentum variables). For more details comparing the classical and the quantum CEs, see [7]. In any case, the fact that $\frac{\partial}{\partial x}v \neq 0$ is possible for a Hamiltonian system is a typical quantum property and the conditions that have to be fulfilled in order to obtain this property will follow from our further discussion.

III. DESCRIPTION OF QUANTUM DYNAMICS VIA TIME PROPAGATOR

A. Feynman kernel via classical Lagrangian

Another way of describing the dynamics of a quantum system is to start with an initial state $\Psi(x',t')$ at time t' (e.g., $t'=0$) and to propagate it into the state at time t with the help of a so-called propagator $K(x,x',t,t')$ via

$$\Psi(x,t) = \int_{-\infty}^{+\infty} dx' K(x,x',t,t')\Psi(x',t'), \tag{17}$$

where the integral kernel $K(x,x',t,t')$ provides the probability for the transition of the initial state at time t' into the state at time t. Since Feynman found a way to derive this kernel (via the so-called path integral method), it is also called Feynman kernel (some authors also call it Feynman propagator, which is incorrect, since the Feynman propagator has a different form and is important in quantum field theory (see, e.g., [8], p. 25)).

According to Feynman [9], the kernel is defined as

$$K_F(x,x',t,t') = \int \exp\left(\frac{i}{\hbar}\int_{t'}^{t} L_{cl}dt''\right)\mathcal{D}x(t), \tag{18}$$

where $\mathcal{D}x(t)$ is the Feynman path differential measure and L_{cl} the classical Lagrangian [in the case of the HO it is in our notation $L_{cl}=\frac{m}{2}(\dot{\eta}^2-\omega^2\eta^2)$] so the integral in the exponent corresponds to the classical action. It is important to mention that L_{cl} depends only on the classical trajectory and the classical velocity (and their initial conditions), but not on any quantum property. Nevertheless, the corresponding kernel K_F is also able to describe the quantum dynamics of the system. Why this is possible, and which quantum aspect still has to be taken into account—and at what point—in order not to miss anything important, shall be clarified subsequently.

For the free motion and the HO (at least with constant frequency ω), analytic expressions for K_F can be obtained; these are [8,9]:

(a) Free motion ($V=0$)

$$K_{F,fr}(x,x',t,t'=0) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \times\exp\left\{\frac{im}{2\hbar t}(x^2-2xx'+x'^2)\right\}, \tag{19}$$

(b) HO $\left(V=\frac{m}{2}\omega^2x^2\right)$

$$K_{F,HO}(x,x',t,t'=0) = \left(\frac{m\omega}{2\pi i\hbar\ \sin\omega t}\right)^{1/2}\exp\left\{\frac{im\omega}{2\hbar\ \sin\omega t} \times\left[(x^2+x'^2)\cos\omega t-2xx'\right]\right\} \tag{20}$$

For an initial WP located (with its maximum) at $\langle x\rangle(t=0)=\eta_0=0$ and with initial momentum p_0 and initial width proportional to $\langle\tilde{x}^2\rangle_0$, i.e.

$$\Psi_{WP}(x',t'=0) = \left(\frac{m\beta_0}{\pi\hbar}\right)^{1/4}\exp\left\{\frac{im}{2\hbar}\left[i\left(\frac{x'}{\alpha_0}\right)^2+2\frac{p_0}{m}x'\right]\right\}, \tag{21}$$

where the abbreviation $\beta_0=\frac{\hbar}{2m\langle\tilde{x}^2\rangle_0}=\frac{1}{\alpha_0^2}$ (with dimension frequency) has been introduced. When the Feynman kernel K_F is inserted into Eq. (17), where $\Psi(x',t')$ is given by Eq. (21), from the explicit time dependence of K_F, as given in Eqs. (19) and (20), together with the initial momentum p_0 occurring in $\Psi(x',t')$, the classical trajectory emerges in the resulting time-dependent WP. Denoting this classical path—as defined below Eq. (2)—by $\eta(t)$, one obtains the time-dependent WP

(a) for the free motion

$$\Psi_{WP,fr}(x,t) = \left(\frac{\beta_0 m}{\pi\hbar}\right)^{1/2}\left(\frac{1}{1+i\beta_0 t}\right)^{1/2} \times\exp\left\{-\frac{[x-\eta(t)]^2}{4\langle\tilde{x}^2\rangle_0[1+(\beta_0 t)^2]} + \frac{im}{2\hbar}\beta_0\frac{(\beta_0 t)}{[1+(\beta_0 t)^2]}(x-\eta)^2 + \frac{i}{\hbar}\left(\langle p\rangle\tilde{x}+\frac{\langle p\rangle\eta}{2}\right)\right\}, \tag{22}$$

where the WP width is spreading quadratically in time, and,

(b) for the HO, where the initial WP corresponds to the ground state with $\beta_0=\omega$, i.e.

$$\Psi_{WP,HO}(x',0) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left\{-\frac{m\omega}{2\hbar}x'^2 + \frac{i}{\hbar}p_0 x'\right\}, \tag{23}$$

one obtains the WP that is usually found in the textbooks (e.g., [10])

$$\Psi_{WP,HO}(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left\{-\frac{m\omega}{2\hbar}(x-\eta)^2 + \frac{i}{\hbar}\left[\langle p\rangle(x-\eta) + \frac{1}{2}\langle p\rangle\eta\right]\right\}, \tag{24}$$

i.e., a WP with constant width $\langle\tilde{x}^2\rangle = \langle\tilde{x}^2\rangle_0 = \frac{\hbar}{2m\omega}$.

One should assume that in the limit $\omega\to 0$, the WP solution (24) of the HO turns into the spreading WP solution (22) of the free motion. However, performing this limit, the WP (24) turns into a plane-wave-type solution

$$\lim_{\omega\to 0}\Psi_{WP,HO}(x,t) \propto \exp\left\{ikx - \frac{i\hbar}{2m}k^2 t\right\}, \tag{25}$$

with $k = \frac{\langle p\rangle}{\hbar}, \langle p\rangle = m\dot{\eta} = \text{const}$, which is only a particular solution of the problem but not the complete one that corresponds to the WP spreading in time. The reason for this inconsistency will become obvious during the course of our discussion.

B. Direct solution for Gaussian WPs

For the quadratic systems with Gaussian WP solutions, the time propagator can also, very simply, be derived directly using a Gaussian ansatz for K, where the time dependence enters via two parameters $\hat{u}$ and $\hat{z}$ that are connected with the Ermakov invariant of Sec. II, as will now be shown. Since, via α and $\dot{\alpha}$ I depends also on the quantum uncertainties, some information about this aspect will enter this form of K.

With the ansatz

$$K_D(x,x',t,t'=0) = \left(\frac{m}{2\pi i\hbar\,\alpha_0\hat{z}}\right)^{1/2} \exp\left\{\frac{im}{2\hbar}\left[\frac{\dot{\hat{z}}}{\hat{z}}x^2 - 2\frac{x}{\hat{z}}\left(\frac{x'}{\alpha_0}\right) + \frac{\hat{u}}{\hat{z}}\left(\frac{x'}{\alpha_0}\right)^2\right]\right\}, \tag{26}$$

where $\alpha_0 = \alpha(t=0)$, and the initial WP, as defined in Eq. (21), one obtains the WP at time t in the form

$$\Psi_{WP}(x,t) = \left(\frac{m}{\pi\hbar}\right)^{1/4}\left(\frac{1}{\hat{u}+i\hat{z}}\right)^{1/2} \times \exp\left\{\frac{im}{2\hbar}\left[\frac{\dot{\hat{z}}}{\hat{z}}x^2 - \frac{\left(x - \frac{p_0\alpha_0}{m}\hat{z}\right)^2}{\hat{z}(\hat{u}+i\hat{z})}\right]\right\}. \tag{27}$$

The ansatz (26) for K_D is valid for the HO as well as for the free motion. In the latter case, one has only to set $\omega=0$ in the equations of motion for $\hat{z}$ and $\hat{u}$ and in any relation derived from them.

Since in the definition of $\Psi(x,t)$ according to Eq. (17), only K actually depends on x and t, the kernel K_D, as defined in Eq. (26), must also fulfill the time-dependent SE. Inserting K_D into the SE, one obtains terms proportional to powers of x and x'. From the terms proportional to x^2, it follows that $\hat{z}$ must fulfill the classical equation of motion for the system. The same also applies to $\hat{u}$; so, for the HO, one obtains

$$\ddot{\hat{z}} + \omega^2\hat{z} = 0 \quad \text{and} \quad \ddot{\hat{u}} + \omega^2\hat{u} = 0. \tag{28}$$

The equation of motion for $\hat{u}$ follows from the fact that $\hat{z}$ and $\hat{u}$ are not independent of each other but must fulfill a relation that is obtained from the terms proportional to x'^2. This relation has formal similarities to a conservation of angular momentum

$$\dot{\hat{z}}\hat{u} - \dot{\hat{u}}\hat{z} = 1. \tag{29}$$

As WP (27) must be identical to the WP solution (2) of the time-dependent SE, direct comparison shows that the following relations exist:

$$\hat{z} = \frac{m}{\alpha_0 p_0}\eta(t) \tag{30}$$

and

$$\frac{2\hbar}{m}y = \frac{\dot{\hat{z}}}{\hat{z}} - \frac{1}{\hat{z}\lambda} = \frac{\dot{\lambda}}{\lambda}, \tag{31}$$

where, in the latter case, the complex quantity

$$\lambda = \hat{u} + i\hat{z} \tag{32}$$

has been introduced and relation (29) has been used. Obviously, λ obeys the classical equation of motion. This can be confirmed by inserting the relation between y and λ into the Riccati equation (4), yielding $\ddot{\lambda} + \omega^2\lambda = 0$.

Expressing $\lambda(t)$ in polar coordinates as

$$\lambda = \alpha e^{i\varphi} \tag{33}$$

allows one to write the Cartesian components as

$$\hat{u} = \alpha\cos\varphi, \quad \hat{z} = \alpha\sin\varphi. \tag{34}$$

Inserting (33) into the rhs of Eq. (31) then yields

$$\frac{2\hbar}{m}y = \frac{\dot{\alpha}}{\alpha} + i\dot{\varphi}. \tag{35}$$

However, inserting Eq. (34) into the conservation law (29) leads to

$$\dot{\varphi} = \frac{1}{\alpha^2}. \tag{36}$$

Comparing now the imaginary part of (35), expressed by (36), with $(2\hbar y_I/m)$, as defined in Eq. (7), shows that the absolute value of λ is identical with α occurring in Eq. (7), which is defined in Eq. (8).

With the help of α, φ, and λ and their time derivatives, the quantum mechanical uncertainties can now be expressed not only via y_I and y_R but also in the following forms:

$$\langle\tilde{x}^2\rangle = \frac{1}{4y_I} = \frac{\hbar}{2m}\alpha^2 = \frac{\hbar}{2m}\lambda\lambda^*, \tag{37}$$

$$\langle \tilde{p}^2 \rangle = \hbar^2 \frac{y_R^2 + y_I^2}{2y_I} = \frac{\hbar m}{2}(\dot{\alpha}^2 + \alpha^2 \dot{\varphi}^2) = \frac{\hbar m}{2}(\dot{\lambda}\dot{\lambda}^*), \quad (38)$$

$$\langle [\tilde{x},\tilde{p}]_+ \rangle = \langle \tilde{x}\tilde{p} + \tilde{p}\tilde{x} \rangle = \hbar \left(\frac{y_R}{y_I} \right) = \hbar \, \dot{\alpha}\alpha = \frac{\hbar}{2}\frac{\partial}{\partial t}(\lambda\lambda^*), \quad (39)$$

and the quantum mechanical energy contribution can be written as

$$\tilde{E} = \frac{\langle \tilde{p}^2 \rangle}{2m} + \frac{m}{2}\omega^2 \langle \tilde{x}^2 \rangle = \frac{\hbar}{4}\{\dot{\alpha}^2 + \alpha^2\dot{\varphi}^2 + \omega^2\alpha^2\} = \frac{\hbar}{4}(\dot{\lambda}\dot{\lambda}^* + \omega^2 \lambda\lambda^*). \quad (40)$$

Using all this information, the WP solution (27) can finally be written in the form

$$\Psi_{WP}(x,t) = \left(\frac{m}{\pi\hbar} \right)^{1/4} \left(\frac{1}{\lambda} \right)^{1/2} \exp\left\{ -\frac{m}{2\hbar}\left(\frac{1}{\alpha^2} - i\frac{\dot{\alpha}}{\alpha} \right)\tilde{x}^2 + \frac{i}{\hbar}\langle p \rangle \tilde{x} + \frac{im}{2\hbar}\dot{\eta}\eta \right\}. \quad (41)$$

The explicit values for $\hat{u}$ and $\hat{z}$ can be determined for our systems of interest and turn out to be

(a) Free motion

$$\hat{u} = \alpha_0 = \text{const}, \quad \hat{z} = \frac{1}{\alpha_0}t, \quad (42)$$

(b) HO

$$\hat{u} = \alpha_0 \cos \omega t, \quad \hat{z} = \frac{1}{\alpha_0 \omega} \sin \omega t. \quad (43)$$

[These values are obtained for $\eta(0)=0$ and $p_0=p_{max}$; different initial conditions for the classical quantities only lead to more cumbersome calculations but do not change the essential results].

Inserting these results into the propagator (26) yields the same analytical expressions as function of time as the Feynman kernels (19) and (20); only in the form (26) of K_D does the initial position uncertainty α_0 occur explicitly, which is connected with the fact that $\hat{u}$ appears instead of $\dot{\eta}$. Later in this section, it will be shown that this not only has to be different by a constant factor, but can be more involved.

What are the consequences for the time-dependent WPs calculated with K_D instead of K_F? The quantities $\hat{u}$ and $\hat{z}$ contain explicitly the initial value α_0 that is identical with the initial value in $\Psi(x',0)$, see Eq. (21). Calculating $\alpha^2(t)$ for the free motion leads to the same results using K_F or K_D. However, calculating α^2 for the HO using K_D does not necessarily lead to the constant WP width according to $\alpha^2 = \frac{2m}{\hbar}\langle \hat{x}^2 \rangle = \alpha_0^2 = \frac{1}{\omega}$, but also to a possibly time-dependent width since

$$\alpha_{HO}^2 = \lambda\lambda^* = \hat{u}^2 + \hat{z}^2 = \alpha_0^2 \left[\cos^2 \omega t + \left(\frac{1}{\alpha_0^2 \omega} \sin \omega t \right)^2 \right] = \alpha_0^2 \left[\cos^2 \omega t + \left(\frac{\beta_0}{\omega} \sin \omega t \right)^2 \right] \neq \text{const} \quad \text{for } \beta_0 \neq \omega. \quad (44)$$

It can now be shown straightforwardly that this oscillating WP width leads to the correct spreading WP width of the free motion WP in the limit $\omega \to 0$, and not only to a plane wave, i.e.

$$\lim_{\omega \to 0} \alpha_{HO}^2(t) = \alpha_0^2[1 + (\beta_0 t)^2] = \alpha_{fr}^2(t). \quad (45)$$

A more explicit explanation for this fact will be given below.

As we mentioned before, the position uncertainty also enters explicitly the Ermakov invariant. Therefore, we now wish to show the connection between the time-dependent parameters $\hat{u}$ and $\hat{z}$, entering the propagator K_D, and this invariant.

The invariant (12) can be rewritten, in terms of $\hat{z} = \frac{m}{\alpha_0 p_0}\eta$, as

$$I = \frac{1}{2}\left(\frac{\alpha_0 p_0}{m} \right)^2 \left[(\dot{\hat{z}}\alpha - \hat{z}\dot{\alpha})^2 + \left(\frac{\hat{z}}{\alpha} \right)^2 \right] = \text{const}. \quad (46)$$

According to Eq. (34), we know that $\frac{\hat{z}}{\alpha} = \sin \varphi$, in order to obtain I=const, $(\dot{\hat{z}}\alpha - \hat{z}\dot{\alpha})^2$ must therefore, be equal to $\left(\frac{\hat{u}}{\alpha}\right)^2 = \cos^2 \varphi$.

So, up to a ± sign, we obtain

$$\hat{u} = \dot{\hat{z}}\alpha^2 - \hat{z}\dot{\alpha}\alpha = \alpha^2 \left(\dot{\hat{z}} - \frac{\dot{\alpha}}{\alpha}\hat{z} \right), \quad (47)$$

or

$$\frac{\hat{u}}{\hat{z}} = \alpha^2 \left(\frac{\dot{\hat{z}}}{\hat{z}} - \frac{\dot{\alpha}}{\alpha} \right), \quad (48)$$

or

$$\frac{\dot{\hat{z}}}{\hat{z}} = \frac{\dot{\eta}}{\eta} = \frac{1}{\alpha^2}\frac{\hat{u}}{\hat{z}} + \frac{\dot{\alpha}}{\alpha}. \quad (49)$$

Proof: Considering the time dependence of $\hat{z}$, and taking into account, that for a time-dependent WP width also α, and hence φ, can be time dependent, one obtains

$$\dot{\hat{z}} = \frac{d}{dt}\hat{z} = \frac{d}{dt}(\alpha \sin \varphi) = \frac{\dot{\alpha}}{\alpha}\hat{z} + \dot{\varphi}\alpha \cos \varphi = \frac{1}{\alpha}(\dot{\alpha}\hat{z} + \cos \varphi), \quad (50)$$

or $\dot{\hat{z}}\alpha - \hat{z}\dot{\alpha} = \cos \varphi = \frac{\hat{u}}{\alpha}$.

The expressions in I more resemble a form that reminds one of a Hamiltonian or Lagrangian, if time derivatives are replaced by derivation with respect to the angle $\varphi(t)$, occurring in $\lambda = \alpha e^{i\varphi}$.

Introducing a new variable $Y = \frac{\hat{z}}{\alpha} = \sin \varphi(t)$ with $\varphi(t) = \int^t \frac{1}{\alpha^2} dt'$ leads to $\hat{u} = \cos \varphi = \frac{d}{d\varphi}Y$, so

$$I=\frac{1}{2}\left(\frac{p_0\alpha_0}{m}\right)^2\left[\left(\frac{dY}{d\varphi}\right)^2+Y^2\right]=\text{const}. \quad (51)$$

This form allows for a comparison with another approach to obtain the time propagator, now based on the Ermakov system of differential equations, which will be presented in the next sub-section.

C. Method of Dhara and Lawande to obtain the time propagator

Dhara and Lawande [2] used an elegant method to derive the time propagator, making use of the Ermakov system. Since the Ermakov system holds also for time dependent ω, the same applies to their method. Starting point is again, as in Feynman's procedure, the classical Lagrangian [where their $x(t)$ equals our $\eta(t)$],

$$L_{cl}=\frac{m}{2}[\dot{\eta}^2-\omega^2(t)\eta^2]=\alpha_0^2\left(\frac{p_0^2}{2m}\right)[\dot{\hat{z}}^2-\omega^2(t)\hat{z}^2]. \quad (52)$$

From Eq. (10) they take

$$\omega^2(t)=\frac{1}{\alpha^4}-\frac{\ddot{\alpha}}{\alpha}=\frac{1}{\alpha^4}-\left(\frac{\dot{\alpha}}{\alpha}\right)^2-\frac{d}{dt}\left(\frac{\dot{\alpha}}{\alpha}\right) \quad (53)$$

to replace $\omega^2(t)$ in Eq. (52), thus introducing explicitly the quantum mechanical uncertainty property, expressed by α and $\dot{\alpha}$, into the method.

After some basic rearrangements, the classical Lagrangian can be expressed in terms of the classical variables [where $\hat{z}$ is proportional $\eta(t),\hat{z}=\frac{m}{\alpha_0 p_0}\eta$] and the quantum variables (α and $\dot{\alpha}$) as

$$L_{cl}=\frac{(\alpha_0p_0)^2}{2m}\left\{\frac{d}{dt}\left(\frac{\dot{\alpha}}{\alpha}\hat{z}^2\right)+\alpha^2\left[\frac{d}{dt}\left(\frac{\hat{z}}{\alpha}\right)\right]^2-\frac{1}{\alpha^2}\left(\frac{\hat{z}}{\alpha}\right)^2\right\}$$
$$=L_\alpha+L_\varphi, \quad (54)$$

with

$$L_\alpha=\frac{m}{2}\left(\frac{\alpha_0p_0}{m}\right)^2\frac{d}{dt}\left(\frac{\dot{\alpha}}{\alpha}\hat{z}^2\right), \quad (55)$$

and

$$L_\varphi=\frac{m}{2}\left(\frac{\alpha_0p_0}{m}\right)^2\left\{\alpha^2\left[\frac{d}{dt}\left(\frac{\hat{z}}{\alpha}\right)\right]^2-\frac{1}{\alpha^2}\left(\frac{\hat{z}}{\alpha}\right)^2\right\}. \quad (56)$$

The α-dependent part can be integrated directly to yield

$$\frac{i}{\hbar}\int_0^t L_\alpha dt'=\frac{im}{2\hbar}\left[\frac{\dot{\alpha}(t)}{\alpha(t)}\hat{z}^2-\left(\frac{\dot{\alpha}}{\alpha}\right)_0\hat{z}'^2(0)\right]\left(\frac{\alpha_0p_0}{m}\right)^2=\frac{i}{\hbar}S_\alpha, \quad (57)$$

where $\left(\frac{\dot{\alpha}}{\alpha}\right)_0$, as well as $\hat{z}(0)$, vanish in the cases considered in this paper.

For the φ-dependent part, they introduce new variables that correspond in our notation to $Y=\frac{\hat{z}}{\alpha}=\sin\varphi$ and $\varphi=\int_0^t\frac{1}{\alpha^2(t')}dt'$ with $\frac{d\varphi}{dt}=\dot{\varphi}=\frac{1}{\alpha^2}$ and hence $d\varphi=\frac{1}{\alpha^2}dt$ or $\frac{d}{dt}=\frac{1}{\alpha^2}\frac{d}{d\varphi}$.

So, L_φ takes the form

$$L_\varphi=\left(\frac{\alpha_0p_0}{m}\right)^2\frac{m}{2}\alpha^2\left\{\left(\frac{d}{dt}Y\right)^2-\left(\frac{1}{\alpha^2}\right)^2Y^2\right\}$$
$$=\left(\frac{\alpha_0p_0}{m}\right)^2\frac{m}{2}\frac{1}{\alpha^2}\left\{\left(\frac{d}{d\varphi}Y\right)^2-Y^2\right\}. \quad (58)$$

From Eq. (58) follows that

$$\int_{t_0}^t L_\varphi dt'=\int_{\varphi_0(t_0)}^{\varphi(t)}\hat{L}_\varphi d\varphi, \quad (59)$$

with

$$\hat{L}_\varphi=\alpha^2L_\varphi[Y(\varphi)], \quad (60)$$

or

$$\hat{L}_\varphi=\left(\frac{\alpha_0p_0}{m}\right)^2\frac{m}{2}\left\{\left(\frac{dY}{d\varphi}\right)^2-Y^2\right\}. \quad (61)$$

This looks very much like a Lagrangian that corresponds to a Hamiltonian of the form (51)—written in velocities instead of momenta—that represents the Ermakov invariant.

On the other hand, this Lagrangian also looks like the classical Lagrangian for the HO, only t is replaced by φ and $\omega=1$.

So, what remains to be calculated is

$$\int_{t_0}^t L_\varphi dt'=\int_{\varphi_0}^{\varphi}\left(\frac{\alpha_0p_0}{m}\right)^2\frac{m}{2}\left[\left(\frac{dY(\varphi')}{d\varphi'}\right)^2-Y^2(\varphi')\right]d\varphi', \quad (62)$$

where $\frac{d}{d\varphi'}Y(\varphi')=\cos\varphi'$ and $Y(\varphi')=\sin\varphi'$.

These calculations have been performed [2,3] and lead to a similar result as in Feynman's case, only $\dot{\eta}$ must be replaced by $\hat{u}$, and instead of x and x', the quantities $\frac{x}{\alpha}$ and $\frac{x'}{\alpha_0}$ occur, which corresponds to $\frac{x''}{\rho''}$ and $\frac{x'}{\rho'}$ in the notation of Dhara and Lawande.

Therefore, their propagator, written in our notation, would read

$$K_{DL}(x,x',t,t'=0)=\left(\frac{1}{2\pi i\hbar\,\alpha_0\hat{z}}\right)^{1/2}\exp\left\{\frac{im}{2\hbar}\frac{\dot{\alpha}}{\alpha}x^2+\frac{im}{2\hbar}\right.$$
$$\left.\times\left[\frac{\hat{u}}{\hat{z}}\left(\frac{x}{\alpha}\right)^2-2\frac{x}{\hat{z}}\left(\frac{x'}{\alpha_0}\right)+\frac{\hat{u}}{\hat{z}}\left(\frac{x'}{\alpha_0}\right)^2\right]\right\}, \quad (63)$$

(with the subscript DL standing for Dhara and Lawande, the authors of [2]) where their $\frac{dY}{d\varphi}$ corresponds to $\frac{d\hat{u}}{dt}$, instead of $\frac{d\eta}{dt}$ in Feynman's case, and the integration with respect to the angle φ instead of time t provides a factor $\frac{1}{\alpha^2}$ (for upper integration limit), or $\frac{1}{\alpha_0^2}$ (for the lower one). Considering Eq. (49) shows that the two terms in the exponent that are proportional to x^2 can be combined to $\frac{\dot{\hat{z}}}{\hat{z}}$, so K_{DL} is *identical* to K_D in Eq. (26), only that in the form (63) the temporal change of $\alpha(t)$ is explicitly pronounced.

This is due to the fact that this method uses the Ermakov system, where $\alpha(t)$ explicitly appears.

D. Influence of the initial uncertainty on the time evolution of the quantum system

We have shown that the time propagator, which has the same analytical dependence on time in all cases considered, can be expressed in different variables, where Feynman's procedure is totally based on the classical variables, whereas the methods that are connected with the Ermakov invariant also take into account the quantum uncertainties, at least at the initial time.

The comparison of the methods is in agreement with Eq. (49). This relation also holds for constant α (e.g., the HO with ground state as initial state) where $\dot{\alpha}=0$ and $\alpha=\alpha_0$. Therefore, relation (49) can be extended to

$$\frac{\dot{\hat{z}}}{\hat{z}} = \frac{1}{\alpha^2(t)}\frac{\hat{u}}{\hat{z}} + \frac{\dot{\alpha}}{\alpha} = \frac{1}{\alpha_0^2}\frac{\hat{u}}{\hat{z}}. \tag{64}$$

This shows that in the case when α is time dependent, not only α_0^2 has to be replaced by $\alpha^2(t)$, but also an additional term $\frac{\dot{\alpha}}{\alpha}$ has to be taken into account. Eq. (64) also allows for the comparison

$$\frac{\dot{\hat{z}}}{\hat{z}} = \frac{\dot{\eta}}{\eta} = \frac{1}{\alpha_0^2}\frac{\hat{u}}{\hat{z}} \quad \text{or} \quad \hat{u} = \alpha_0^2\dot{\hat{z}}, \tag{65}$$

which is in agreement with the fact that $K_D \equiv K_F$, if $\frac{\hat{u}}{\alpha_0^2}$ is replaced by $\dot{\hat{z}}$ in K_D.

Inserting (65) into the expression for $\alpha^2(t)$, i.e.,

$$\alpha^2(t) = \hat{u}^2 + \hat{z}^2 = \left(\frac{m}{\alpha_0 p_0}\right)^2(\alpha_0^4\dot{\eta}^2 + \eta^2) = \alpha_0^2\left[\frac{\dot{\eta}^2}{v_0^2} + \frac{1}{\alpha_0^4}\left(\frac{\eta}{v_0}\right)^2\right],$$

leads to

$$\alpha^2(t) = \frac{\alpha_0^2}{v_0^2}\left[\dot{\eta}^2 + \beta_0^2\eta^2\right] = \frac{2m}{\hbar}\langle\tilde{x}^2\rangle. \tag{66}$$

This shows that the *quantum mechanical uncertainty* of position (at any time t) can be *expressed solely* in terms of the *classical trajectory* $\eta(t)$ and the *corresponding velocity* $\dot{\eta}(t)$, if the *initial velocity* v_0 *and* the *initial position uncertainty*, expressed by α_0, or $\beta_0 = \frac{1}{\alpha_0^2}$, are known. Note that in our case, the initial position was chosen to be zero, but a different choice would not change the essential result. So, even for a complete knowledge of the classical situation, the quantum mechanical aspect of initial (even minimum) uncertainty would not allow a deterministic description that is more accurate than the initial uncertainty. To the contrary, the uncertainty can even become larger due to a possible time dependence.

This explains why Feynman's procedure, based only on the classical Lagrangian, provides the correct time evolution of the system since the time dependence enters only via the classical variables. On the other hand, the importance of the initial position uncertainty should not be underestimated. In Feynman's procedure it does not occur explicitly, therefore, the use of the ground state wave function of the HO as initial function does not provide the complete solution of the time-dependent problem. The influence of the initial uncertainty becomes clear if one inserts the expressions for $\eta(t)$ and $\dot{\eta}(t)$ into Eq. (66).

(a) For the free motion, one obtains with $\eta = v_0 t$, $\dot{\eta} = v_0$

$$\alpha^2 = \alpha_0^2[1 + (\beta t)^2]; \tag{67}$$

(b) for the HO with $\eta = \frac{v_0}{\omega}\sin\omega t$ and $\dot{\eta} = v_0\cos\omega t$, Eq. (66) yields

$$\alpha^2 = \alpha_0^2\left\{\cos^2\omega t + \left(\frac{\beta_0}{\omega}\sin\omega t\right)^2\right\} = \alpha_0^2\left\{1 + \left[\left(\frac{\beta_0^2 - \omega^2}{\omega^2}\right)\sin^2\omega t\right]\right\}. \tag{68}$$

(Note the formal similarity with transition rates connected with Fermi's Golden Rule).

This shows, obviously, that for the case where $\beta_0 = \frac{\hbar}{2m\langle\tilde{x}^2\rangle_0}$ is not identical with the ground state value $\frac{\hbar}{2m\langle\tilde{x}^2\rangle_{GS}} = \omega$, the WP width is oscillating.

That the time dependence of α depends on the difference between the initial state's position uncertainty and its ground state uncertainty, connected with a parameter occurring in the potential of the system, becomes even more obvious, regarding the time derivative of α^2

$$\dot{\alpha}\alpha = \frac{\alpha_0^2}{v_0^2}\dot{\eta}[\beta_0^2\eta + \ddot{\eta}] = \frac{1}{\hbar}\langle[\tilde{x},\tilde{p}]_+\rangle. \tag{69}$$

For $V=0$ follows $\ddot{\eta}=0$ and only the initial uncertainty, entering via β_0, matters. For the HO with $\ddot{\eta} = -\omega^2\eta$, it follows

$$\dot{\alpha}\alpha = \frac{\alpha_0^2}{v_0^2}\eta\dot{\eta}[\beta_0^2 - \omega^2]. \tag{70}$$

This obviously vanishes if $\beta_0 = \omega$, which corresponds to $\langle\tilde{x}^2\rangle_0 = \langle\tilde{x}^2\rangle_{GS}$. In all other cases, even for constant ω, α is time dependent. Obviously, for $\omega = \omega(t)$ also a time dependent α follows, since β_0 is constant. In the case of the HO with constant frequency $\omega = \omega_0$, the WP width oscillates with the frequency $2\omega_0$ and an amplitude that depends on the absolute value of the difference between β_0^2 and ω^2.

Finally, it should be pointed out that for $\dot{\alpha} \neq 0$, and hence $\langle[\tilde{x},\tilde{p}]_+\rangle \neq 0$, the term $\frac{\partial}{\partial x}v$ in the CE is also not vanishing, which gives rise to the above-mentioned difference between the classical and the quantum mechanical case.

IV. CONNECTION BETWEEN THE ERMAKOV INVARIANT AND THE WIGNER FUNCTION

Another way of describing quantum systems that comes closest to the classical phase space description, is with the Wigner function. Despite its well-known problematic aspects [11], it is often a useful tool, and is particularly applied recently in studies concerning the transition from quantum to classical physics.

In order to keep the paper self-contained, the first part of this section will present a short overview of the connection between the Wigner function and the Ermakov invariant that has been communicated by one of the authors recently [12]. In the second part, the particular consequences of time-dependent uncertainties for the description of the dynamics based on the Wigner function will be analyzed.

To obtain the Wigner function corresponding to our WP solution (2) of the time-dependent SE, one has to perform the transformation [11]

$$W(x,p,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dq e^{ipq/\hbar}\Psi_{WP}^{*}\left(x+\frac{q}{2},t\right)\Psi_{WP}\left(x-\frac{q}{2}t\right). \tag{71}$$

Using our notation with $\langle x\rangle = \eta$, $\tilde{x}=x-\eta$, $\tilde{p}=p-m\dot{\eta}$, and $y(t)=y_R+iy_I$, the Wigner function can be expressed as

$$W(x,p,t) = \frac{1}{\pi\hbar}\exp\left\{-2\left(\frac{y_I^2+y_R^2}{y_I}\right)\tilde{x}^2 - \frac{\tilde{p}^2}{2\hbar^2 y_I} + \frac{2}{\hbar}\left(\frac{y_R}{y_I}\right)\tilde{x}\tilde{p}\right\}. \tag{72}$$

In order to elucidate the physical meaning of this expression, it is helpful to rewrite y_I and y_R in terms of the position and momentum uncertainties, according to Eqs. (37)–(39). Using these relations, $W(x,p,t)$ can be expressed as

$$W(x,p,t) = \frac{1}{\pi\hbar}\exp\left\{-\frac{2}{\hbar^2}\left[\langle\tilde{p}^2\rangle\tilde{x}^2 - \langle[\tilde{x},\tilde{p}]_+\rangle\tilde{x}\tilde{p} + \langle\tilde{x}^2\rangle\tilde{p}^2\right]\right\}. \tag{73}$$

The connection with the Ermakov invariant is found easily, providing the uncertainties are now expressed in terms of α and $\dot{\alpha}$ instead of y_I and y_R, again according to Eqs. (37)–(39).

The Wigner function then reads

$$W(x,p,t) = \frac{1}{\pi\hbar}\exp\left\{-\frac{m}{\hbar}\left[\left(\dot{\alpha}^2+\frac{1}{\alpha^2}\right)\tilde{x}^2 - 2\alpha\dot{\alpha}\frac{\tilde{p}}{m}\tilde{x} + \alpha^2\frac{\tilde{p}^2}{m^2}\right]\right\}. \tag{74}$$

The term in square brackets can be rewritten to yield $W(x,p,t)$ as

$$W(x,p,t) = \frac{1}{\pi\hbar}\exp\left\{-\frac{m}{\hbar}\left[\left(\dot{\alpha}\tilde{x}-\alpha\frac{\tilde{p}}{m}\right)^2 + \left(\frac{\tilde{x}}{\alpha}\right)^2\right]\right\}. \tag{75}$$

The form of the exponent of $W(x,p,t)$ already looks very similar to the Ermakov invariant and, particularly at the origin of the phase space, i.e., for $x=0$ an $p=0$, the square bracket in the exponent of Eq. (75) is, up to a constant factor, identical with the Ermakov invariant, i.e.

$$W(0,0,t) = \frac{1}{\pi\hbar}\exp\left\{-\frac{m}{\hbar}\left[(\dot{\eta}\alpha-\eta\dot{\alpha})^2 + \left(\frac{\eta}{\alpha}\right)^2\right]\right\} = \frac{1}{\pi\hbar}\exp\left\{-\frac{2m}{\hbar}I\right\} = \text{const}. \tag{76}$$

Since the time dependence of position and momentum enters the exponent of $W(x,p,t)$ only via $\langle x\rangle = \eta(t)$ in $\tilde{x}$ or $\langle p\rangle = m\dot{\eta}$ in $\tilde{p}$ (and $\langle x\rangle$ and $\langle p\rangle$ occur only quadratic or bilinear, so the negative sign in $\tilde{x}$ and $\tilde{p}$ does not matter), also the exponent of $W(x,p,t)$ is a temporal invariant. This agrees with the facts (i) that the normalization factor of $W(x,p,t)$, $\frac{1}{\pi\hbar}$, is time independent and (ii) that for quadratic potentials, $W(x,p,t)=W_0(x_0(x,p,t),p_0(x,p,t))$, where $W_0(x,p)$ is the Wigner function at time $t=0$, and $(x_0(x,p,t),p_0(x,p,t))$ is the phase space point at which a classical particle would have to start at time $t=0$ in order to reach the point (x,p) at time t (see [13]).

Inserting the Wigner function (73) into the corresponding equation of motion

$$\frac{\partial}{\partial t}W(x,p,t) = -\frac{p}{m}\frac{\partial W}{\partial x} + \frac{\partial V}{\partial x}\frac{\partial W}{\partial p}, \tag{77}$$

with $V=0$ or $V=\frac{m}{2}\omega^2 x^2$, leads to terms that are quadratic, bilinear, or linear in the shifted variables $\tilde{x}$ and $\tilde{p}$, or independent of them. The time dependence of the position uncertainty, entering $\langle[\tilde{x},\tilde{p}]_+\rangle = \hbar\dot{\alpha}\alpha$, only affects the terms that are quadratic or bilinear in the variables. Since the equations of motion for the classical trajectory and velocity (or momentum) are only connected with the terms linear in the variables, they are not affected by the value of $\langle[\tilde{x},\tilde{p}]_+\rangle$. However, the time dependence of position—and momentum—uncertainty strongly depends on this term.

So, ignoring the time dependence of α, or taking $\langle[\tilde{x},\tilde{p}]_+\rangle=0$, respectively, would lead to the following results that contradict the well-known established ones:

(a) Free motion

(i) $\frac{\langle\tilde{p}^2\rangle}{2m}=0$ instead of being constant $\neq 0$;

(ii) $\frac{\partial}{\partial t}\langle\tilde{x}^2\rangle=0$, i.e., $\langle\tilde{x}^2\rangle=$ const instead of having a time dependence describing the spreading of the WP.

(b) HO:

(i) $\frac{\langle\tilde{p}^2\rangle}{2m}=\frac{m}{2}\omega^2\langle\tilde{x}^2\rangle$, which is only valid for WPs with constant width, but as we have seen, there exist also WPs with oscillating width;

(ii) $\frac{\partial}{\partial t}\langle\tilde{x}^2\rangle=0$, which is not valid in the general case, as discussed above;

(iii) $\frac{\partial}{\partial t}\langle\tilde{p}^2\rangle=0$, here the same applies as stated above for $\frac{\partial}{\partial t}\langle\tilde{x}^2\rangle=0$.

Therefore, the time dependence of α is crucial in order to obtain the proper dynamical behavior of the quantum mechanical aspect of the system.

V. CONCLUSIONS AND PERSPECTIVES

One major result of our investigation is that the Ermakov invariant is the central quantity that connects different forms for the description of the dynamics of quantum systems, such as the time-dependent SE, the time propagator, or Feynman kernel, respectively, and the time-dependent Wigner function.

Unlike the classical Hamiltonian or Lagrangian, this invariant not only depends on the classical variables such as position and momentum or velocity (and their initial conditions), but also on the quantum uncertainties contained in α and $\dot{\alpha}$. Therefore, the initial conditions of these quantities are also somehow taken into account in a description using the variables $\hat{u}$ and $\hat{z}$ or $\alpha=\sqrt{\hat{u}^2+\hat{z}^2}$, occurring in I. So, time-dependent phenomena such as the possible time dependence of the WP width become directly obvious, whereas they are somewhat hidden (although present in principle) in the usual description according to Feynman, based only on the classical variables.

The importance of the initial value of the uncertainty has been demonstrated [so far, in particular of the position uncertainty or α_0, respectively, since in the cases considered $\left(\frac{\dot{\alpha}}{\alpha}\right)_0=0$ is valid, that might be different for different systems], since it determines whether α is time dependent or not. For $\dot{\alpha}\neq 0$, a term $\frac{\partial}{\partial x}v\neq 0$ occurs in the CE (13) for the quantum mechanical probability density that is not present in the classical description of the same Hamiltonian system. This term is related to a nonclassical contribution to the probability current $j=\rho v$, since v is given by $v=\dot{\eta}+\frac{\dot{\alpha}}{\alpha}\tilde{x}$, and therefore related to transport phenomena such as conduction or resistance properties of the system. The effect of an oscillating $\alpha(t)$ might also be measurable. Since a similar effect also occurs for the motion of a quantum system in a magnetic field (even with dissipation), one of us proposed a possible experiment in an earlier work [14].

In particular, it is interesting that the time-dependent effect can arise when the system is initially in a state that does not match a parameter occurring in the potential, since $\dot{\alpha}$ is, according to Eq. (69), proportional to the term $\beta_0\eta+\ddot{\eta}=\beta^2\eta-\frac{\partial}{\partial\eta}V(\eta)$.

So, for the free motion, no influence of any potential is present, which leads to the same WP with the time-dependent spreading width using the Feynman kernel or the methods based on the Ermakov invariant. For the HO, however, using the latter methods, it immediately becomes obvious that $\dot{\alpha}\neq 0$ if the system is initially not in the ground state, but, e.g., in an excited state.

It also becomes immediately clear from Eq. (70), that for a system with time dependent $\omega(t)$, such as for example single atoms or ions caught in a Paul trap [15], $\dot{\alpha}\neq 0$ holds.

In the case of the Wigner function, the occurrence of $\dot{\alpha}\neq 0$ or $\dot{\alpha}\alpha=\frac{1}{\hbar}\langle[\tilde{x},\tilde{p}]_+\rangle\neq 0$, respectively, has no effect on the classical aspect, i.e., the equation of motion of the classical trajectory, but it has an important influence on the dynamics of the position and momentum uncertainties. The possibility of $\dot{\alpha}\neq 0$, and hence $\frac{\partial}{\partial x}v\neq 0$, shows, that the coupling of position and momentum, expressed by $\langle\tilde{x}\tilde{p}+\tilde{p}\tilde{x}\rangle\neq 0$, is obviously a phenomenon, that is not present in the classical phase space description.

Another major result is the fact that, according to Eq. (69), the *time evolution* of a typical *quantum mechanical* property such as the position uncertainty or $\alpha^2(t)$, respectively, can be *totally described*, if one only knows the *classical* trajectory $\eta(t)$ and the *classical* velocity $\dot{\eta}(t)$ (including their initial conditions) plus the *initial* position uncertainty, expressed by α_0 or $\beta_0=\frac{1}{\alpha_0^2}$ but without the knowledge of any dynamical quantum degree of freedom. In other words, the dynamics of the quantum system can be described totally in terms of the classical degrees of freedom, if only the initial uncertainty of the position measurement is given. This is a very surprising result, since it traces quantum dynamics entirely back to the classical one plus the existence of an uncertainty principle. However, this is, so far, demonstrated only for the model system treated in this work. It seems to be an interesting question if this property is valid for all dynamical quantum systems, or, at least, for which other systems apart from the ones considered above. This question is, however, beyond the scope of this work and will be discussed elsewhere. It shall only be mentioned that an extension to include polynomial Hamiltonians might be possible based on the work of Sarlet [16], who showed that in this case it is sometimes possible to reduce these Hamiltonians to a quadratic form by carrying out canonical transformations.

Finally, it should be mentioned that further developments to include dissipative effects should be possible. A way of reaching this goal might be the use of a logarithmic nonlinear SE for the effective description of dissipative systems with irreversible dynamics [17] since, particularly for this equation, also an exact dynamical invariant of Ermakov type exists [18]. Work in this direction is in progress.

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