Geometric phase for open quantum systems and stochastic unravelings

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We analyze the geometric phase for an open quantum system when computed by resorting to a stochastic unraveling of the reduced density matrix (quantum jump approach or stochastic Schrödinger equations). We show that the resulting phase strongly depends on the type of unraveling used for the calculations: as such, this phase is *not* a geometric object since it depends on nonphysical parameters, which are not related to the path followed by the density matrix during the evolution of the system.

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I. INTRODUCTION

The geometric phase is a property of a physical system that depends only on the path the system follows during its evolution, not on the details of the dynamics. Since the work of Berry [1], geometric phases have acquired a primary role in our understanding of many physical phenomena [2], and have been subject to several experimental verifications [3]. The original idea, framed within the context of adiabatic and cyclic evolutions of isolated systems, has been generalized in various directions [4–8]; of particular importance are all those efforts [5–9] aiming at defining a geometric phase for open quantum systems, which are not mathematically described by a pure state $|\psi_t\rangle$ but in terms of a reduced density matrix ρ_t undergoing a nonunitary evolution. Such proposals, besides being interesting on their own, could be important for possible applications, e.g., to quantum computation [10].

The mathematical framework within which the geometric phase for an open quantum system is defined is the following:

(i) One assumes that the effect of the environment on the quantum system is such that, under suitable approximations, the system can be *effectively* treated as an isolated system undergoing a nonunitary type of *linear*[1] evolution,

$$\Sigma_t:\rho_0 \to \Sigma_t[\rho_0] \equiv \rho_t, \tag{1}$$

which takes into account both the internal dynamics of the system and its interaction with the environment. Under reasonable assumptions [12], which have nevertheless been questioned in the literature (see, e.g., [13]), the map Σ_t can be taken of the quantum-dynamical-semigroup type, generated by the following class of equations:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] - \frac{\lambda^2}{2}\sum_{n=1}^{N}\{L_n^\dagger L_n\rho + \rho L_n^\dagger L_n - 2L_n\rho L_n^\dagger\}; \tag{2}$$

the self-adjoint operator H is usually identified with the standard Hamiltonian of the system, while the operators L_n, together with the positive constant λ, summarize the effect of the environment on the system.

A general consequence of this type of approach is that a pure state $|\psi_t\rangle$ is usually mapped into a statistical mixture ρ_t, so the problem arises of how to identify a geometric phase for the evolution of a density matrix ρ_t.

(ii) A common strategy that is used in the literature for associating a geometric phase to the evolution of ρ_t is to formally map the density matrix into a statistical mixture of pure states $|\psi_t^n\rangle$, each of which is weighted with a probability $p_n(t)$,

$$\rho_t \to \{(|\psi_t^n\rangle, p_n(t))\}: \quad \rho_t = \sum_n p_n(t)|\psi_t^n\rangle\langle\psi_t^n|; \tag{3}$$

one can then use the standard definition of geometric phase for a pure state,

$$\gamma_t^{\text{geo}} = \gamma_t^{\text{tot}} - \gamma_t^{\text{dyn}} = \arg\langle\psi_0|\psi_t\rangle - \text{Im}\int_0^t \langle\psi_t|d|\psi_t\rangle, \tag{4}$$

to associate a geometric phase also to ρ_t; this strategy has given fruitful results in the case of mixed states undergoing a unitary evolution [6], while the case of nonunitary evolutions, in particular those associated to open quantum systems, is still under debate. In this second case, a tentative definition of geometric phase has been given via *state purification* [7] and the *quantum-jump approach* [9].

A well-known characteristic property of relation (3) is that the association between a density matrix ρ_t and an ensemble $\{(|\psi_t^n\rangle, p_n(t))\}$ is not one-to-one, but one-to-many [14]: in general, there are different ensembles, containing different vectors and different probabilities, which give rise to the same density matrix; moreover, such vectors and the corresponding probabilities evolve in completely different ways. In the state purification approach, this property of density

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[1]As discussed in Ref. [11], the evolution equation for the statistical operator must be linear, otherwise it can give origin to superluminal effects.

matrices is related to the fact that there are different Kraus representations of the dynamical evolution of ρ_t [15]. In the quantum-jump approach, the same property reflects the fact that there are different stochastic unravelings which end up reproducing the same statistical operator [16].

In this paper, we analyze the consequences of such a feature of density matrices for the definition of geometric phases, within the context of the stochastic unraveling formalism, to which the quantum-jump approach belongs. We will show that the approach followed in [9] to identify a geometric phase strongly depends on the type of unraveling of ρ_t: as a consequence, such a phase is not a geometric object, because it depends also on non-physical parameters that are not related to the path followed by the density matrix during its evolution.

II. STOCHASTIC UNRAVELINGS

A stochastic unraveling of a linear evolution Σ_t for a density matrix ρ_t is defined as follows. Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let us consider a stochastic evolution for state vectors,

$$T_t(\omega): |\psi_0\rangle \to |\psi_t(\omega)\rangle, \tag{5}$$

which, for each different sample element $\omega \in \Omega$, associates a different state vector $|\psi_t(\omega)\rangle$ to the same initial state $|\psi_0\rangle$. One can then define the density matrix,

$$\tilde{\rho}_t \equiv \sum_n p_n \mathbb{E}_{\mathbb{P}}[|\psi_t^n(\omega)\rangle\langle\psi_t^n(\omega)|], \tag{6}$$

where the symbol $\mathbb{E}_{\mathbb{P}}$ denotes the average value with respect to the probability measure $\mathbb{P}$; here we have assumed that the initial state of the system is represented by a statistical mixture $\{(|\psi_0^n\rangle, p_n)\}$, so an extra sum over the possible initial states appears in the definition of $\tilde{\rho}_t$. The above relation defines a map,

$$\tilde{\Sigma}_t: \rho_0 \to \tilde{\Sigma}_t[\rho_0] \equiv \tilde{\rho}_t, \tag{7}$$

where $\tilde{\rho}_t$ is given by Eq. (6). Now, if the map $\tilde{\Sigma}_t$ defined by Eqs. (7) and (6) coincides with the linear map Σ_t, we say that $T_t(\omega)$ is a *stochastic unraveling* of Σ_t.

Among the other things, the above definition of stochastic unraveling implies that, when computing observable quantities of a system that evolves according to the map Σ_t, e.g., the expectation value of a self-adjoint operator, $\langle O\rangle_t = \mathrm{Tr}[O\rho_t]$, one can start with a stochastic unraveling (5) of Σ_t, then compute the quantum expectation $\langle\psi_t(\omega)|O|\psi_t(\omega)\rangle$, and finally average over the noise; this sequence of operations is legitimate since, by the definition of stochastic unraveling, one trivially has

$$\mathbb{E}_{\mathbb{P}}[\langle\psi_t|O|\psi_t\rangle] = \mathrm{Tr}\{O\Sigma_t[|\psi_0\rangle\langle\psi_0|]\}; \tag{8}$$

of course, if the initial state is a mixed state, an extra sum over the possible initial states, weighted with the corresponding probability distribution, has to be added at the left-hand side of Eq. (8).

In the literature, two types of stochastic unravelings have been proposed, one discrete and one continuous. The first one is the *quantum jump approach* [16] which has been used in Ref. [9] to associate a geometric phase to the evolution of an open quantum system; the second one is given in terms of *stochastic Schrödinger equations* [17]. The two approaches are similar and, in the following, we will resort to the second one, because it is more elegant from the mathematical point of view and easy to handle. We now briefly review it.

The idea is simple: the stochastic evolution $T_t(\omega)$ is assumed to be generated by a stochastic Schrödinger equation, whose typical structure is the following [17–19]:

$$d|\psi_t\rangle = \left[-\frac{i}{\hbar}Hdt + \lambda\sum_{n=1}^{N}(L_n - r_{n,t})dW_t^n - \frac{\lambda^2}{2}\sum_{n=1}^{N}(L_n^\dagger L_n - 2L_n r_{n,t} + r_{n,t}^2)dt \right]|\psi_t\rangle, \tag{9}$$

where

$$r_{n,t} = \frac{1}{2}\langle\psi_t|[L_n^\dagger + L_n]|\psi_t\rangle, \tag{10}$$

with H and L_n defined as in Eq. (2); W_t^n $(n=1,\ldots,N)$ are N independent standard Wiener processes with respect to the measure $\mathbb{P}$, which make Eq. (9) a stochastic differential equation.

Such kinds of equations have been used in several contexts: within the *theory of quantum measurement*, to describe the effects of a repeated measurement on the evolution of a quantum system [20]; within *collapse models*, to provide a solution to the measurement problem [21]; and within the theory of an *open quantum system*, as a mathematical tool to efficiently simulate the evolution of an open system [22].

One of the fundamental properties [17] of Eq. (9) is that the density matrix $\rho_t \equiv \mathbb{E}_{\mathbb{P}}[|\psi_t\rangle\langle\psi_t|]$ solves Eq. (2), i.e., Eq. (9) represents a stochastic unraveling of the Lindblad-type equation (2).

Note that Eq. (9) is nonlinear, but it preserves the norm of the state vector. There is a well-known way [17] to linearize the equation, at the price of relinquishing the normalization condition; consider the following stochastic differential equation:

$$d|\phi_t\rangle = \left[-\frac{i}{\hbar}Hdt + \lambda\sum_{n=1}^{N}L_n d\xi_t^n - \frac{\lambda^2}{2}\sum_{n=1}^{N}L_n^\dagger L_n dt \right]|\phi_t\rangle; \tag{11}$$

the stochastic processes ξ_t^n are standard Wiener processes with respect to a new probability measure $\mathbb{Q}$, whose relation to $\mathbb{P}$ will soon be established.

The connection between the linear Eq. (11) and the nonlinear Eq. (9) is the following; given the solution $|\phi_t\rangle$ of Eq. (11) for an initial condition $|\phi_0\rangle$, if one performs the following two operations:

(i) normalize the solution: $|\phi_t\rangle \to |\psi_t\rangle = |\phi_t\rangle / \||\phi_t\rangle\|$,

(ii) make the substitution,

$$\xi_t^n \to W_t^n = \xi_t^n - 2\lambda \int_0^t r_{n,t} ds, \tag{12}$$

then the wave function $|\psi_t\rangle$ so defined is the solution of Eq. (17) for the same initial condition $|\psi_0\rangle = |\phi_0\rangle$. Moreover, one can further show that the two probability measures $\mathbb{P}$ and $\mathbb{Q}$ are related as follows [17]:

$$\mathbb{E}_{\mathbb{P}}[X_t] \equiv \mathbb{E}_{\mathbb{Q}}[\langle \phi_t | \phi_t \rangle X_t], \tag{13}$$

where X_t is a stochastic process.

A. Equivalent stochastic unravelings

Two stochastic unravelings $T_t^{(1)}$ and $T_t^{(2)}$ are said to be *equivalent* if they unravel the same evolution Σ_t. A very remarkable property is that there are *infinite* different but equivalent stochastic unravelings for practically all physically interesting Σ_t; within the quantum jump approach, this issue is addressed, e.g., in [16]; within the stochastic Schrödinger formalism, such a feature is less known, still very easy to show. As a matter of fact, suppose in Eq. (9) we change the Lindblad operators L_n as follows:

$$L_n \to c_n L_n, \tag{14}$$

where $c_n = e^{i\varphi_n}$ are arbitrary phase factors; clearly, Eq. (2) does not change, while Eq. (9) does change, since terms appear that are not proportional to $L_n^\dagger L_n$. Such a change is not as trivial as it may seem: as we shall see, stochastic equations with different values of φ_n entail completely different evolutions for the state vector. Of course, there are other possible unravelings of Eq. (2) besides those which can be obtained by a phase shift in the Lindblad operators, but for simplicity we consider here only these, since they are sufficient for the subsequent analysis.

III. GEOMETRIC PHASE AND STOCHASTIC UNRAVELINGS: AN EXAMPLE

As already remarked, in Ref. [9] the stochastic-unraveling approach has been used to associate a geometric phase to an open system; in this section, we discuss how the existence of different equivalent stochastic unravelings affects the computation of the geometric phase. As an example, we now calculate the geometric phase associated with the evolution of a spin particle in a constant magnetic field directed along the z axis of a chosen reference frame, while the spin is subject to dephasing. The quantum Hamiltonian is $H = -\mu B \sigma_z$, and the effect of the environment is described by one Lindblad operator $L = \sigma_z$; the corresponding Lindblad equation is ($\hbar = 1$)

$$\frac{d}{dt}\rho_t = i\mu B[\sigma_z, \rho_t] - \frac{\lambda^2}{2}[\sigma_z, [\sigma_z, \rho_t]]. \tag{15}$$

The initial spin state is taken equal to

$$|\psi_0\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle, \tag{16}$$

where $|+\rangle$ and $|-\rangle$ are the two eigenstates of σ_z. Equation (9) becomes

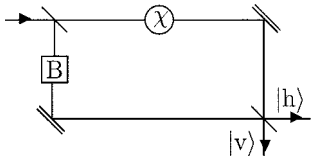

FIG. 1. Interferometric scheme for measuring the total phase. $|h\rangle$ corresponds to the beam traveling in the horizontal direction, while $|v\rangle$ corresponds to the beam traveling in the vertical direction. χ is a variable phase shifter and B a magnetic field.

$$d|\psi_t\rangle = \left[i\mu B\sigma_z dt + \lambda(c\sigma_z - \cos\varphi\langle\sigma_z\rangle_t)dW_t - \frac{\lambda^2}{2}(\sigma_z^2 - 2c\cos\varphi\langle\sigma_z\rangle_t\sigma_z + \cos^2\varphi\langle\sigma_z\rangle_t^2)\right]|\psi_t\rangle, \tag{17}$$

with $\langle\sigma_z\rangle_t = \langle\psi_t|\sigma_z|\psi_t\rangle$. In the above equation, we have included also the arbitrary phase factor $c = e^{i\varphi}$ which, as already discussed, does not appear in Eq. (15) for the density matrix ρ_t. We now compute the total and dynamical phases associated with the ensemble of vectors $\{|\psi_t\rangle \equiv |\psi_t(\omega)\rangle, \omega \in \Omega\}$ generated by Eq. (17).

A. Total phase

To derive the correct formula for the total phase, we resort to an interferometric scheme like the one depicted in Fig. 1, i.e., a Mach-Zehnder interferometer with a variable phase shifter χ in one of the two arms and the magnetic field in the other. We call $|h\rangle$ and $|v\rangle$ the spatial part of the wave function when the beam travels along the horizontal and vertical direction, respectively; in the subsequent analysis, we will neglect the contribution to the phase given by the free part of the evolution, since we assume that the length of the two arms of the interferometer is the same. The initial state of the beam is then

$$|\Psi_0\rangle = |\psi_0\rangle \otimes |h\rangle = \left[\cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle\right] \otimes |h\rangle. \tag{18}$$

To find the evolution from the initial time $t=0$ to the final time $t=t_F$ when the beam comes out through the interferometer, we first consider the linear version of Eq. (17), which is[2]

$$d|\Phi_t\rangle = \left[i\mu B\sigma_z dt + \lambda c\sigma_z d\xi_t - \frac{\lambda^2}{2}\sigma_z^2 dt\right]|\Phi_t\rangle. \tag{19}$$

By taking into account the effects of the two mirrors and partial beam splitters, one finds for the beam at time t_F

[2]In the following when we write $|\Psi_t\rangle$ and $|\Phi_t\rangle$, we mean that Eqs. (17) and (19) include not only the spin degree of freedom, but also the spatial one; when on the other hand we write $|\psi_t\rangle$ and $|\phi_t\rangle$, we mean that we are taking into account only the spin degree of freedom.

$$|\Phi_{t_F}\rangle = \frac{i}{2}\Bigg[(e^{i\chi} + e^{i\mu BT}) e^{\lambda c \xi_{t_F} - (\lambda^2/2)(1+c^2)t_F} \cos\frac{\theta}{2} |+\rangle + (e^{i\chi} + e^{-i\mu BT}) e^{-\lambda c \xi_{t_F} - (\lambda^2/2)(1+c^2)t_F} \sin\frac{\theta}{2} |-\rangle \Bigg] |h\rangle + \frac{1}{2}\Bigg[(e^{i\chi} - e^{i\mu BT}) e^{\lambda c \xi_{t_F} - (\lambda^2/2)(1+c^2)t_F} \cos\frac{\theta}{2} |+\rangle + (e^{i\chi} - e^{-i\mu BT}) e^{-\lambda c \xi_{t_F} - (\lambda^2/2)(1+c^2)t_F} \sin\frac{\theta}{2} |-\rangle \Bigg] |v\rangle. \tag{20}$$

One can now compute the output intensity along $|h\rangle$,

$$I_{t_F}(\omega) = \langle \Psi_{t_F} | [|h\rangle\langle h| \otimes \mathbb{I}_{\mathrm{SPIN}}] | \Psi_{t_F} \rangle = \frac{1}{2} + \frac{1}{2} |f_{t_F}(\omega)| \cos\{\chi + \arg[f_{t_F}(\omega)]\}, \tag{21}$$

where $|\Psi_{t_F}\rangle = |\Phi_{t_F}\rangle / \| |\Phi_{t_F}\rangle \|$ and

$$f_{t_F}(\omega) = \frac{e^{-2\lambda^2 t_F \cos^2\varphi}}{\langle \Phi_{t_F} | \Phi_{t_F} \rangle} \Bigg[e^{2\lambda \xi_{t_F} \cos\varphi - i\mu BT} \cos^2\frac{\theta}{2} + e^{-2\lambda \xi_{t_F} \cos\varphi + i\mu BT} \sin^2\frac{\theta}{2} \Bigg]. \tag{22}$$

One can then identify the total phase, for each realization of the noise, as

$$\gamma_t^{\mathrm{tot}}(\omega) \equiv \arg[f_t(\omega)], \tag{23}$$

which depends not only on ω, but also on φ, i.e., on the type of unraveling of the master equation (15). Anyway, $\gamma_t^{\mathrm{tot}}(\omega)$ as such is *not* a physical quantity because it cannot be observed. The final outcome—i.e., the interference pattern—consists of many repetitions of the experiment, accordingly the observable physical quantity is the *average* intensity $I_t \equiv \mathbb{E}_{\mathrm{P}}[I_t(\omega)]$, which can be easily computed by taking into account relation (13),

$$I_T = \frac{1}{2} + \frac{1}{2} \nu_T \cos(\chi - \gamma_T^{\mathrm{tot}}), \tag{24}$$

where

$$\nu_T \equiv |\mathbb{E}_{\mathrm{P}}[f_{t_F}(\omega)]| = \left| e^{i\mu BT} \cos^2\frac{\theta}{2} + e^{-i\mu BT} \sin^2\frac{\theta}{2} \right| \tag{25}$$

is the output visibility, while

$$\gamma_T^{\mathrm{tot}} = \arg \mathbb{E}_{\mathrm{P}}[f_{t_F}(\omega)] = \arg\left[e^{i\mu BT} \cos^2\frac{\theta}{2} + e^{-i\mu BT} \sin^2\frac{\theta}{2} \right] = \arg \mathbb{E}_{\mathrm{Q}}[\langle \phi_0 | \phi_t \rangle] \tag{26}$$

is the total phase difference; in particular, if B acts on the spin for a time $T = \pi/\mu B$, we have the standard result $\gamma_T^{\mathrm{tot}} = \pi$.

Note that the total phase γ_T^{tot} does not depend on φ, i.e., it does not depend on the specific choice of the unraveling used to make the calculations. Such a result is a consequence of property (8) of stochastic unravelings, i.e., of the fact that Eq. (21) for the output intensity, when also the average over the noise is taken into account, can be expressed as a function of the density matrix ($I_t \equiv \mathbb{E}_{\mathrm{P}}[I_t(\omega)] = \mathrm{Tr}[(|h\rangle\langle h| \otimes \mathbb{I}_{\mathrm{SPIN}})\rho_t]$), so that any dependence on the unraveling disappears. Such a result is then not a peculiar byproduct of the specific model taken into account, but a *necessary* mathematical consequence of the formalism. This fact can be seen in a different way: the total phase difference between the two arms of the interferometer is a physical quantity which can be experimentally measured; like all physical quantities, it must be deducible from the master equation (15), so it does not have to depend on φ.

As a final observation, we note that both the average visibility and the average total phase do not depend on λ. This specific fact is a consequence of our simple model of open quantum system, according to which the noise is perfectly correlated among the two arms of the interferometer. Of course such an assumption is not realistic, and it has been made only to simplify the calculations, since it does not affect the conclusion of our work.

B. Dynamical phase

We now compute the dynamical phase γ_t^{dyn} induced by the precession of the spin-system when interacting with the magnetic field; by using Itô calculus [18], one finds from Eq. (17)

$$\langle \psi(t) | d | \psi(t) \rangle = i\mu B \langle \sigma_z \rangle_t dt + i\lambda \sin\varphi \langle \sigma_z \rangle_t dW_t - \frac{\lambda^2}{2} [1 - (2c\cos\varphi - \cos^2\varphi) \langle \sigma_z \rangle_t^2] dt, \tag{27}$$

where only the imaginary part has to be taken into account. For each realization of the noise, the dynamical phase is

$$\gamma_t^{\mathrm{dyn}}(\omega) = \mu B \int_0^t \langle \sigma_z \rangle_s ds + \lambda \sin\varphi \int_0^t \langle \sigma_z \rangle_s dW_s + \lambda^2 \sin\varphi \cos\varphi \int_0^t \langle \sigma_z \rangle_s^2 ds, \tag{28}$$

which, like the total phase $\gamma_t^{\mathrm{tot}}(\omega)$, depends not only on ω, but also on the unraveling of the master equation.

We now compute the stochastic average γ_t^{dyn} of $\gamma_t^{\mathrm{dyn}}(\omega)$, for which we need to know the statistical properties of both $\langle \sigma_z \rangle_t$ and $\langle \sigma_z \rangle_t^2$: these can be easily computed by writing the corresponding stochastic differential equations, both of which can be quite easily derived from Eq. (17). The equation for $\langle \sigma_z \rangle_t$ is

$$d\langle \sigma_z \rangle_t = 2\lambda \cos\varphi [1 - \langle \sigma_z \rangle_t^2] dW_t, \tag{29}$$

which tells us that since the Brownian increment dW_t has zero mean, the average value of $\langle \sigma_z \rangle_t$ does not change in time: $\mathbb{E}_{\mathrm{P}}[\langle \sigma_z \rangle_t] = \mathbb{E}_{\mathrm{P}}[\langle \sigma_z \rangle_0] = \cos\theta$.

The stochastic differential equation for $\langle \sigma_z \rangle_t^2$ instead is

$$d\langle\sigma_z\rangle_t^2 = 4\lambda\cos\varphi\langle\sigma_z\rangle_t[1-\langle\sigma_z\rangle_t^2]dW_t + 4\lambda^2\cos^2\varphi[1-\langle\sigma_z\rangle_t^2]^2dt; \quad (30)$$

the first term on the right-hand side does not contribute to the stochastic average, so one has

$$\frac{d}{dt}\mathbb{E}_{\mathbb{P}}[\langle\sigma_z\rangle_t^2] = 4\lambda^2\cos^2\varphi\,\mathbb{E}_{\mathbb{P}}[1-\langle\sigma_z\rangle_t^2]^2 \geq 0, \quad (31)$$

which implies that $\mathbb{E}_{\mathbb{P}}[\langle\sigma_z\rangle_t^2]$ constantly increases in time, and in general (unless $\varphi=\pi/2+k\pi, k\in\mathbf{Z}$) it stops increasing only when $\mathbb{E}_{\mathbb{P}}[1-\langle\sigma_z\rangle_t^2]^2=0$, i.e., when $\langle\sigma_z\rangle_t^2=1$, with the possible exception of a set of points $\omega\in\Omega$ of measure 0.

Concluding, the average dynamical phase $\gamma_t^{\mathrm{dyn}} \equiv \mathbb{E}_{\mathbb{P}}[\gamma_t^{\mathrm{dyn}}(\omega)]$, after a time T, is equal to

$$\gamma_T^{\mathrm{dyn}} = \mu BT\cos\theta + \lambda^2\sin\varphi\cos\varphi\int_0^T \mathbb{E}_{\mathbb{P}}[\langle\sigma_z\rangle_t^2]dt, \quad (32)$$

which, contrary to what happens to the average total phase, still depends on φ, i.e., on the specific stochastic unraveling of the master equation.

One could argue that, when computing the average dynamical phase, we should not average over the phase, but over the phase factor, i.e., we should compute $\mathbb{E}_{\mathbb{P}}(\exp[i\gamma_t^{\mathrm{dyn}}(\omega)])$ in place of $\mathbb{E}_{\mathbb{P}}[\gamma_t^{\mathrm{dyn}}(\omega)]$, and then extract the argument; in this way we would take into account the fact that a phase is defined modulus 2π. The stochastic differential of $\exp[i\gamma_t^{\mathrm{tot}}(\omega)]$ is

$$de^{i\gamma_t^{\mathrm{tot}}(\omega)} = \left[i\mu B\langle\sigma_z\rangle_t + i\lambda^2\sin\varphi\cos\varphi\langle\sigma_z\rangle_t^2 - \frac{\lambda^2}{2}\sin^2\varphi\langle\sigma_z\rangle_t^2\right]e^{i\gamma_t^{\mathrm{tot}}(\omega)}dt + [i\lambda\sin\varphi\langle\sigma_z\rangle_t]e^{i\gamma_t^{\mathrm{tot}}(\omega)}dW_t. \quad (33)$$

Its average value cannot be explicitly computed, due to the dependence of both $\langle\sigma_z\rangle_t$ and $\langle\sigma_z\rangle_t^2$ on the noise; anyway, when taking the average, the dependence on φ in general does not disappear. For example, if we take the trivial case in which the initial state is $|\psi_0\rangle=|+\rangle$, so that $\langle\sigma_z\rangle_0=1$, then Eqs. (29) and (31) tell us that both $\langle\sigma_z\rangle_t$ and $\langle\sigma_z\rangle_t^2$ remain equal to 1 for each realization of the noise; in such a case, the average value of $\exp[i\gamma_t^{\mathrm{tot}}(\omega)]$ at time T is

$$\mathbb{E}_{\mathbb{P}}(\exp[i\gamma_T^{\mathrm{tot}}(\omega)]) = e^{-(\lambda^2/2)\sin^2\varphi T + i(\mu B+\lambda^2\sin\varphi\cos\varphi)T}, \quad (34)$$

and its argument clearly depends on φ.

C. Geometric phase

The geometric phase γ_t^{geo} is the difference between the total and the dynamical phase. For each realization of the stochastic process W_t, one has from Eqs. (23) and (28),[3]

$$\gamma_T^{\mathrm{geo}}(\omega) = \arg[f_t(\omega)] - \mu B\int_0^T\langle\sigma_z\rangle_s ds - \lambda\sin\varphi\int_0^T\langle\sigma_z\rangle_s dW_s - \lambda^2\sin\varphi\cos\varphi\int_0^T\langle\sigma_z\rangle_s^2 ds \quad (35)$$

[$f_t(\omega)$ is defined in Eq. (22)], which clearly depends on the type of unraveling. Its average value is

$$\gamma_T^{\mathrm{geo}} = \arg\mathbb{E}_{\mathbb{P}}[f_t(\omega)] - \begin{cases}\mathbb{E}_{\mathbb{P}}[\gamma_T^{\mathrm{dyn}}(\omega)] \text{ or}\\ \arg(\mathbb{E}_{\mathbb{P}}[\exp\{i\gamma_T^{\mathrm{dyn}}(\omega)\}])\end{cases}, \quad (36)$$

Also the average geometric phase depends on the type of stochastic unraveling of the master equation, whichever way the average dynamical phase is computed. This is the main result of our paper.

IV. DISCUSSION AND CONCLUSIONS

The geometric phase of an open quantum system should be a quantity depending only on the path followed by the density matrix ρ_t in its state space; we have seen that such a phase, when computed by means of stochastic unravelings—as done in Ref. [9]—depends on the type of unraveling, both for single realizations of the noise [Eq. (35)] and for its average value [Eq. (36)]. This fact has two important consequences.

(i) First of all, the phase defined in Eqs. (35) and (36) is *not* a geometric object, since it depends also on φ, which has nothing to do with the path followed by ρ_t during its evolution.

(ii) Worse than this, such a phase is not even an object somehow related to a physical quantity, because φ itself has no physical meaning since it only selects one of the infinitely many equivalent stochastic unravelings that can be used.

The conclusion is that the stochastic-unraveling method does not lead to a sensible definition of geometric phase.

One could say that different stochastic unravelings might correspond to different ways to perform the measurement or to monitor the environment [16] and then that different values for the phase correspond to different ways to measure the system; anyway, this is not the case here: in our example, we have taken a standard interferometer where the output intensity is measured in a standard and unique way. Nevertheless, different unravelings can still be taken into account.

Note that the dependence on φ comes only from the dynamical phase, not from the total phase. As already remarked, this is not a consequence of the specific model of open quantum system we are considering here, but a direct consequence of the fact that a total-phase difference is a measurable quantity and as such must be deducible from the density matrix ρ_t, which does not depend on φ. On the contrary, the dynamical phase and thus also the geometrical phase are not directly observable, so—at least from the mathematical point of view—it can depend on φ, as it happens here.

What is the mathematical origin of the dependence of $\gamma_t^{\mathrm{geo}}(\omega)$ and γ_t^{geo} on φ? Its unraveling dependence does not come from the total phase but from the dynamical compo-

[3] In Eqs. (35) and (36), t refers to the time during which the beam travels through the interferometer, while T is the time during which the spin interacts with the magnetic field.

nent for the following reason: by definition, $\gamma_t^{\mathrm{dyn}}(\omega)$ is not a function of $|\psi_t\rangle$ at the considered time, but a function of the *whole* history of $|\psi_s\rangle$, from $s=t_0$ to $s=t$, i.e., it depends on the whole trajectory followed by the state vector. Now, it is easy to see that for different unravelings, the trajectories followed by the state vector are radically different. For example, when $\varphi=\pi/2+k\pi$ with $k\in\mathrm{Z}$, Eq. (29) shows that $\langle\sigma_z\rangle$ is constant in time, for *each* realization of the stochastic process: this implies that the projection of the spin vector along the magnetic field does not change in time, i.e., the vector rotates always along the same circle on the Bloch sphere. On the other hand, when $\varphi\neq\pi/2+k\pi$ with $k\in\mathbf{Z}$, then, as we have already discussed in connection with Eq. (31), $\langle\sigma_z\rangle^2$ approaches the value 1 for $t\to\infty$, i.e., the variance $\mathrm{var}[\sigma_z]\equiv\langle\sigma_z^2\rangle-\langle\sigma_z\rangle^2=1-\langle\sigma_z\rangle^2$ of the operator σ_z approaches zero: this means that the state vector is driven toward one of the two eigenstates of σ_z, thus changing the projection of the spin vector along the magnetic field [23]. As a consequence, since the trajectories followed by the state vector so strongly depend on the kind of unraveling, there is no need for the dynamical phase to be unraveling-independent, as it actually occurs.

A different way to see what happens is the following: the relation $\langle\psi_t|d|\psi_t\rangle=0$ defines the parallel transport condition, and for different unravelings one has different inequivalent parallel transport conditions, thus different definitions of a geometric phase.

Another interesting question is about the physical reason for such a dependence of the geometric phase on φ. It has been surmised that the problem arises because the master equation used to model the effect of the environment is of the Lindblad type. Since the Lindblad equation is only an effective equation approximating an otherwise too complex system, its validity is limited and it could not be suitable for computing the geometric phase. We think that this is not the case: the source of all troubles derives from the fact that there are different equivalent stochastic unravelings associated with the same evolution Σ_t which determines different evolutions for the state vector, thus different parallel transport conditions; such a feature is not an exclusive property of the Lindblad equation (it is not even a mathematical consequence of it), but has a more general character.

To summarize, the stochastic-unraveling approach used in [9] does not produce a phase which is geometric, i.e., which depends only on the trajectory followed by ρ_t during the evolution; it depends also on the specific choice of the unraveling used for the calculations, which by itself has no particular physical meaning. This difficulty in principle can be overcome by fixing the unraveling to be used for computing the geometric phase, as often implicitly done in the literature, but this procedure cannot be satisfactory for two reasons: first, it obviously does not remove the fact that the definition is mathematically unraveling-dependent; second, there is no fundamental physical reason to choose one unraveling in place of another, since they are all on the same footing.

In Ref. [8] it has been stated that, within the state purification approach of [7], different Kraus representations may lead to different values for the geometric phase; if so, then our criticism applies also to the approach of [7].

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