Nonclassical photon statistics

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Some nonclassical properties of a particular state of light are demonstrated by computing the Wigner function. A quantum model of generating this state is proposed. A criterion called the distance variation difference is introduced to measure how far any state is from its corresponding coherent state.

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I. INTRODUCTION

We recently demonstrated that a random point process (RPP) derived from a Poisson process (PP) can show nonclassical properties. This RPP, a random distribution of time instants ϑ_i, denoted here by E2, is of a renewal type and called an Erlang process of order 2. An example is given by the times of detection of photons. The E2 process can simply be generated by regularly deleting one point over two of the initial PP. The criteria utilized to characterize its nonclassical properties were established from semiclassical theory, namely, the second-order statistics of the random variables (RVs) (i) $N(\vartheta_i;T)$, the number of points registered within a fixed finite time interval $[\vartheta_i,\vartheta_i+T]$, and (ii) $t_i=\vartheta_{i+1}-\vartheta_i$, the time intervals.

The purpose of this paper is mainly to treat the E2 process quantum mechanically.

It is well known that in order to characterize nonclassical properties of a state, the criterion that is widely utilized in the literature is the Wigner function defined in the phase space. For a nonclassical state, this function is a quasiprobability. That is, although it is a probability density function (PDF), it has a negative part that the Q function does not have.

In Sec. II, we first briefly review the basic equations of phase space in quantum mechanics. We then write the density matrix in the number representation taking into account only diagonal elements.

In Sec. III, we calculate the relevant Wigner and Q functions numerically and approximate them analytically for our particular model. In Sec. IV we deal with the nondiagonal elements of the density matrix and the corresponding Wigner and Q functions, in order to obtain interference contributions. The analytical and numerical results are summarized in several curves. A quantum model yielding the diagonal elements of the density matrix is briefly analyzed in Section V. Finally, the variation distance difference, a criterion to measure the deviation of any state from its corresponding coherent state, is introduced in Sec. VI. This criterion is calculated for some classical and nonclassical states including entangled ones such as the superposition of two coherent states.

II. BASIC FORMALISM

Let $\mathcal{H}$ be a Hilbert space of dimension $D\to\infty$ spanned by $\{|n\rangle\}$ a set of infinitely denumerable vector states which form a complete orthonormal basis ($\langle m|n\rangle=\delta_{m,n}$, $\Sigma_{n=0}^{\infty}|n\rangle\langle n|=\hat{1}$), where $\hat{1}$ is the identity operator. These $\{|n\rangle\}$ are the eigenstates of the number operator $\hat{N}$.

A. Density operator

We consider a positive, Hermitian, and trace class density matrix having the canonical decomposition

$$\hat{\rho}=\sum_{n=1}^{\infty}|\psi_n\rangle\gamma_n\langle\psi_n|,\quad \gamma_n\geqslant 0,\quad \sum_{n=1}^{\infty}\gamma_n=1, \tag{2.1}$$

where $\{|\psi_n\rangle\}$ forms a complete orthonormal basis. Each $|\psi_n\rangle$ appearing in Eq. (2.1) admits the expansion $|\psi_n\rangle=\Sigma_{q=0}^{\infty}\psi_{qn}|n\rangle$, so that

$$\hat{\rho}=\sum_{m=0}^{\infty}\sum_{n>m}^{\infty}P(m,n)|m\rangle\langle n|+\sum_{n=0}^{\infty}p(n)|n\rangle\langle n|+\text{conj}. \tag{2.2}$$

where $P(m,n)=\Sigma_{q=1}^{\infty}\psi_{qm}\gamma_q\psi_{qn}^{*}=\langle m|\hat{\rho}|n\rangle$, $\Sigma_{m,n}P(m,n)=1$. For a pure state, $\text{Tr}\,\hat{\rho}^2=1$; otherwise $\text{Tr}\,\hat{\rho}^2<1$.

An important set of mixed states are those given by $\hat{\rho}=\Sigma_{n=0}^{\infty}\gamma_n\hat{\rho}_n$ where the set of pure-state density matrices are those based on $|n\rangle$ such that $\hat{\rho}_n=|n\rangle\langle n|$ are projection operators $\hat{\rho}_n^2=\hat{\rho}_n$.

Another representation based on the coherent state uses the so-called P function such that

$$\hat{\rho}=\frac{1}{\pi}\int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha, \tag{2.3}$$

where $(1/\pi)\int|\alpha\rangle\langle\alpha|d^2\alpha=1$. The relations connecting both representations are

$$|\alpha\rangle=\sum_n e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}|n\rangle,\quad |n\rangle=\frac{1}{\pi}\int e^{-|\alpha|^2/2}\frac{\alpha^{*n}}{\sqrt{n!}}|\alpha\rangle d^2\alpha. \tag{2.4}$$

In the following, we will be using both representations. We start with the number representation of the density matrix in order to study two types of quasiprobability functions, namely, the Q and the Wigner functions [1,2].

B. $Q(\alpha)$ function

The Q function can be defined as

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$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle = Q_d(\alpha) + Q_i(\alpha) \quad (2.5)$$

where we set

$$Q_d(\alpha) = \frac{1}{\pi}e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}p(n), \quad (2.6)$$

$$Q_i(\alpha) = \frac{2}{\pi}e^{-|\alpha|^2}\sum_{m,n}P(m,n)\frac{|\alpha|^{(m+n)}}{\sqrt{m!\,n!}}\cos[\varphi(m-n)]. \quad (2.7)$$

These functions have the main properties $Q(\alpha) \geqslant 0$, $\int Q_d(\alpha)d^2\alpha = 1$, $\int Q_i(\alpha)d^2\alpha = 0$, $d^2\alpha = |\alpha|d|\alpha|d\varphi$ where $\alpha = |\alpha|e^{i\varphi}$, $|\alpha| \in [0,\infty[$. The phase has the PDF $p(\varphi)d\varphi = d\varphi/2\pi$, $\varphi \in [0,2\pi]$. In the following, most of the calculations are restricted to the case $\varphi = 0$.

C. Wigner function

The form that will be used here is given by

$$W(x,p) = \frac{2}{\pi}\int d\xi\, e^{-2ip\xi}\langle x+\xi|\hat{\rho}|x-\xi\rangle, \quad (2.8)$$

such that $\int W(x,p)dx\,dp = 1$, $\mathrm{Im}(W) = 0$. Note that we may obtain the elements of the density operator using the inverse Fourier transform of $W(x,p)$,

$$\langle x+\xi|\hat{\rho}|x-\xi\rangle = \int dp\; W(\xi,p)e^{2ip\xi}. \quad (2.9)$$

After inserting (2.2) into (2.8), the Wigner function becomes

$$W(x,p) = \frac{2}{\pi}\sum_{m\neq n}P(m,n)\int d\xi\, e^{-2ip\xi}\langle x+\xi|m\rangle\langle n|x-\xi\rangle. \quad (2.10)$$

The scalar product appearing in the integral is known [3]:

$$\langle z|n\rangle = \frac{1}{\sqrt{2^n n!\,\sqrt{\pi}}}e^{-z^2/2}H_n(z) \quad (2.11)$$

where $x^2 + p^2 = 2|\alpha|^2$ and $H_n(z)$ are the Hermite polynomials. Therefore

$$W(\alpha) = W_d(\alpha) + W_i(\alpha), \quad (2.12)$$

$$W_d(\alpha) = \frac{2e^{-2|\alpha|^2}}{\pi}\sum_{n=0}^{\infty}(-1)^n p(n)L_n(4|\alpha|^2), \quad (2.13)$$

$$W_i(\alpha) = \frac{4e^{-2|\alpha|^2}}{\pi}\sum_{m=0}^{\infty}\sum_{n>m}^{\infty}P(m,n)(-1)^m\sqrt{\frac{m!}{n!}}(2|\alpha|)^{n-m} \times L_m^{n-m}(4|\alpha|^2)\cos[\varphi(n-m)], \quad (2.14)$$

where [4]

$$L_\ell^r(z) = \sum_{k=0}^{\ell}\frac{(-z)^k}{(\ell-k)!}\frac{(\ell+r)!}{(r+k)!\,k!} \quad (2.15)$$

are the associated Laguerre polynomials $L_\ell(z) = L_\ell^0(z)$ and where $\int W_i(\alpha)d^2\alpha = 0$.

Clearly, both $W_d(\alpha)$ and $W_i(\alpha)$ can be negative because of the $(-1)^n$ and $(-1)^m$ terms in Eqs. (2.13) and (2.14). Due to this negativity, the Wigner function is a quasiprobability. It then characterizes the nonclassical properties of the state.

III. DIAGONAL APPROXIMATION

Within this approximation, only the diagonal terms of the density operator are taken into account,

$$\hat{\rho} = \sum_{n=0}^{\infty}p(n)|n\rangle\langle n|, \quad (3.1)$$

yielding (2.6) and (2.13).

As soon as $p(n)$ is given, e.g., for a particular state of light, the calculations of both functions are straightforward. This has been done for various types of states including the squeezed and the entangled states which, as known, have nonclassical properties.

Now, among the states that exhibit nonclassical properties, we wish to examine the particular state denoted here by E2 and defined by its PDF $p(n)$ [5].

A. Erlang state of order 2

This state is most conveniently defined by its density matrix in the number presentation. Hence, the needed photon number probability density is expressed as [5]

$$p(n) = \left[\frac{\lambda^{2n}}{(2n)!} + \frac{1}{2}\left(\frac{\lambda^{2n+1}}{(2n+1)!} + \frac{\lambda^{2n-1}}{(2n-1)!}\right)\right]e^{-\lambda}. \quad (3.2)$$

We recently calculated the first two moments of the RV n obeying (3.2). We obtained

$$E[n] = \frac{\lambda}{2}, \quad (3.3)$$

$$E[n^2] = \frac{\lambda(\lambda+2)(\lambda^2+6\lambda+6)(\lambda^2+\lambda+e^{-\lambda}\sinh\lambda)}{(2\lambda^2+4\lambda+3)^2+4(4\lambda-1)(\lambda+3/2)^2}, \quad (3.4)$$

and pointed out that $h \triangleq E[n^2] - E[n]^2 - E[n] \simeq -\lambda^2/12$, for $\lambda \ll 1$, and $h \simeq -\lambda/4$ for $\lambda \gg 1$, showing a nonclassical property.

Let us examine the quasiprobability functions. For $\lambda \lesssim 4$ and up to $|\alpha|^4$, the $Q_d(\alpha)$ function is easily shown to be

$$Q_d(\alpha) \simeq \frac{e^{-|\alpha|^2-\lambda}}{\pi}(a_0 + a_1|\alpha|^2 + a_2|\alpha|^4) = Q_d^a(\alpha), \quad (3.5)$$

where $a_0 = 1 + \lambda/2$, $a_1 = (\lambda/2)(1+\lambda)$, and $a_2 = (\lambda^3/12)(1+\lambda/2)$. As expected, $Q_d \geqslant 0\ \forall\ \alpha$.

Note that this very simple approximation appears excellent $\forall\ \alpha, \lambda$. Note also that Q_d is symmetrical against α, and

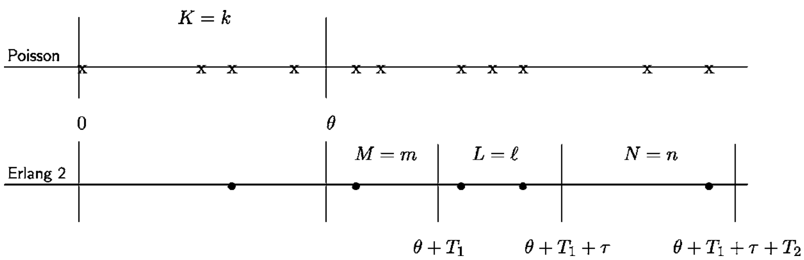

FIG. 1. Scheme for generating an Erlang process of order 2 from the initial Poisson process and notations for multicounting.

exhibits an extremum for $\alpha=0$ and a maximum for $\alpha_m \simeq \pm\sqrt{2-3/2a_2}$ which has been evaluated for moderate λ. With the help of a Gaussian approximation of the PDF, the asymptotic form of Q_d for $\lambda \gg 1$ can be expressed as

$$Q_d(\alpha) = \frac{1}{\Gamma\left(\frac{\lambda+1}{2}\right)}|\alpha|^{\lambda} e^{-|\alpha|^2}, \tag{3.6}$$

where $\Gamma(z+1/2)=(\sqrt{\pi}/2^z)(2z-1)!!$. Note that even in the limit of large λ, $Q_d(\alpha)$ differs from the classical limit (Gaussian shape).

On the other hand $W_d(\alpha)$, which is symmetrical in α and has a minimum for $\alpha=0$, is

$$W_d(\alpha) = \frac{2e^{-2|\alpha|^2}}{\pi}\left[e^{-\lambda}\left(1+\frac{\lambda}{2}\right) + \sum_{n=1}^{\infty}(-1)^n p(n) L_n(4|\alpha|^2)\right], \tag{3.7}$$

where $p(n)$ is given by (3.2). To evaluate the conditions required for $W_d(\alpha)$ to have a negative region, it is convenient to approximate it for $\lambda \lesssim 2$ as

$$\begin{aligned} W_d(\alpha) &\simeq \frac{2e^{-2|\alpha|^2-\lambda}}{\pi}\left[1 + 2\tan\lambda\frac{6+12\lambda+7\lambda^2+\lambda^3}{6+6\lambda+\lambda^2}|\alpha|^2 \right. \\ &\quad \left. - 2\left(1+\frac{\lambda}{2}\right)(\lambda-\tan\lambda)|\alpha|^4\right]\cos\lambda \\ &= W_d^a(\alpha), \end{aligned} \tag{3.8}$$

which is obtained from the expansion up to α^4 of W_d given by Eq. (3.7) and for $\alpha \in \mathrm{Re}$. Now, it can be shown that there is an area $-\Delta \leqslant \alpha \leqslant \Delta$ where $W_d(\alpha)$ is negative. The value of Δ can be approximated by α_0 which is such that $W_d^a(\alpha)=0$. We obtain $\alpha_0 \simeq 1/\sqrt{\lambda}$ for $\lambda > \pi/2$. Note finally that (3.8) is an excellent approximation of the exact $W_d(\alpha)$ in the conditions of the present study.

Now, the best asymptotic form ($\lambda \gg 1$) we obtained for W_d is

$$\begin{aligned} W_d(\alpha) &\simeq \sqrt{\frac{2e}{\pi}}e^{-2|\alpha|^2-\lambda} I_0(2\sqrt{2}\lambda|\alpha|) \\ &\simeq \frac{e^{-\lambda}}{\pi\sqrt{\pi\sqrt{2\lambda}}}\frac{e^{-2|\alpha|^2+2|\alpha|\sqrt{2\lambda}}}{\sqrt{|\alpha|}}, \end{aligned} \tag{3.9}$$

where $I_0(z)=\Sigma_{k=0}^{\infty}(z/2)^{2k}/(k!)^2$ is the modified Bessel function of the first kind. However, this approximation, although not of a Gaussian shape, does not appear good enough.

On the other hand, it is of interest to point out that the nonclassicity is verified by the generalized Erlang processes. However, the extension to order r (r points are omitted in the initial PP instead of one as in Fig. 1) is simple only in calculations involving the diagonal representation. The photon number distribution has recently been expressed as [5]

$$p_r(n) = \frac{e^{-\lambda}}{r}\left(\sum_{\ell=rn}^{rn+r-1}(rn+r-\ell)\frac{\lambda^{\ell}}{\ell!} + \sum_{\ell=rn-r+1}^{rn-1}(\ell-rn+r)\frac{\lambda^{\ell}}{\ell!}\right). \tag{3.10}$$

The results for W_d, the semiclassical part of the Wigner function, are drawn in Fig. 6 for the density $\lambda=9$. It is seen that the negativity of $W_d(\alpha)$ holds for all values of $r>1$.

IV. INTERFERENCE CONTRIBUTIONS

To complete the expressions established for E2, it is necessary to take into account terms obtained from off-diagonal elements of the density operator. To do so, we need to extend the results recently published [5] for the relaxed distribution $p(n)$ to the twofold relaxed distribution $P(m,n)$.

A. Twofold relaxed distribution $P(m,n)$

The twofold relaxed distribution is more easily established for successive times of registration of the E2 with the help of the scheme and diagram shown in Figs. 1 and 2.

We start with the stationary Poisson process of density λ. This process is registered within $\Theta \in [0,\infty[$. We set $\omega=\lambda\theta$,

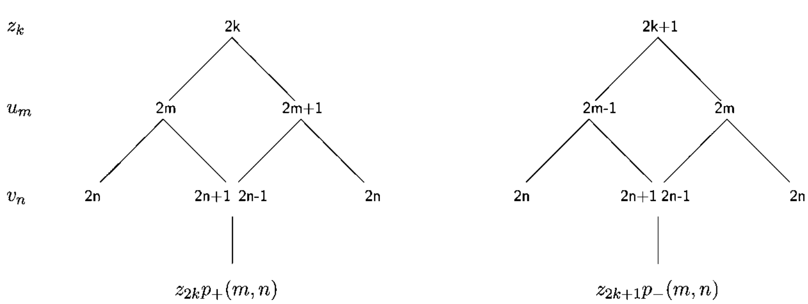

FIG. 2. Diagram for enumerating the terms of the twofold PDF.

$\mu=\lambda T_1$, $\nu=\lambda\tau$, $\eta=\lambda T_2$, $z_\ell=e^{-\omega}\omega^\ell/\ell!$, $u_\ell=e^{-\mu}\mu^\ell/\ell!$, $v_\ell=e^{-\nu}\nu^\ell/\ell!$, and $w_\ell=e^{-\eta}\eta^\ell/\ell!$. Most of the calculations will be done here for $\mu=\eta$.

Let k be the number of points within $[0,\theta[$ taken by the RV K of the random Poisson process (P). The E2 process is obtained by omitting one point on the previous process as depicted in Fig. 1. We have

$$P(m,n|\ell)=\lim_{\theta\to\infty}\sum_{k=0}^{\infty}P(m,n|k,\ell)=\lim_{\theta\to\infty}\sum_{k=0}^{\infty}z_{2k}P_+(m,n|\ell)+z_{2k+1}P_-(m,n|\ell)=\frac{1}{2}[P_+(m,n|\ell)+P_-(m,n|\ell)], \quad (4.1)$$

which depends on τ, the time interval where ℓ points have been registered (Fig. 1).

To simplify this expression, we limit τ to specific values, namely, $\tau=0$ and $\tau\to\infty$. These values yield constant values of $g(\tau)$, the time coincidence function (TCF). To calculate this TCF, it is better to use the method based on Laplace transforms.

Thus, let $f(s)=\int_0^\infty d\tau\, e^{-s\tau}w(\tau)=(\lambda/\lambda+s)^2$, $0\leqslant f(s)\leqslant 1\ \forall\ \lambda,s\in\mathrm{Re}$, be the Laplace transform of $w(\tau)=\lambda^2\tau e^{-\lambda\tau}$, the PDF of τ. Now, from the definition of $g(\tau)=\sum_{k=0}^{\infty}w_k(\tau)$ [6], where $w_k(\tau)$ is the PDF of the sum of k independent intervals such that $w_k(\tau)=w(\tau)\star w(\tau)\star\cdots\star w(\tau)$ (the symbol $\star$ denotes the product of convolution, applied here k times). Hence the inverse Laplace transform of $F(s)=\sum_{k=0}^{\infty}f^k(s)=f(s)/[1-f(s)]=1/[(1+s/\lambda)^2-1]$ is the TCF $g(\tau)=(\lambda/2)(1-e^{-2\lambda\tau})$ [5]. This TCF, which is a monotonically increasing function of τ, differs significantly from the so-called covariance function $c(\tau)=-(\lambda^2/4)(1-e^{-2\lambda\tau})\leqslant 0\ \forall\ \tau$ [7].

1. $\tau=0$

In what follows, the PDF $P(m,n|\tau=0)$ is denoted $P(m,n)$ and $P_+(m,n|\ell)$ and $P_-(m,n|\ell)$ are denoted by $p_+(m,n)$, and $p_-(m,n)$, respectively. The terms of $P(m,n)$ are conveniently numbered using the diagram (Fig. 2). We have

$$p_+(m,n)=u_{2m}(v_{2n}+v_{2n+1})+u_{2m+1}(v_{2n}+v_{2n-1}), \quad (4.2)$$

$$p_-(m,n)=u_{2m-1}(v_{2n}+v_{2n+1})+u_{2m}(v_{2n}+v_{2n-1}), \quad (4.3)$$

yielding for $m,n\geqslant 0$, $u_{-1}=v_{-1}=0$,

$$P(m,n)=\frac{1}{2}[v_{2n}(2u_{2m}+u_{2m+1}+u_{2m-1})+v_{2n+1}(u_{2m}+u_{2m-1})+v_{2n-1}(u_{2m}+u_{2m+1})]. \quad (4.4)$$

Hence

$$p(m)=\sum_{n=0}^{\infty}P(m,n)=\frac{1}{2}(2u_{2m}+u_{2m+1}+u_{2m-1}). \quad (4.5)$$

Hence, the expression used in this paper for $\mu=\eta=\lambda$ is

$$P(m,n)=\frac{e^{-2\lambda}}{2}\left[\frac{\lambda^{2m-1}}{(2m-1)!}\left(\frac{\lambda^{2n}}{(2n)!}+\frac{\lambda^{2n+1}}{(2n+1)!}\right)+\frac{\lambda^{2m}}{(2m)!}\left(\frac{\lambda^{2n-1}}{(2n-1)!}+2\frac{\lambda^{2n}}{(2n)!}+\frac{\lambda^{2n+1}}{(2n+1)!}\right)+\frac{\lambda^{2m+1}}{(2m+1)!}\left(\frac{\lambda^{2n-1}}{(2n-1)!}+\frac{\lambda^{2n}}{(2n)!}\right)\right]. \quad (4.6)$$

2. Arbitrary τ

The conditional distribution is given by

$$\pi(n|m)=\frac{P(m,n)}{p(m)}=v_{2n}+v_{2n+1}\frac{u_{2m}+u_{2m-1}}{2u_{2m}+u_{2m+1}+u_{2m-1}}+v_{2n-1}\frac{u_{2m}+u_{2m+1}}{2u_{2m}+u_{2m+1}+u_{2m-1}} \quad (4.7)$$

yielding

$$\pi(n|0)=v_{2n}+v_{2n+1}\frac{1}{2+\mu}+v_{2n-1}\frac{1+\mu}{2+\mu}, \quad (4.8)$$

where $\pi(n|0)=p(n)$ for $\tau\to 0$. Again from (4.7), we have for $\mu=\lambda$

$$\pi(n|1)(\tau)=e^{-\nu}\left(v_{2n}+v_{2n+1}\frac{2+\lambda}{2(1+\lambda+\lambda^2/6)}+v_{2n-1}\frac{\lambda(3+\lambda)}{6(1+\lambda+\lambda^2/6)}\right), \quad (4.9)$$

yielding $\pi(n|1)(0)=v_{2n}+v_{2n+1}=q(n)$, which is the triggered distribution (see [8]). Furthermore, from (4.9) and for $\lambda=1$, and taking $\nu>0$ as a parameter, we obtain

$$\pi(1|1)(\tau)\triangleq G(\tau)=\frac{\nu}{6}e^{-\nu}\left(3\nu+\frac{\tau^2+(\tau/2)(6+\nu^2)+\nu^2}{\tau^2/6+\tau+1}\right), \quad (4.10)$$

which increases for $\nu<\sqrt{6}$ and decreases for $\nu>\sqrt{6}$.

The function $G(\tau)$ is not a TCF; it is a delayed triggered PDF [9]. However, because it can increase or decrease, in contrast to $g(\tau)$ which always increases, it contains more information than $g(\tau)$ about the process of measurement of the state.

3. $\tau\to\infty$

In this case, the PDFs of m and n factorize such that

$$P(m,n)=\frac{1}{4}(2u_{2m}+u_{2m+1}+u_{2m-1})(2v_{2n}+v_{2n+1}+v_{2n-1}), \quad (4.11)$$

and the interference contributions to Q and W are Q_d^2 and W_d^2.

Let us concentrate on the case $\tau=0$.

B. $Q(\alpha)$ function

We consider first the off-diagonal part given by (2.7) for $\varphi=0$. All terms involving λ^{-1} are set to 0. It is simple to

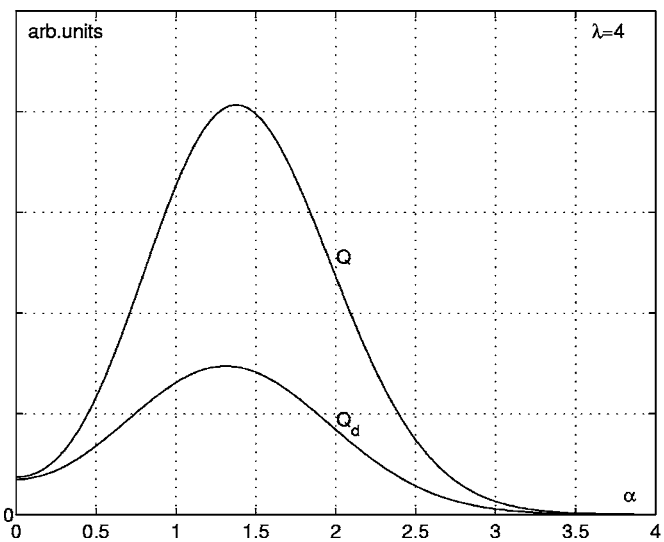

FIG. 3. $Q(\alpha)$ functions versus $\alpha \in \mathrm{Re}$ in dimensionless units for Erlang process of order 2 with density $\lambda=4$. The function $Q_d(\alpha)$ is obtained taking into account only diagonal elements of the density operator.

prove that an excellent approximation of $Q(\alpha)$, given by (2.7), is

$$Q_{\mathrm{ap}}(\alpha) = Q_d^a(\alpha) + \frac{e^{-|\alpha|^2-2\lambda}}{\pi}(b_0 + b_1|\alpha| + b_2|\alpha|^2 + b_3|\alpha|^3) \tag{4.12}$$

where $b_0=2+5\lambda/2, b_1=\lambda^2+5\lambda^3/3, b_2=\lambda^3+5\mu^4/6, b_4=\lambda^5+\lambda^6/6$ and Q_d^a is given by (3.5).

Here again $Q \geqslant 0\ \forall\ \alpha$. Note also that $Q(\alpha)$ exhibits extrema at abscissas values very close to those of Q_d (Fig. 3). Therefore, if we are interested only in the form of Q, it is enough to limit (2.5) to its semiclassical counterpart Q_d.

C. $W(\alpha)$ function

We consider now the off-diagonal part for $\varphi=0$. As above, all terms involving λ^{-1} in $P(m,n)$ are set to 0. For $\lambda \lesssim 1$, it is simple to prove a reasonably good approximation of $W_i(\alpha)$ is given by

$$W_{\mathrm{ap}}(\alpha) = W_d^a(\alpha) + \frac{e^{-|\alpha|^2-2\lambda}}{\pi}[c_1|\alpha| + c_2|\alpha|^2(2|\alpha|^2 - 1)] \tag{4.13}$$

where $W_d^a(\alpha)$ is given by (3.8) with $c_1=4\lambda$, $c_2=\lambda^5\sqrt{2}$. The second term of (4.13) has been obtained by taking into account only the terms corresponding to $m=0, n=1$ and to $m=1, n=2$ in (2.14). Here again, if we are interested only in the form of W, it is enough to limit (2.12) to its semiclassical counterpart W_d (Fig. 4). For large values of λ, it can be seen in Fig. 5 that $W(\alpha)$ oscillates but as λ increases it tends to be positive $\forall\ \alpha \in \mathrm{Re}$. The oscillatory behavior becomes more pronounced when λ is fixed but the order r is increased as is shown in Fig. 6.

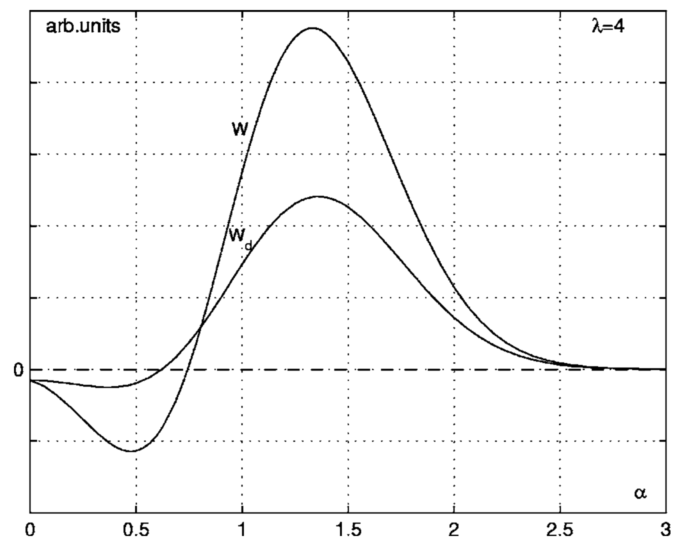

FIG. 4. Erlang process of order 2. The Wigner function $W(\alpha)$ is plotted versus α in dimensionless units with $\varphi=0$ for the average value of the photon number $\lambda=4$. The function $W_d(\alpha)$ is obtained when taking into account only diagonal elements of the density operator.

Let us now introduce a quantum model that can help to interpret the E2 process in terms of the state representation.

V. QUANTUM MODEL

Let $\mathcal{A}$ be a one-dimensional harmonic oscillator in the Hilbert space $\mathcal{H}_{\mathcal{A}}$ spanned by the set of the number states $\{|n\rangle_a\}$, and defined by the creation and annihilation operators $\hat{a}$ and $\hat{a}^\dagger$ with $[\hat{a},\hat{a}^\dagger]=1$ of Hamiltonian $\hat{H}_{\mathcal{A}}=\hbar\hat{a}^\dagger\hat{a}$. Let $\mathcal{B}$ be another one-dimensional harmonic oscillator in the Hilbert space $\mathcal{H}_{\mathcal{B}}:\{|k\rangle_b\}$ of Hamiltonian $\hat{H}_{\mathcal{B}}=\hbar\hat{b}^\dagger\hat{b}$. The systems $\mathcal{A}$ and $\mathcal{B}$ can be, for example, two modes of the harmonic oscillator.

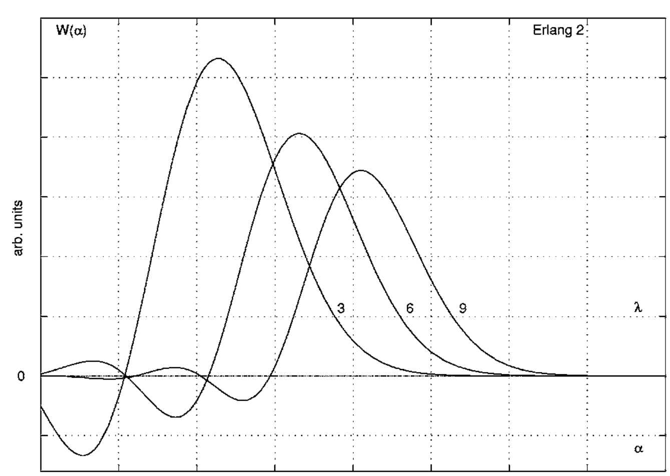

FIG. 5. Erlang processes of order 2. The Wigner function $W(\alpha)$ is plotted versus α in dimensionless units with $\varphi=0$ for various average values of the photon number $\lambda=3,6,9$.

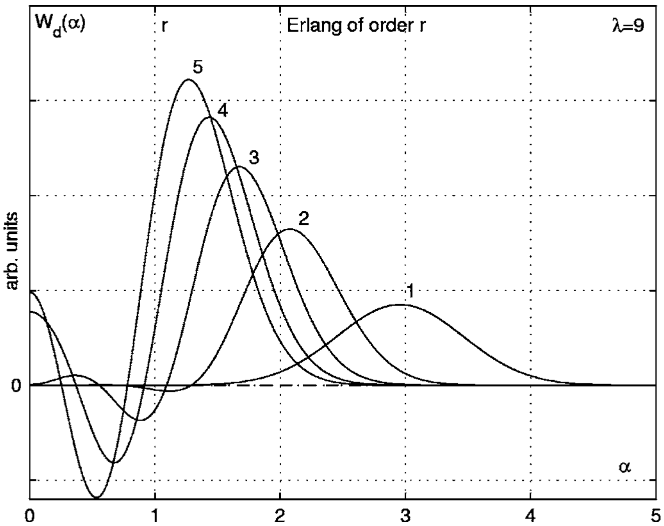

FIG. 6. Erlang processes of order r. $W_d(\alpha)$ functions versus α in dimensionless units where $\varphi=0$, the average value of the photon number $\lambda=9$, and for various values of $r=1,\ldots,5$. The function $W_d(\alpha)$ is obtained taking into account only diagonal elements of the density operator.

The Hamiltonian of the interaction in the Hilbert space $\mathcal{H}_{\mathcal{W}}:\mathcal{H}_{\mathcal{A}}\otimes\mathcal{H}_{\mathcal{B}}$, is modeled by

$$\hat{W}_{\mathcal{AB}}=\epsilon\hbar\,\hat{a}(\hat{b}^{\dagger})^k. \tag{5.1}$$

This model of interaction has already been suggested by Yuen for number states amplification [10]. We start with the initial state

$$|\psi(0)\rangle=|n\rangle_a\otimes|0\rangle_b \tag{5.2}$$

so that at time t, the state becomes

$$|\psi(t)\rangle=e^{-i\hat{H}t/\hbar}|\psi(0)\rangle, \tag{5.3}$$

$$\hat{H}=\hat{H}_{\mathcal{A}}+\hat{H}_{\mathcal{B}}+\hat{W}_{\mathcal{AB}}. \tag{5.4}$$

For $\hat{H}=\hat{W}_{\mathcal{AB}}$ and expanding the exponential in (5.3) up to the second order in t, it can be seen that

$$|\psi(t)\rangle=|n\rangle_a\otimes|0\rangle_b-i\epsilon t\sqrt{n_a}\sqrt{k_b!}|n-1\rangle_a\otimes|k\rangle_b - \frac{\epsilon^2t^2}{2}\sqrt{n_a(n_a-1)}\sqrt{(2k_b)!}|n-2\rangle_a\otimes|2k\rangle_b+\cdots. \tag{5.5}$$

In order to simplify the presentation, we limit ourselves to real functions. From $|\psi(t)\rangle$, it is therefore possible to generate the new states

$$|\phi(t)\rangle={}_a\langle n-2|\psi(t)\rangle. \tag{5.6}$$

Noticing that $\sqrt{(2k)!}=1/\sqrt{e}\langle\alpha_1|2k\rangle$ where $|\alpha_1\rangle$ is the coherent state of amplitude $\alpha=1$, we can form the final states

$$|\nu_k\rangle=\sqrt{e}\langle\alpha_1|2k\rangle|\phi\rangle=|2k\rangle \tag{5.7}$$

at a very short time $t_0=\sqrt{2}/\epsilon n_a\ll 1$ ($n_a\gg 1$), (the subscript b referring to the harmonic oscillator $\mathcal{B}$ being omitted). On the other hand, given the operators $\hat{c}$ and $\hat{c}^\dagger$ acting on the space $\mathcal{H}_{\mathcal{B}}$, we can evaluate $r_k=\langle k|\hat{c}^\dagger\hat{c}|k\rangle$.

These scalar values being calculated, we then simultaneously measure the operators $\hat{c}^\dagger\hat{c},\hat{c}^\dagger$, and $\hat{c}$ such that

$$\frac{1}{r_{2k}^2}(\langle\alpha|\hat{c}^\dagger\hat{c}|\nu_k\rangle)^2=e^{-\lambda}\frac{\mu^{2k}}{(2k)!}=p_1(k), \tag{5.8}$$

$$\frac{1}{r_{2k}}(\langle\alpha|\hat{c}|\nu_k\rangle)^2=e^{-\lambda}\frac{\mu^{2k-1}}{(2k-1)!}=p_2(k), \tag{5.9}$$

$$\frac{1}{r_{2k+1}}(\langle\alpha|\hat{c}^\dagger|\nu_k\rangle)^2=e^{-\lambda}\frac{\lambda^{2k+1}}{(2k+1)!}=p_3(k), \tag{5.10}$$

with $\lambda=\alpha^2$. Now since for an incoherent process probabilities add, we obtain

$$p(k)=p_1(k)+\frac{1}{2}[p_2(k)+p_3(k)] \tag{5.11}$$

which is the expression given by the Eq. (3.2).

VI. DISTANCE VARIATION DIFFERENCE

Several functions have been defined to measure the difference between classical and quantum systems. A distance function has recently been proposed [11]. We analyzed another criterion [12], called the distance variation difference which we derived from the one introduced in [13] for different purposes.

A. Definition and some properties

The distance variation difference $D(\lambda)$ is based on the variation distance of PDFs as

$$D(p_\psi;p_\alpha)\equiv D(\lambda)=\sum_n(\sqrt{p_\psi(n;\lambda)}-\sqrt{p_\alpha(n;\lambda)}), \tag{6.1}$$

where all the previous PDFs must be seen as functions of the average photon number $E[n]=\lambda$ (T, the interval of registering the RV n, is set to unity). Here $p_\psi(n;\lambda)$ is the PDF of n in the state $|\psi\rangle$, and $p_\alpha(n;\lambda)$ is the PDF of n in the coherent state $|\alpha\rangle$. This distance was constructed in analogy with the entropy difference criterion such that another convex function $\sqrt{f(z)}$ is substituted for $-f(z)\ln[f(z)]$ where $0\leqslant f(z)\leqslant 1$. Although $D(\lambda)$ obeys two properties of a distance function, it is not a distance because it can be negative. On the other hand, we can easily show that

(i) $$D(0)=0, \tag{6.2}$$

(ii) $D(\lambda)$ is a monotonically increasing (decreasing) function with λ ($p_\psi\neq p_\alpha$), (6.3)

(iii) $$\lim_{\lambda\to 0}\left(\sum_n\frac{\partial\sqrt{p_\alpha(n;\lambda)}}{\partial\lambda}\right)-\frac{1}{2\sqrt{\lambda}}=\frac{\sqrt{2}-1}{2}\triangleq d_\alpha, \tag{6.4}$$

(iv) $$D(\lambda)\geqslant\frac{1}{2}\|p_\psi(n;\lambda)-p_\alpha(n;\lambda)\|,\quad D(\lambda)\geqslant 0, \tag{6.5}$$

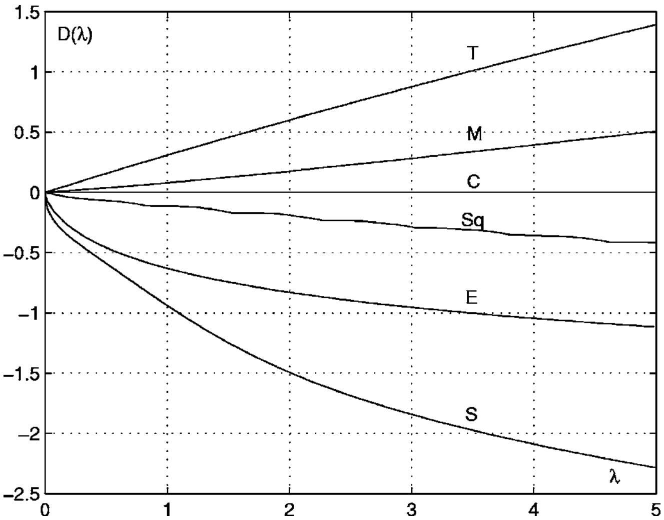

FIG. 7. Variation distance function D versus λ, the average value of the photon number, in dimensionless units for various states which are denoted T for thermal, M for mixture of coherent and thermal ($\varrho=0.1$), C for coherent, Sq for squeezed, E for Erlang of order 2, and S for superposition of two coherent states.

$$\text{(v)}\quad D(\lambda) \leqslant -\frac{1}{2}\|p_{\psi}(n;\lambda)-p_{\alpha}(n;\lambda)\|, \quad D(\lambda)\leqslant 0, \tag{6.6}$$

where $\|f\|$ denotes the norm ℓ_1 of f. The inequalities (iv) and (v) have been derived from the results demonstrated in [13].

Therefore for

$$d_{\psi} \triangleq \lim_{\lambda\to 0}\left(\sum_n \frac{\partial\sqrt{p_{\psi}(n;\lambda)}}{\partial\lambda}\right) - \frac{1}{2\sqrt{\lambda}},$$

we can state the criterion of nonclassicity as

$$C = d_{\psi} - d_{\alpha} < 0. \tag{6.7}$$

Let us apply this criterion to some classical and nonclassical examples of $|\psi\rangle$.

B. Classical states

As the first example of a classical state $|\psi\rangle$, let us consider a thermal light. The PDFs needed for the calculations are well known, $p_{\psi}(n;\lambda)=\lambda^n/(1+\lambda)^{n+1}$ and $p_{\alpha}(n;\lambda)=(\lambda^n/n!)e^{-\lambda}$. The exact results are plotted in Fig. 7 and the approximate expressions are obtained as

$$D_t(\lambda) \simeq \left(1-\frac{\sqrt{2}}{2}\right)\lambda + \frac{1}{2}\left(1-\frac{\sqrt{6}}{3}\right)\lambda^{3/2} + \frac{1}{4}\left(\sqrt{2}-1-\frac{\sqrt{6}}{3}\right)\lambda^2, \quad \lambda \lesssim 1. \tag{6.8}$$

It can easily be shown that Eq. (6.7) is positive.

Another example for which (6.7) is positive is given by a mixture of coherent and thermal states of PDF $p_{\psi}(n;\lambda)=(1-b)b^n e^{-\lambda(1-b)}L_n(-z)$ (see, for example, [14]) where we set $b=N_b/(1+N_b)$, $z=(1-b)^2\lambda/b$ and where the $L_n(z)$ are the Laguerre polynomials of degree n and of the variable z. Here, we need to choose $N_b=\varrho\lambda_c$ $(0\leqslant\varrho\leqslant 1)$, λ_c being the coherent part of the light. For moderate values of thermal component $\left(\varrho\lesssim\frac{1}{3}\right)$, it can be shown that

$$D_m(\lambda) \simeq \varrho\frac{\sqrt{2}}{2}\lambda + \varrho\left(\frac{\sqrt{6}}{2}-1\right)\lambda^{3/2}, \quad \lambda\lesssim 1. \tag{6.9}$$

Asymptotic behaviors in both cases vary as $\sqrt{\lambda}$. Results are plotted in Fig. 7 for such a mixture with a thermal part of moderate average photon number $\varrho=0.1$.

The deviation from a coherent state increases with the average value λ in both cases.

C. Nonclassical states

The first example of a nonclassical state we wish to treat is the E2 for which $p_{|\psi\rangle}(n;\lambda)$ is given by (3.2). The approximate expression for weak values of λ can easily be calculated as

$$D_e(\lambda) \simeq -\frac{\sqrt{2}}{2}\lambda + \left(\frac{1}{2}+\frac{\sqrt{6}}{3}\right)\lambda^{3/2} - \frac{1}{4}\left(1+\frac{\sqrt{6}}{3}-\sqrt{2}\right)\lambda^2, \tag{6.10}$$

for $\lambda \lesssim \frac{1}{4}$.

For a comparison with the previous case, we take for $|\psi\rangle$ a squeezed state with a squeezing parameter σ chosen so that the photon number variance $V_n=\lambda_c e^{-2\sigma}+(\sinh^2 2\sigma)/2$ is minimum, i.e., $\sigma=\sigma_s\simeq\frac{1}{6}\text{arg}\sinh(2\lambda_c)$, where λ_c is the average photon number of the coherent part and $\lambda=\lambda_c+\sinh^2\sigma_s$. We thus satisfy the condition $D(0)=0$. The corresponding PDF is [15]

$$p_{|\psi\rangle}(n;\lambda) = \frac{e^{-\lambda_c(1+\tanh\sigma_s)}}{\cosh\sigma_s}\frac{\left(\frac{\tanh\sigma_s}{2}\right)^n}{n!}H_n^2(z)$$

with $z^2=\lambda_c e^{\sigma_s}/\sinh(2\sigma_s)$ and where the $H_n(z)$ are the Hermite polynomials of degree n and of the variable $z\in\text{Re}$. An approximate expression has been obtained as

$$D_{sq}(\lambda) \simeq -\frac{\sqrt{2}}{6}\lambda + \frac{1}{3}\left(1-\frac{\sqrt{6}}{2}\right)\lambda^{\frac{3}{2}} + \frac{1}{2}\left(\sqrt{2}-\frac{1}{3}\right)\lambda^2, \quad \lambda\lesssim\frac{1}{4}. \tag{6.11}$$

The expression (6.11) has been obtained by limiting the sum of $\sqrt{p_{|\psi\rangle}(n;\lambda)}$ to only a few values of n and for $\sigma_s\simeq\lambda/3-2\lambda^3/9$.

The third example of a nonclassical state we deal with is the superposition of two coherent states. Its PDF is given by $p_{|\psi\rangle}(n;\lambda)=\{[1+(-1)^n]\lambda^n/(1+e^{-2\lambda})n!\}e^{-\lambda}$. Here again approximations are simple. One easily obtains

$$D_s(\lambda) \simeq -\sqrt{\lambda} + \frac{1}{2}\lambda + \frac{1}{2}\left(1 - \frac{\sqrt{6}}{3}\right)\lambda^{3/2} - \frac{1}{4}\left(\sqrt{2} + \frac{3}{2}\right)\lambda^2, \tag{6.12}$$

for $\lambda \lesssim \frac{1}{5}$.

On the other hand, for $\lambda \gg 1$, all the functions $D(\lambda)$ describing the nonclassical states considered here vary like $-\sqrt{\lambda}$.

The exact results obtained from numerical computations are displayed in Fig. 7 for all the cases.

The calculation of the C (6.7) yields the coefficients which are also given by $\lim_{\lambda\to 0}(\partial/\partial\lambda)D_\psi(\lambda)$. We get $C_c=0$, $C_t=1-\sqrt{2}/2$, $C_m=\varrho\sqrt{2}/2$, $C_{sq}=-\sqrt{2}/6$, $C_e=-\sqrt{2}/2$, and $C_s=-1$, for the coherent state, the thermal state, the mixture state, the squeezed state, the nonclassical state of Erlang type, and the superposition state, respectively: $C_s \leqslant C_e \leqslant C_{sq} \leqslant C_c \leqslant C_m \leqslant C_t$.

Finally, we want to point out that the approximations given here are reasonably good and at least they clearly show the sign and the variation of $D(\lambda)$.

VII. CONCLUDING REMARKS

A simple model for the generation of a state exhibiting nonclassical statistical properties is presented. The demonstration, both numerical and analytical, is based on the properties of a Wigner function which has a negative part. A model of the quantum density matrix is proposed to express the semiclassical approximation which has been found to be good enough to estimate the main features of both the Q and the Wigner functions. A criterion for testing the nonclassicity of the state is introduced. This criterion provides a scale for several types of states that have been selected to illustrate the theoretical results.

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