Stability, gain, and robustness in quantum feedback networks

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In this paper we are concerned with the problem of stability for quantum feedback networks. We demonstrate in the context of quantum optics how stability of quantum feedback networks can be guaranteed using only simple gain inequalities for network components and algebraic relationships determined by the network. Quantum feedback networks are shown to be stable if the loop gain is less than one—this is an extension of the famous small gain theorem of classical control theory. We illustrate the simplicity and power of the small gain approach with applications to important problems of robust stability and robust stabilization.

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I. INTRODUCTION

Stable operation is a fundamental prerequisite for the proper functioning of any technological system. Instability can cause some system variables to grow in magnitude without bound (or at least saturate or oscillate), with detrimental effects on performance and even damage. Consequently, methods for stability analysis and design have played an important role in the development of classical technologies. A significant early example was Watt's steam engine governor in the 1780s (subsequently analyzed by Maxwell in 1868) [1]. Indeed, one of the chief applications of feedback (but by no means the only application) is to stabilize systems that would otherwise be unstable. A striking example of this is the X29 plane [2], which has forward-swept wings and requires the use of a stabilizing feedback control system.

However, feedback *per se* does not guarantee stability: indeed, feedback can be "degenerative or regenerative—either stabilizing or destabilizing" [3]. In particular, when interconnections of stable components include components with active elements, instability can occur (such as when a microphone is placed too close to a loudspeaker). An additional requirement of considerable practical importance is that stable operation be maintained in the presence of uncertainty (e.g., due to model error and approximation, altered operating conditions, etc.) and noise—this is a basic robustness requirement.

Feedback is increasingly being used in the design of new technologies that include quantum components, e.g., [4–12]. In fact, a wide range of quantum technologies can be considered as networks of quantum and classical components that include cascade (feedforward [13], Chap. 12) and feedback interconnections. Since these networks may include components that are active, e.g., optical amplifiers or classical amplifiers, questions of network stability are of considerable importance. Quantum input-output theory started developing in the 1980s [14,15], however, general methods for stability analysis and design for quantum networks still do not appear to be readily available in the literature. Our purpose in this paper is to begin to address this gap; in particular, we show how the *small gain theorem* developed in the classical context by Sandberg, Zames, and others in the early 1960s (see, e.g., [3,16,17,19]) can be employed with ease and considerable power for stability analysis and the design of quantum networks.

There are many methods for stability analysis available for classical systems, as a glance at any control textbook will confirm [16–18]. For example, it is well known that linear continuous time invariant classical systems are asymptotically stable if all the poles of their transfer functions have negative real parts. The Nyquist criteria and root locus methods are widely used for stability analysis in simple single input single output (SISO) linear systems. Lyapunov methods are extremely powerful and apply when detailed state space (phase space) models (either linear or nonlinear) are available. The state-space representation uses a differential equation model of the system under consideration. A complementary approach, which employs input-output representations, treats network elements as black boxes, and describes the relationship between the inputs and outputs of each element. The small gain methods focus on the input-output properties of systems and form the basis for much of the work that has been done on robust control system design. The focus on signals entering and leaving network components rather than on the details of each component makes this technique extremely valuable for analyzing complex feedback networks.

In the small gain feedback stability framework, each component system (of the type shown in Fig. 1) is stable in the sense that bounded input signals produce bounded output signals. This concept of input-output stability is quantified using the notion of a (time-invariant) *gain* g. Roughly, if $\|x\|$ is a measure of the "size" of a time dependent signal $x(t)$, we say that the system is *bounded input bounded output (BIBO) stable* if $\|y\| \leq g\|u\|$, where u and y are, respectively, the input and output signals. Gains g less than one correspond to attenuation, while gains greater than one mean amplification. When BIBO stable components are interconnected in a feed-

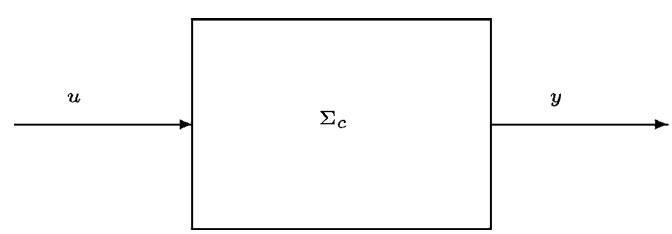

FIG. 1. A classical system with input u and output y.

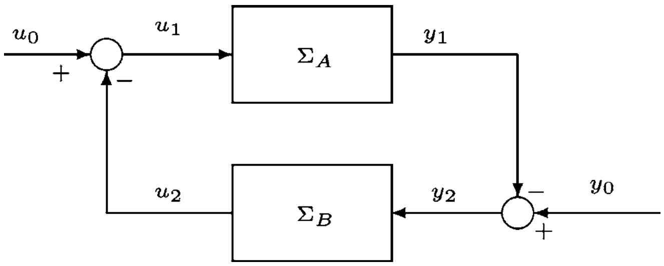

FIG. 2. A classical feedback network with inputs u_0, y_0 and internal network signals u_1, u_2, y_1, y_2.

back network, such as in the prototypical example network shown in Fig. 2, it is of importance to know when the network is BIBO stable, considered as a single system, with respect to inputs u_0, y_0, and outputs u_1, y_1, u_2, y_2, the internal network signals. The *small gain theorem* says that the network will be stable if the *loop gain* is less than one (the loop gain is the product of all the component gains going around the loop). The loop gain condition is therefore a sufficient but not, in general, necessary (due to the absence of phase information) criterion for stability.

In this paper we consider feedback networks of simple elements taken from quantum optics to illustrate the underlying principles of the small gain methodology in an important quantum technology setting. Our aim is to demonstrate how the small gain theorem can guarantee the stability of a complex quantum network using only simple gain inequalities for network components and algebraic relationships determined by the network. The quantum networks may include classical components. Loop gain analysis techniques are simple and powerful, and we can expect these and other methods for stability analysis and design to have many applications in future quantum technologies. One such application is to *robust stability*, which refers to the ability of a feedback network to remain stable in the presence of uncertainty, noise, and environmental influences (which may cause decoherence). Here, the environment is considered as a possibly active network component, parts of which may include unknown model errors, unmodeled dynamics, and noise sources.

The models we use are quantum stochastic differential equations [13,20], which provide excellent approximations to quantum optical systems and offer considerable power, with clear conceptions of input and output fields. The complete network and noise sources are described by an overall unitary evolution (interaction picture with respect to the noises). Signals are viewed in "ball and stick" terms, [21], and the signal size is described in mean square terms (average length).

The paper is organized as follows. In Sec. II we provide a review of the small gain theorem for classical systems, both deterministic and stochastic. As preparation for the small gain analysis of quantum optical networks, a discussion of the signals and components to be used is given in Sec. III, with particular emphasis on mean square gains. The main ideas concerning the small gain methodology for quantum networks are described in Sec. IV. Then, in Sec. V we give two examples illustrating important applications of the small gain theorem to robust stability analysis and design. In particular, we show in the second example how feedback can be used to increase robustness, so that the effect of environmental influences is reduced.

II. CLASSICAL SYSTEMS

In this section we review the meaning of mean square gain for classical systems and summarize the content of the small gain theorem for a classical feedback network.

A. Gain for classical systems

Consider the classical system Σ_c shown in Fig. 1, with input u and output y. The system could be nonlinear. The signals u and y are vector valued functions of time. Here we consider an *input-output* description, which does not include internal details; the system is an operator or function mapping input signals to output signals.

The system Σ_c is said to have *mean square gain* $g>0$ if g is finite and there exists a constant $\mu \geq 0$ (called the *bias*), such that

$$\int_0^t |y(s)|^2\, ds \leq \mu + g^2 \int_0^t |u(s)|^2\, ds, \tag{1}$$

for all input signals that are mean square finite (square integrable $u \in L^2[0,t]$) on the time interval $[0,t]$, and this should hold for all $t \geq 0$. If the system has an internal state variable x, then the bias is a function of the initial state x_0 in which case we may write $\mu(x_0)$.

The significance of the mean square gain property (1) is that it captures the important BIBO stability property of the system ([16], Chap. 6; [22], Chap. 4). In particular, if any mean square finite signal is applied to the system, then the system responds with a mean square finite output ($u \in L^2$ implies $y \in L^2$). If the system has an internal state x, then with additional properties like observability or detectability, the stability of $x(t)$ as $t \to \infty$ can be inferred.

B. The classical small gain theorem

Consider the classical feedback network shown in Fig. 2. The network has inputs u_0, y_0 and internal network signals u_1, u_2, y_1, y_2 (the term *internal* here refers to the network, not the internal details of each constituent component system). The classical components A and B are of the type Σ_c and satisfy the mean square gain inequality (1) with gains g_A, g_B and biases μ_A, μ_B, respectively.

We are interested in the *internal stability* of the network in the BIBO sense, meaning that mean square bounded input signals should produce mean square bounded internal signals. The *small gain theorem*, ([3]; [16], Theorem 6.6.1-1; [22], Theorem 4.15), addresses this question, and asserts that the network will be internally stable if the *loop gain* is strictly smaller than one. That is, if

$$g_A g_B < 1, \tag{2}$$

then

$$(1 - g_A g_B)\int_0^t |z(s)|^2\, ds$$

$$\leqslant c_1 + c_2\int_0^t (|u_0(s)|^2 + |y_0(s)|^2)ds, \qquad (3)$$

for suitable positive constants c_1 and c_2. Here, z is any of the internal signals u_1, u_2, y_1, y_2. Note that inequality (3) provides a meaningful bound on the internal signals only when the loop gain condition (2) holds. Inequality (3) is a quantification of the BIBO stability of the feedback network. In this section, we have only explicitly considered a network with two elements, but this generalizes in a straightforward way to any number of elements; similarly the results in this paper also apply more generally. Indeed, the small-gain theorem applies to multiple input multiple output (MIMO) systems, so that, in general, the signals u,y are signal vectors.

C. The classical stochastic case

In the case where Σ_c is a stochastic system and u and y are random signals, the mean square gain property (1) can be written in terms of expectations,

$$\mathbf{E}\left(\int_0^t |y(s)|^2\, ds\right) \leqslant \mathbf{E}\left(\mu + \lambda t + g^2\int_0^t |u(s)|^2\, ds\right), \qquad (4)$$

where $\lambda \geqslant 0$ is a non-negative constant (related to the variance of the signals). The small gain theorem also applies to a network of stochastic systems and guarantees internal stability under the same loop gain condition (2) ([23], Sec. III). In this situation, as $t \to \infty$ we must divide the inequality (4) by t to obtain a bound on the output signal power in terms of the input signal power.

In inequality (4) we have used $\mathbf{E}$ to denote the classical expectation. However, in the remainder of the paper we use the notation $\langle\cdot\rangle$ for both classical and quantum expectations.

III. QUANTUM OPTICAL NETWORK SIGNALS AND COMPONENTS

In this paper we are interested in feedback interconnections of quantum systems, such as the fully quantum feedback loop of Fig. 9, later, or the quantum-classical network of Fig. 10, later. Our purpose in this section is to provide a careful description of the signals and components in these networks. This description will focus on the input-output relations that are needed to facilitate the stability analyses given in subsequent sections. Standard models will be used to derive these relations. This material may be familiar to some readers.

Electromagnetic fields will be used as the basic carriers of quantum information between quantum network components. We use the quantum stochastic models as in [13], [20] to describe the fields, which in some situations below also serve to model heat baths. The models are defined on a Hilbert space, capturing all components and signals. Dynamics are described by a unitary operator $\mathbf{U}(t)$, so that if X is a system operator of a network component, it evolves in time according to $X(t)=\mathbf{U}^\dagger(t)X\mathbf{U}(t)$, which solves a quantum Langevin equation (QLE); see, e.g., [13], Eq. (11.2.33). System operators for distinct components will commute ($[X_1(t),X_2(t)]=\mathbf{U}^\dagger(t)[X_1,X_2]\mathbf{U}(t)=0$), and satisfy the nondemolition condition [24], [13], [25]

$$[B_{noise}(t),X] = [B^\dagger_{noise}(t),X] = 0, \quad \forall\, t \geqslant 0, \qquad (5)$$

where B_{noise} is any purely input noise term in the network or applied to the network.

Typically, internal quantum signals [e.g., of the form (12) later] will be comprised of a zero-mean noise term, and another term, denoted $\beta(t)$, which may contain system operators from network components. The operator function $\beta(t)$, in general, satisfies commutation relations of the form

$$[\beta(t),\beta^\dagger(t)] = c, \qquad (6)$$

for a suitable number c.

We use a common notation for quadratures:

$$\beta_r = \beta + \beta^*, \quad \beta_i = \frac{\beta - \beta^*}{i}. \qquad (7)$$

The commutation relation (6) implies $[\beta_r,\beta_i]=2ci$, and

$$|\beta(t)|^2 \overset{\triangle}{=} \beta_r^2(t) + \beta_i^2(t) = 4\beta^\dagger(t)\beta(t) + 2c, \qquad (8)$$

where the (rectangular) modulus notation on the left hand side is defined by the right. In the special case that β is a complex number, $|\beta|^2=\beta_r^2+\beta_i^2=4\beta^*\beta$, a consequence of the convention (7).

We often need to take expectations of the various quantities appearing in the networks. Expectations will always be taken with respect to the full state. In general, we do not make explicit the state in the notation. We will also use classical signals, such as electric currents, when classical components are used.

A. Signals and fields

1. Vacuum fields

Calculations (such as expectations) involving fields in this paper will be all be carried out relative to underlying vacuum fields (more generally squeezed states with positive temperature could also be considered, though we do not here for simplicity). These fields will be represented by annihilation operators $b(t)$, which satisfy the canonical commutation relations (CCR),

$$[b(t),b^\dagger(t')] = \delta(t-t'). \qquad (9)$$

We write

$$B(t) = \int_0^t b(s)ds, \qquad (10)$$

and use the Ito sense increment $dB(t)=B(t+dt)-B(t)=b(t)dt$. In the vacuum state $|0\rangle$, all Ito products are zero, except $dB(t)dB^\dagger(t)=dt$ ([13], Chap. 11).

The real and imaginary quadratures of the field are defined as

$$Q(t) = B(t) + B^\dagger(t),$$

$$P(t) = \frac{B(t) - B^\dagger(t)}{i}. \tag{11}$$

Both of these quadratures have zero mean and variance t [Ito's rule here reads as $dQ(t)dQ(t)=dt$, $dP(t)dP(t)=dt$, $dQ(t)dP(t)=i\,dt$, $dP(t)dQ(t)=-i\,dt$].

Each noise quadrature is self-adjoint [e.g., $Q^\dagger(t)=Q(t)$], and self-commutative (e.g., $[Q(t),Q(s)]=0$ for all s,t). As a consequence, each quadrature is individually stochastically equivalent to a classical Wiener process ([26], Sec. 5.2.1). The quadratures $Q(t)$ and $P(t)$, of course, do not commute.

2. Coherent and displaced fields

A coherent state $|\beta\rangle$ of a field is characterized by a classical complex-valued function of time $\beta(t)$. The coherent state is determined by $|\beta\rangle = W(\beta)|0\rangle_{in}$, where $W(\beta)$ is the Weyl displacement operator, [20,25]. To facilitate simple computations involving QLEs, the field in the coherent state is equivalent to an effective field b_{eff} displaced relative to a field operator $b(t)$ in the vacuum state, and so we write

$$B_{\text{eff}}(t) = \int_0^t \beta(s)ds + B(t); \tag{12}$$

see, e.g., [13], Eq. (9.2.47).

More generally, we will need to consider fields of the form (12), but with quantum operator β, as mentioned above. Note that (12) is a basic "signal or message plus noise" model, or "ball and stick" model [21] (semimartingale in mathematical terms).

The field quadratures are defined as in (11), and satisfy

$$dQ_{\text{eff}}(t) = \beta_r(t)dt + dQ(t),$$

$$dP_{\text{eff}}(t) = \beta_i(t)dt + dP(t), \tag{13}$$

where β_r and β_i are given by (7).

The root mean square size of the field b_{eff} on a time interval $[0,t]$ is defined to be

$$\|\beta\|_t = \sqrt{\int_0^t \langle|\beta(s)|^2\rangle ds}, \tag{14}$$

which, for an optical field, is a measure of the intensity. The quantum expectation takes into account the fluctuations due to quantum noise.

3. Classical signals

Classical signals, such as currents and voltages, occur in electro-optical networks. These signals are of the "ball and stick" form,

$$dy(t) = \beta_y(t)dt + dw_y(t), \tag{15}$$

where $\beta_y(t)$ is a classical stochastic process and $w_y(t)$ is a Wiener process. This signal has root mean square size,

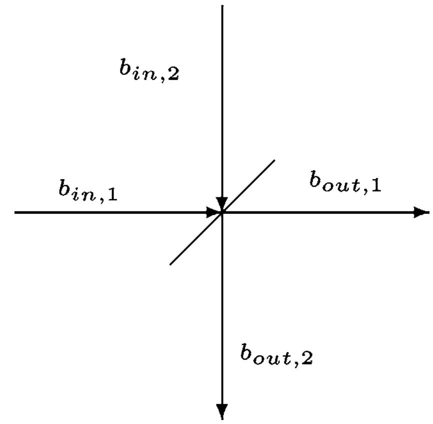

FIG. 3. A beamsplitter with inputs $b_{in,1}$, $b_{in,2}$ and outputs $b_{out,1}$, $b_{out,2}$.

$$\|\beta_y\|_t = \sqrt{\int_0^t \langle|\beta_y(s)|^2\rangle ds}, \tag{16}$$

on a time interval $[0,t]$. We remind the reader that we use $\langle\cdot\rangle$ to denote both quantum and classical expectations.

B. Some quantum network components and their gains

In this section we review some components that are commonly used in quantum optics, and pay particular attention to their gain properties. These devices will be used in the networks considered later in the paper.

1. Beamsplitters

A beamsplitter is shown in Fig. 3. The input-output relations are ([21], Sec. 5.1),

$$b_{out,1} = \varepsilon b_{in,1} - \delta b_{in,2}$$

$$b_{out,2} = \delta b_{in,1} + \varepsilon b_{in,2}, \tag{17}$$

where

$$\varepsilon^2 + \delta^2 = 1. \tag{18}$$

The parameters ε and δ describe the levels of transmission and attenuation for the field channels. Note that the output fields satisfy the CCRs (9).

2. Cavities

We consider resonant optical cavities, with a simple first order dynamical model ([21], Sec. 5.3; [13], Chap. 12; [27], Sec. III). The cavity is weakly coupled to a resonant external field, with parameter γ, about a nominal frequency. The QLE for the cavity annihilation operator $a(t)$ in the interaction picture is

$$da(t) = -\frac{\gamma}{2}a(t)dt - \sqrt{\gamma}dB_{in}(t). \tag{19}$$

If the input field b_{in} is taken to be a displaced field, of the form (12), with operator-valued $\beta(t)$,

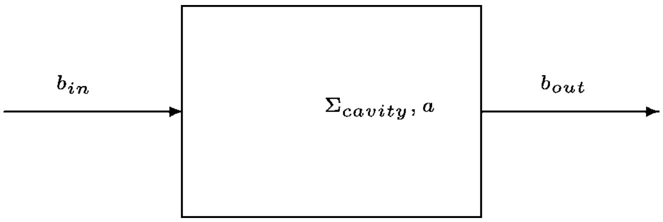

FIG. 4. A network representation of an optical cavity with annihilation operator a, showing input b_{in} and output b_{out} fields.

$$B_{in}(t) = \int_0^t \beta(s)ds + B(t), \tag{20}$$

i.e.,

$$dB_{in}(t) = \beta(t)dt + dB(t),$$

then the output field is given by

$$dB_{out}(t) = \sqrt{\gamma}a(t)dt + dB_{in}(t) = \beta_{out}(t)dt + dB(t), \tag{21}$$

where the output operator β_{out} is given by

$$\beta_{out}(t) = \sqrt{\gamma}a(t) + \beta(t). \tag{22}$$

Consider the cavity quadratures

$$q = a + a^{\dagger}, \quad p = \frac{a - a^{\dagger}}{i},$$

and the output field quadratures

$$dQ_{out}(t) = \beta_{out,r}(t) + dQ(t),$$

$$dP_{out}(t) = \beta_{out,i}(t) + dP(t), \tag{23}$$

where

$$\beta_{out,r}(t) = \sqrt{\gamma}q(t) + \beta_r(t),$$

$$\beta_{out,i}(t) = \sqrt{\gamma}p(t) + \beta_i(t). \tag{24}$$

Then

$$dq(t) = \left(-\frac{\gamma}{2}q(t) - \sqrt{\gamma}\beta_r(t)\right)dt - \sqrt{\gamma}dQ(t), \tag{25}$$

and similarly,

$$dp(t) = \left(-\frac{\gamma}{2}p(t) - \sqrt{\gamma}\beta_i(t)\right)dt - \sqrt{\gamma}dP(t). \tag{26}$$

Consider now the mean squares of cavity quadratures. By the completion of squares in the above, we have

$$\begin{aligned} dq^2(t) &= [dq(t)]q(t) + q(t)dq(t) + dq(t)dq(t) \\ &= [-\beta_{out,r}^2(t) + \beta_r^2(t) + \gamma]dt - 2\sqrt{\gamma}q(t)dQ(t). \end{aligned} \tag{27}$$

(see Fig. 4) Combining this with the analogous expression for $dp^2(t)$, and taking the integral of the expectations gives

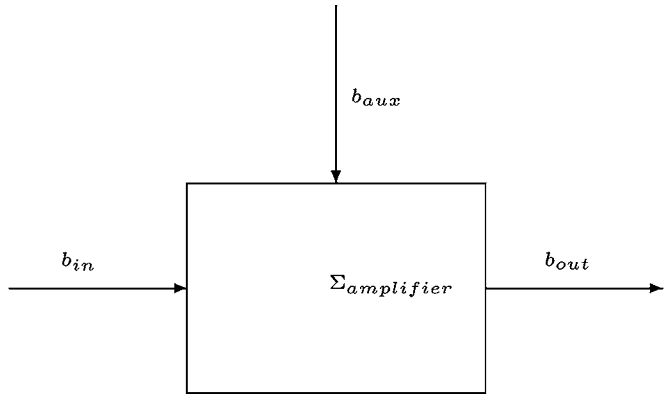

FIG. 5. An optical amplifier or attenuator with gain g, showing input b_{in} and output b_{out} fields, and the auxiliary input b_{aux}.

$$\begin{aligned} &\langle q^2(t) + p^2(t)\rangle + \int_0^t \langle|\beta_{out}(s)|^2\rangle ds \\ &\quad = \langle q^2(0) + p^2(0)\rangle + \int_0^t \langle|\beta(s)|^2\rangle ds + \lambda t, \end{aligned} \tag{28}$$

where $\lambda = 2\gamma$ [recall the modulus notation (8)]. This is equivalent to a statement of energy conservation for the cavity, and implies

$$\|\beta_{out}\|_t^2 \leq \mu_0 + \lambda t + \|\beta\|_t^2, \tag{29}$$

where $\mu_0 = \langle q^2(0) + p^2(0)\rangle$. This inequality is of the form of the classical gain inequality (4), with gain $g = 1$.

3. Amplifiers and attenuators

An optical amplifier or attenuator is shown in Fig. 5. We consider first the amplifier, and consider its gain. We use the inverted temperature model described in [13], Chap. 7 for a linear amplifier. The model consists of a gain medium in an optical cavity, described by an inverted heat bath b_{aux}, which provides the gain. The input field b_{in} is also coupled to the cavity mode and an amplified output field b_{out} is produced. Of necessity, this process introduces noise, as documented in [13], Chap. 7 and [28]. In general, this includes additive noise due to fluctuations in the inverted heat bath, which is independent of the input signal, as well as multiplicative noise, which amplifies the fluctuations in the input signal.

For an input field b_{in} of the form (20), the amplifier model is described by

$$da(t) = \frac{\gamma}{2}a(t)dt - \sqrt{\gamma}dB_{aux}(t) - \frac{\kappa}{2}a(t)dt - \sqrt{\kappa}dB_{in}(t) \tag{30}$$

and

$$dB_{out}(t) = \beta_{out}(t)dt + dB(t), \tag{31}$$

where

$$\beta_{out}(t) = \sqrt{\kappa}a(t) + \beta(t). \tag{32}$$

The energy required for the gain is provided by the inverted heat bath field b_{aux}, which has a nonzero Ito product,

$$dB_{aux}^{\dagger}(t)dB_{aux}(t) = dt. \quad (33)$$

We assume $\kappa > \gamma > 0$ for stable amplifier operation.

Calculations analogous to those in Sec. III B 2 show that

$$d\langle q^2(t) + p^2(t)\rangle = \{\langle -(\kappa-\gamma)[q^2(t)+p^2(t)] - \sqrt{\kappa}[\beta_r(t)q(t) + q(t)\beta_r(t)] + \beta_i(t)p(t) + p(t)\beta_i(t)\rangle + \lambda\}dt, \quad (34)$$

where $\lambda = 2(\kappa+\gamma)$. Define

$$\tilde{\beta}_{out}(t) = \sqrt{\kappa-\gamma}\,a(t) + \sqrt{\frac{\kappa}{\kappa-\gamma}}\beta(t),$$

which is related to the output operator β_{out} by

$$\beta_{out}(t) = \sqrt{\frac{\kappa}{\kappa-\gamma}}\tilde{\beta}_{out}(t) - \frac{\gamma}{\kappa-\gamma}\beta(t). \quad (35)$$

Then, by completion of squares, (34) implies

$$\int_0^t \langle|\tilde{\beta}_{out}(s)|^2\rangle ds \leqslant \langle q^2(0) + p^2(0)\rangle + \frac{\kappa}{\kappa-\gamma}\int_0^t \langle|\beta(s)|^2\rangle ds + \lambda t, \quad (36)$$

and using the relation (35) and some calculations we arrive at the mean square gain inequality cf. (4),

$$\|\beta_{out}\|_t^2 \leqslant \mu + \lambda t + g^2\|\beta\|_t^2, \quad (37)$$

where the gain g is given by

$$g = \frac{\kappa+\gamma}{\kappa-\gamma} > 1, \quad (38)$$

and μ and λ are suitable constants, [33]. The inequality (37) is an upper bound on the size of the output signal, with the additional noise amplified by $\nu = \sqrt{g^2-1}$. We note that a lower bound on the output signal has been derived previously, e.g., [28], which highlights the quantum limit on noise in linear amplifiers.

Attenuators can be analyzed similarly. The auxiliary field b_{aux} is a standard (noninverted) heat bath, to facilitate loss. A mean square gain inequality of the form (37) also holds, but with gain

$$g = \left|\frac{\gamma-\kappa}{\gamma+\kappa}\right| < 1, \quad (39)$$

and noise amplification $\nu = \sqrt{1-g^2}$. When, as here, the gain is smaller than one, attenuation occurs.

We mention also that idealized static models for amplifiers and attenuators are described in [13], Chap. 7, and take the form

$$b_{out}(t) = g b_{in}(t) - \nu b_{aux}(t), \quad (40)$$

where

$$g^2 + \sigma\nu^2 = 1, \quad (41)$$

and

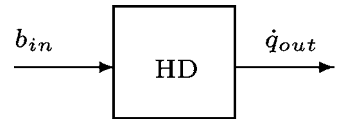

FIG. 6. Homodyne detection of the real quadrature q_{in} of an input field b_{in} produces a current $\dot{q}_{out}$.

$$\sigma = \begin{cases} +1, & \text{if } 0 < g < 1 \text{ (attenuator)}, \\ -1, & \text{if } g > 1 \text{ (amplifier)}. \end{cases} \quad (42)$$

In the case that $g=1$, this model reduces to that for a resonant optical cavity.

4. Quadrature measurement

A schematic representation of the measurement of the real quadrature $Q_{in}(t)$ of the field $B_{in}(t)$ given by (12) is shown in Fig. 6.

The detection scheme (homodyne) produces a photocurrent $\dot{q}_{out}$, which can be described by the Ito equation,

$$dq_{out}(t) = \tilde{\beta}_r(t)dt + d\tilde{w}(t), \quad (43)$$

where $\tilde{w}(t)$ is a standard Wiener process and $\tilde{\beta}_r(t)$ is related to the real quadrature $\beta_r(t)$ in the input field (12). The basic inequality for root mean square values before (for the entire field β) and after (just for the measured quadrature $\tilde{\beta}_r$) measurement is

$$\|\tilde{\beta}_r\|_t \leqslant \|\beta\|_t. \quad (44)$$

Hence the gain of the quadrature measurement is not more than one.

We remark that the photocurrent can be represented in two ways. We consider $B_{in}(t)$ to be an output of some quantum component. The quadrature $Q_{in}(t)$ is a self-commutative quantum stochastic process, statistically equivalent to the classical quantity $q_{out}(t)$. The driving noise quadrature $Q(t)$ is also self-commutative, and statistically equivalent to a standard classical Wiener processes. So $\tilde{w}(t)$ and $\tilde{\beta}_r(t)$ can be interpreted as the classical statistical equivalents of dQ and β_r. This gives the first of the two representations. The second representation depends on quantum filtering and stochastic master equation considerations, and views $\tilde{w}(t)$ as the innovation process, also a standard Wiener process but distinct from the one in the first interpretation. In this case, the process $\tilde{\beta}_r(t)$ is a quantity $\check{\beta}_r(t)$ obtained from a conditional stochastic master equation. [34]

5. Modulators

The final component we use in this paper is the modulator, shown in Fig. 7 ([21], Sec. 5.4.3).

We shall assume that the modulator has gain no more than one:

$$\|\beta\|_{t,(out)} \leqslant \|\beta\|_{t,(in)}. \quad (45)$$

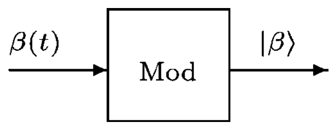

FIG. 7. An electro-optical modulator produces a field b_{out} in a coherent state $|\beta\rangle$ (with intensity $|\beta|^2$) from a classical signal $\beta(t)$.

IV. THE SMALL GAIN METHODOLOGY FOR QUANTUM OPTICAL NETWORKS

In this section we describe the small gain methodology for stability analysis to quantum optical networks. The procedure is to first determine the gain of each of the components, and the algebraic relationships among the signals. Each component could be of the form shown in Fig. 8, with inputs and outputs being displaced signals of the form

$$dU(t) = \beta_u(t)dt + dB_u(t), \tag{46}$$

$$dY(t) = \beta_y(t)dt + dB_y(t). \tag{47}$$

What is important is a mean square gain inequality of the form

$$\|\beta_y\|_t^2 \leqslant \mu + \lambda t + g^2 \|\beta_u\|_t^2, \tag{48}$$

which emphasizes input-output properties of the components. The dynamics of the component are unspecified, and need not be linear—the only stipulation is that the dynamics are those of a valid quantum system. In general, the nonnegative number λ depends on the noise variances. Note that components may be active elements, or may dissipate energy, and while connections to external heat baths would be involved, these are not necessarily shown explicitly.

For a closed network the loop gain is the product of the gains going around a loop. We will demonstrate in this section that if the loop gain is less than one then all internal network signals are mean square bounded in terms of the mean square inputs. The feedback network is then called *internally stable*.

If one is also interested in the behavior of internal component variables (system operators), then additional information is needed. This information is often available from explicit physical models.

A. A quantum feedback network

We consider now the stability of the fully quantum feedback loop shown in Fig. 9, consisting of two components with gains g_A, g_B, respectively, linked by two beamsplitters

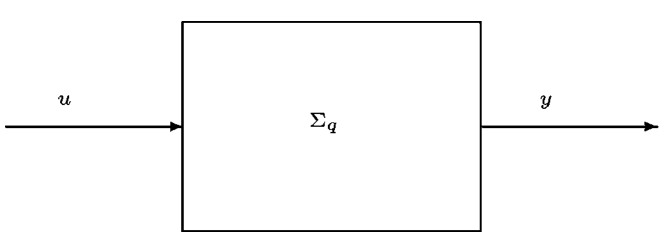

FIG. 8. A quantum system with input u and output y.

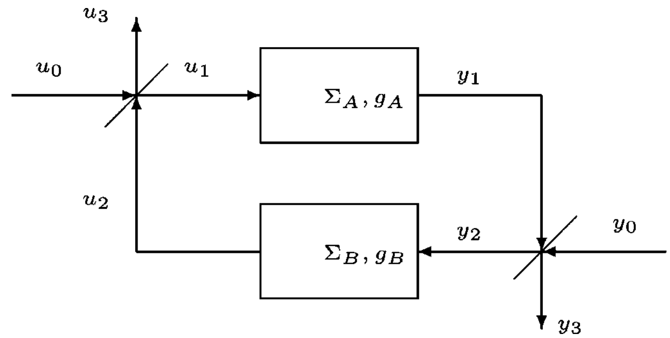

FIG. 9. A fully quantum feedback loop, consisting of two quantum components with gains g_A and g_B linked by two beamsplitters.

with attenuation parameters δ_A, δ_B, and transmissivity parameters ε_A, ε_B, respectively. The small gain technique will be used to show that the network is *internally stable* when the loop gain is strictly less than one. The meaning of this statement will become clear in what follows.

1. Network inputs and internal signals

The input fields u_0, y_0 are displaced fields given by

$$U_0(t) = \int_0^t \beta_{u_0}(s)ds + B_{u_0}(t), \tag{49}$$

$$Y_0(t) = \int_0^t \beta_{y_0}(s)ds + B_{y_0}(t), \tag{50}$$

where β_{u_0}, β_{y_0} are operator valued and B_{u_0}, B_{y_0} are vacuum fields.

The internal network signals, u_1, u_2, y_1, y_2, will be displaced signals of the form (12), and we use notation analogous to (49), (50).

2. Small gain theorem

We now prove a quantum version of the small gain theorem for the network of Fig. 9, assuming that the following loop gain condition holds:

$$\delta_A \delta_B g_A g_B < 1. \tag{51}$$

The loop gain here is the product $\delta_A g_A \delta_B g_B$, which takes into account the attenuation due to the beamsplitters. We shall show that

$$\|\beta\|_t \leqslant C(1 + \sqrt{t} + \|\beta_{u_0}\|_t + \|\beta_{y_0}\|_t), \tag{52}$$

for some positive constant C, where β corresponds to any of the internal network signals β_{u_1}, β_{y_2}, β_{y_1}, β_{u_2}.

3. Network analysis

The beamsplitter equations (17) imply

$$u_1 = \varepsilon_A u_0 - \delta_A u_2,$$

$$u_3 = \delta_A u_0 + \varepsilon_A u_2, \tag{53}$$

and

$$y_2 = \varepsilon_B y_0 - \delta_B y_1,$$

$$y_3 = \delta_B y_0 + \varepsilon_B y_1. \quad (54)$$

We go around the loop clockwise. The signal u_1 exiting the top-left beamsplitter gives

$$U_1(t) = \varepsilon_A U_0(t) - \delta_A U_2(t) = \int_0^t \beta_{u_1}(s)ds + B_{u_1}(t), \quad (55)$$

where

$$\beta_{u_1}(t) = \varepsilon_A \beta_{u_0}(t) - \delta_A \beta_{u_2}(t), \quad (56)$$

$$dB_{u_1}(t) = \varepsilon_A\, dB_{u_0}(t) - \delta_A\, dB_{u_2}(t). \quad (57)$$

Note that β_{u_1} is a quantum signal since β_{u_0} and β_{u_2} are quantum signals. The output of component Σ_A is of the form

$$Y_1(t) = \int_0^t \beta_{y_1}(s)ds + B_{y_1}(t), \quad (58)$$

and is related to the input by the gain inequality

$$\| \beta_{y_1} \|_t^2 \leqslant \mu_A + \lambda_A t + g_A^2 \| \beta_{u_1} \|_t^2. \quad (59)$$

Similarly, considering the bottom-right beamsplitter and component Σ_B, we have

$$\| \beta_{u_2} \|_t^2 \leqslant \mu_B + \lambda_B t + g_B^2 \| \beta_{y_2} \|_t^2, \quad (60)$$

where

$$\beta_{y_2}(t) = \varepsilon_B \beta_{y_0} - \delta_B \beta_{y_1}(t). \quad (61)$$

4. *Small gain calculations*

Continuing with the derivation of the small gain bounds, we first consider the field u_1. From (56), and using the triangle inequality

$$\| c_1\beta_1 + c_2\beta_2 \|_t \leqslant |c_1|\, \| \beta_1 \|_t + |c_2|\, \| \beta_2 \|_t, \quad (62)$$

we have

$$\| \beta_{u_1} \|_t \leqslant \epsilon_A \| \beta_{u_0} \|_t + \delta_A \| \beta_{u_2} \|_t. \quad (63)$$

Going around the loop anticlockwise, we consider the field u_2. From (60), and using the inequality

$$|c|^2 + g^2 \| \beta \|_t^2 \leqslant (|c| + g \| \beta \|_t)^2, \quad (64)$$

to obtain the root mean square, we have

$$\| \beta_{u_2} \|_t \leqslant c_B(t) + g_B \| \beta_{y_2} \|_t, \quad (65)$$

where $c_B(t) = \sqrt{\mu_B + \lambda_B t}$. Substituting (65) into (63),

$$\| \beta_{u_1} \|_t \leqslant \epsilon_A \| \beta_{u_0} \|_t + \delta_A c_B(t) + \delta_A g_B \| \beta_{y_2} \|_t. \quad (66)$$

Next, we consider the field y_2. Using the relations (61) and (62), we have

$$\| \beta_{y_2} \|_t \leqslant \epsilon_B \| \beta_{y_0} \|_t + \delta_B \| \beta_{y_1} \|_t. \quad (67)$$

Substituting (67) into (66) gives

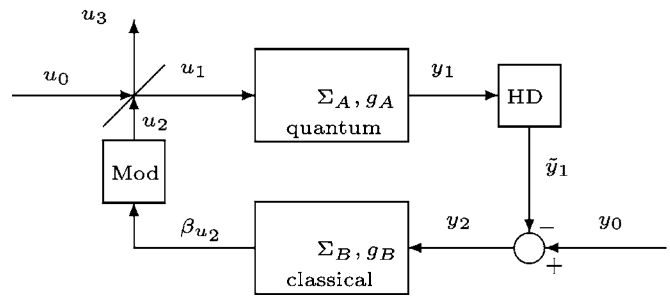

FIG. 10. A quantum-classical feedback loop, consisting of a quantum component Σ_A with gain g_A and a classical (e.g., electronic) device Σ_B with gain g_B linked using optoelectronic connections. The quantum to classical transition (measurement) is via homodyne detection (HD), while the modulator (Mod) turns the classical signal β_{u_2} into a coherent state $|\beta_{u_2}\rangle$, shown as the field u_2.

$$\| \beta_{u_1} \|_t \leqslant \delta_A c_B(t) + \epsilon_A \| \beta_{u_0} \|_t + \delta_A \epsilon_B g_B \| \beta_{y_0} \|_t + \delta_A \delta_B g_B \| \beta_{y_1} \|_t. \quad (68)$$

Finally, we consider the field y_1. From (59) and (64), we have

$$\| \beta_{y_1} \|_t \leqslant c_A(t) + g_A \| \beta_{u_1} \|_t, \quad (69)$$

where $c_A(t) = \sqrt{\mu_A + \lambda_A t}$. Substituting (69) into (68) and rearranging, yields the desired bound (52) for field u_1,

$$(1 - \delta_A \delta_B g_A g_B) \| \beta_{u_1} \|_t \leqslant \delta_A [c_B(t) + \delta_B g_B c_A(t)] + \epsilon_A \| \beta_{u_0} \|_t + \delta_A \epsilon_B g_B \| \beta_{y_0} \|_t. \quad (70)$$

The other internal fields in Fig. 9 can be bounded in a similar manner.

B. A quantum-classical feedback network

In this section we derive a small gain theorem for the quantum-classical feedback network shown in Fig. 10. Stability will be assessed relative to a quantum input field u_0 and a classical noise source y_0.

1. *Network inputs*

We begin by considering the two inputs. The input field u_0 is a displaced field given by (49), where B_{u_0} is a vacuum field. The classical input signal $\dot{y}_0$ is given by

$$dy_0(t) = \beta_{y_0}(t)dt + dw_{y_0}(t), \quad (71)$$

where β_{y_0} is a classical real-valued signal, and w_{y_0} is a standard Wiener process.

2. *Network analysis*

Next, we present explicitly the equations and inequalities for the network, working around the loop clockwise, beginning with the beamsplitter. The output field of the modulator is given by

$$U_2(t) = \int_0^t \beta_{u_2}(s)ds + B_{u_2}(t), \qquad (72)$$

where $\beta_{u_2}(t)$ is the amplitude of the coherent state $|\beta_{u_2}\rangle$ created by the classical output of Σ_B, and B_{u_2} is a vacuum field. The signal u_1 from the beamsplitter is

$$U_1(t) = \varepsilon U_0(t) - \delta U_2(t) = \int_0^t \beta_{u_1}(s)ds + B_{u_1}(t) \qquad (73)$$

where

$$\beta_{u_1}(t) = \varepsilon\beta_{u_0}(t) - \delta\beta_{u_2}(t), \qquad (74)$$

$$dB_{u_1}(t) = \varepsilon dB_{u_0}(t) - \delta dB_{u_2}(t). \qquad (75)$$

Note that β_{u_1} is, in general, operator valued. The output of the quantum component Σ_A is of the form (58) with gain inequality (59).

Next, from (43) the output of the quadrature measurement (HD) is the classical signal

$$d\tilde{y}_1(t) = \tilde{\beta}_{y_1,r}(t)dt + dw_1(t), \qquad (76)$$

where w_1 is a standard Wiener process $[dw_1(t)dw_1(t)=dt]$. From (44) the corresponding root mean square inequality is

$$\|\tilde{\beta}_{y_1,r}\|_t \leq \|\beta_{y_1}\|_t. \qquad (77)$$

The classical system Σ_B has input

$$dy_2(t) = -d\tilde{y}_1(t) + dy_0(t) = \beta_{y_2}(t)dt + dB_{y_2}(t), \qquad (78)$$

where

$$\beta_{y_2}(t) = \beta_{y_0}(t) - \tilde{\beta}_{y_1,r}(t), \qquad (79)$$

$$dB_{y_2}(t) = dw_{y_0}(t) - dw_1(t), \qquad (80)$$

and is assumed to have mean square gain g_B:

$$\|\beta_{u_2}\|_t^2 \leq \mu_B + \lambda_B t + g_B^2 \|\beta_{y_2}\|_t^2. \qquad (81)$$

The final part of the loop is the modulator (Mod), for which we have

$$\|\beta_{u_2}\|_{t,(out)} \leq \|\beta_{u_2}\|_{t,(in)}, \qquad (82)$$

using (45).

3. Small gain theorem

We will show that the loop gain condition,

$$\delta g_A g_B < 1, \qquad (83)$$

implies that if the inputs u_0 and y_0 are mean square bounded, then any signal β in the feedback loop of Fig. 10 will be mean square bounded, in the sense that there exists a positive constant $C>0$, such that

$$\|\beta\|_t \leq C(1 + \sqrt{t} + \|\beta_{u_0}\|_t + \|\beta_{y_0}\|_t). \qquad (84)$$

We first consider the field u_1. Using the triangle inequality (62) and applying the modulator inequality (82), we have

$$\|\beta_{u_1}\|_t \leq \epsilon \|\beta_{u_0}\|_t + \delta \|\beta_{u_2}\|_t. \qquad (85)$$

Going around the loop anticlockwise, we consider the classical signal β_{u_2}. From (81), and using the inequality (64) to obtain the root mean square, we have

$$\|\beta_{u_2}\|_t \leq c_B(t) + g_B \|\beta_{y_2}\|_t, \qquad (86)$$

where $c_B(t)=\sqrt{\mu_B+\lambda_B t}$. Substituting (86) into (85),

$$\|\beta_{u_1}\|_t \leq \epsilon \|\beta_{u_0}\|_t + \delta c_B(t) + \delta g_B \|\beta_{y_2}\|_t. \qquad (87)$$

Next, we consider the classical signal y_2. Using the inequality (62) to bound (79), we have

$$\|\beta_{y_2}\|_t \leq \|\beta_{y_0}\|_t + \|\tilde{\beta}_{y_1,r}\|_t. \qquad (88)$$

Substituting (88) into (87), and applying the measurement inequality (77) gives

$$\|\beta_{u_1}\|_t \leq \delta c_B(t) + \epsilon \|\beta_{u_0}\|_t + \delta g_B \|\beta_{y_0}\|_t + \delta g_B \|\beta_{y_1}\|_t. \qquad (89)$$

Finally, we consider the field y_1, which is the output of the quantum component Σ_A, with mean square gain inequality (59). From (59) and the inequality (64), we have

$$\|\beta_{y_1}\|_t \leq c_A(t) + g_A \|\beta_{u_1}\|_t, \qquad (90)$$

where $c_A(t)=\sqrt{\mu_A+\lambda_A t}$. Substituting (90) into (89) and rearranging yields the required bound (84) for field u_1,

$$(1 - \delta g_A g_B) \|\beta_{u_1}\|_t \leq \delta[c_B(t) + g_B c_A(t)] + \epsilon \|\beta_{u_0}\|_t + \delta g_B \|\beta_{y_0}\|_t. \qquad (91)$$

The other internal fields in Fig. 10 can be bounded in a similar manner.

V. APPLICATIONS TO ROBUST STABILITY ANALYSIS AND DESIGN

In this section we illustrate two of the important applications of the small gain methodology, namely robust stability analysis and design. The basic issue addressed is *robust stability*. The first example (Sec. V A) analyzes the condition on the maximum allowable external disturbance for which a nominal feedback network remains stable. In the second example (Sec. V B), we design a controller that improves the robustness of an open oscillator e.g., an atom trapped in an optical cavity.

A. A simple quantum feedback loop

Consider the feedback loop shown in Fig. 11, known as the *nominal system*. The network consists of a quantum component Σ_q with gain g [i.e., a mean square inequality of the form (48) holds]. The algebraic relationship of the network signals is determined by the beamsplitter as

$$u_1 = \varepsilon u_0 - \delta y_1, \qquad (92)$$

where, as usual, $\varepsilon^2+\delta^2=1$. The loop gain is $g\delta$, and so we assume these parameters have been chosen so that

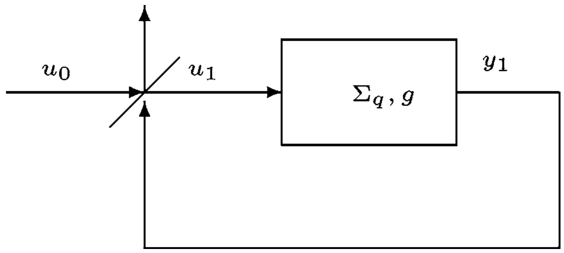

FIG. 11. The nominal quantum feedback loop.

$$g\delta < 1. \tag{93}$$

By the small gain theorem, this means that this system is internally stable.

In reality, such a system will interact with an external environment. One way of modeling the *actual system* is shown in Fig. 12. The system Σ_Δ is meant to represent the way in which the environment interacts with the system Σ_q, specifically via the feedback loop, as shown, in a way that may depend on the variables of Σ_q. We assume that Σ_Δ itself is mean square stable with gain g_Δ. The specific robust stability question here is as follows: does the feedback loop remain stable under the influence of the environment?

We use the small gain theorem to answer this question. First, the network signals are related by the equations

$$\tilde{u}_1 = \varepsilon u_0 - \delta \tilde{u}_2,$$

$$u_1 = \varepsilon_u \tilde{u}_1 - \delta_u u_2,$$

$$y_2 = \varepsilon_y y_1 - \delta_y b_y,$$

$$\tilde{u}_2 = \delta_y y_1 + \varepsilon_y b_y. \tag{94}$$

Next, the gain of the lower part of the network in Fig. 12, from input u_2 to output y_2, can be determined to be

$$g_{u_2 \to y_2} \leqslant \frac{\delta_u \epsilon_y g}{1 - \epsilon_u \delta_y (g\delta)} = g_{max}. \tag{95}$$

The small gain theorem implies the internal stability of the actual quantum feedback loop if

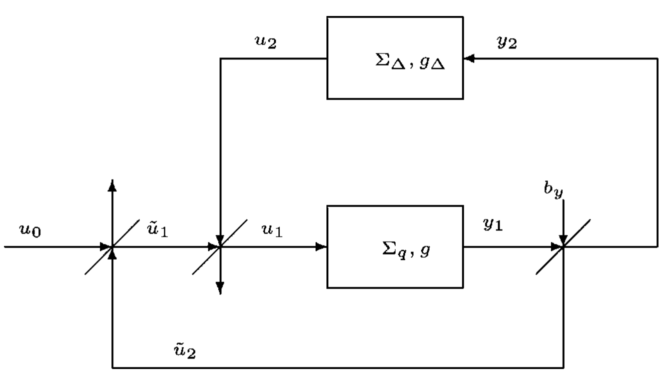

FIG. 12. The "actual" quantum feedback loop, showing the effect of the environment Σ_Δ.

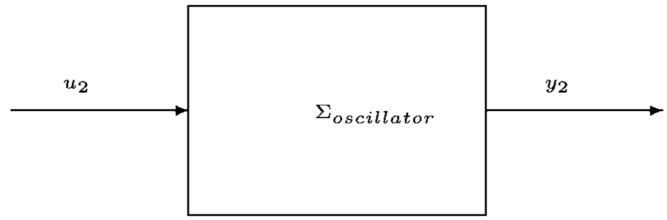

FIG. 13. The open harmonic oscillator, showing the input u_2 and output y_2 fields that provide the coupling to an environment Σ_Δ (not shown).

$$g_\Delta \cdot g_{u_2 \to y_2} < 1, \tag{96}$$

i.e.,

$$g_\Delta < (g_{max})^{-1}. \tag{97}$$

Inequality (97) gives a bound on the maximum allowable influence of the environment Σ_Δ on the quantum feedback loop that preserves stability, in terms of g, δ, ε_u, δ_u, ε_y, δ_y. We note that if the feedback loop is removed i.e., let $\delta=0$, so that $\tilde{u}_2$ has no effect on Σ_q, then (97) reduces to $g_\Delta < (\delta_u \varepsilon_y g)^{-1}$, as expected, by applying the small gain theorem to the loop formed by Σ_q and Σ_Δ.

In general, however, the beamsplitter parameters ε_u, δ_u, ε_y, δ_y, specifying how the environment interacts with the feedback loop may not be known. This can be dealt with by setting a stricter condition than (97) on the maximum allowable gain of the environment,

$$g_\Delta < \frac{1 - g\delta}{g} < (g_{max})^{-1}, \tag{98}$$

which depends only on the nominal parameters g and δ. This implies that robust stability is assured, regardless of beamsplitter parameters, provided the environmental influence has a gain g_Δ satisfying condition (98).

B. Robust stabilization of an open harmonic oscillator

In this section we consider the open quantum harmonic oscillator shown in Fig. 13.

The harmonic oscillator has a Hamiltonian,

$$H = q^2 + p^2, \tag{99}$$

where p, q are the oscillator quadratures, with the same form as the cavity quadratures defined in Sec. III B 2. We have chosen units for which $\hbar=1$, and set other parameters to unity for simplicity. The real quadrature of the oscillator, q, is coupled to an external field u_2, by an operator $L_2=\sqrt{\kappa}q$. The resulting QLEs for the oscillator quadratures are

$$dq(t) = 4p(t)dt,$$

$$dp(t) = -4q(t)dt - 2\sqrt{\kappa}dU_{2,i}(t), \tag{100}$$

and the commutation relations are, as before, $[q,p]=2i$, $[a,a^\dagger]=1$. These are the equations of motion of a noisy oscillation, with average motion of frequency 2 rad per second (when u_2 is zero mean). Such models have been used in the

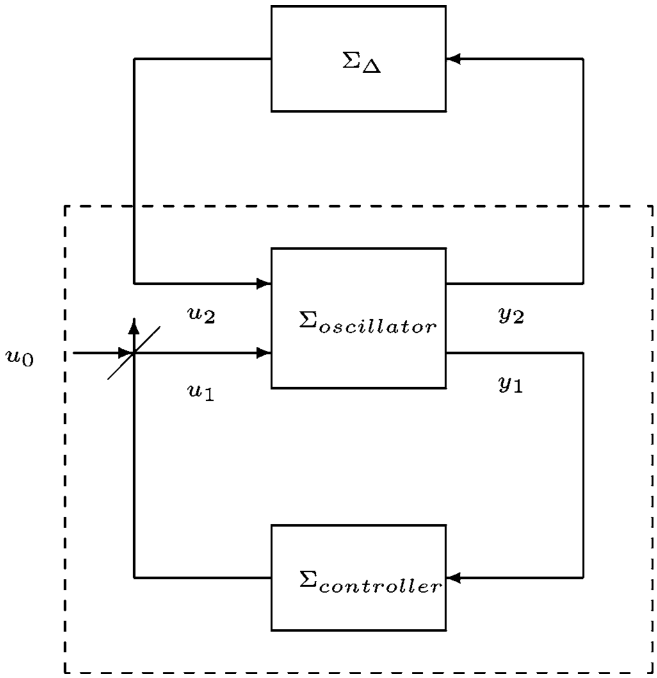

FIG. 14. The open harmonic oscillator, with coupling to the environment Σ_Δ and quantum feedback control. The dashed box indicates the feedback loop that we are constructing to improve the robustness of the oscillator.

literature e.g., to model an atom trapped in an optical cavity [7,30]. It is important to note that this coupled oscillator has only marginally stable dynamics (its poles lie on the imaginary axis), so it does not have a finite mean square gain. Therefore this system is very susceptible to environmental influence.

Suppose that the input u_2 due to the environment is a displaced field of the form

$$dU_2(t) = \beta_{u_2}(t)dt + dB_{u_2}(t). \tag{101}$$

The corresponding output channel y_2 is given by

$$dY_2(t) = \beta_{y_2}(t)dt + dB_{u_2}(t), \tag{102}$$

where

$$\beta_{y_2}(t) = \sqrt{\kappa}q(t) + \beta_{u_2}(t). \tag{103}$$

In addition to the effect of the quantum noise dB_{u_2}, we need to address the potentially disruptive influence of the operator β_{u_2}, which may depend in some way on oscillator quadratures, via feedback from the environment Σ_Δ. We therefore consider the problem of robustly stabilizing this open oscillator. By this we mean the problem of constructing a feedback loop that will ensure stability, and tolerate as much environmental influence as possible.

As a first step, we need to establish a channel to mediate the desired feedback, which employs a quantum system as a controller, and a beamsplitter, as shown in Fig. 14.

We assume that it is possible to establish a coupling to a second field channel u_1 via the operator $L_1=\sqrt{\gamma}a$, where $2a=q+ip$. The field u_0 entering the beamsplitter is assumed to be in the vacuum state. We seek a controller such that the gain of the oscillator-controller system from u_2 to y_2 is "small"—by the small gain theorem, this allows for a larger gain of the environment that ensures the system remains mean square stable. We increase robustness in this way, by reducing the effect of the environment on the oscillator.

The equations of motion for the oscillator quadratures coupled to both external fields u_1, u_2, but with no controller present (i.e., no feedback loop, $\beta_{u_1}=0$), are

$$dq(t) = \left(-\frac{\gamma}{2}q(t) + 4p(t)\right)dt - \sqrt{\gamma}dU_{1,r}(t),$$

$$dp(t) = \left(-\frac{\gamma}{2}p(t) - 4q(t)\right)dt - \sqrt{\gamma}dU_{1,i}(t) - 2\sqrt{\kappa}dU_{2,i}(t). \tag{104}$$

It can be seen that the second coupling to u_1 provides damping, which stabilizes the oscillator and implies that the feedback loop has a finite mean square gain. Indeed, we can calculate

$$d\langle q^2(t) + p^2(t)\rangle = \langle -\gamma[q^2(t) + p^2(t)] - 2\sqrt{\kappa}[\beta_{u_2,i}(t)p(t) + p(t)\beta_{u_2,i}(t)] + \lambda_0\rangle dt, \tag{105}$$

where $\lambda_0=4\kappa+2\gamma$, which shows the damping effect of this coupling. The gain from u_2 to y_2 without any controller is estimated from (102) and (105) to be

$$g_{u_2\to y_2,(\text{no fb})} \leqslant \frac{\kappa}{\gamma} + 1. \tag{106}$$

As expected, this inequality captures the intuitive idea that the effect of the environment depends on the strength of the two couplings, and in fact is proportional to the ratio κ/γ. In particular, if the control channel coupling γ is small relative to the environment channel coupling κ, this gain will be large, which results in low robustness since the maximum gain of the environment allowable for stable operation will be small.

We now consider the effect of including feedback control. The output channel y_1 corresponding to an input $dU_1 = \beta_{u_1} dt + dB_{u_1}$ is given by

$$dY_1(t) = \beta_{y_1}(t)dt + dB_{u_1}(t), \tag{107}$$

where

$$\beta_{y_1}(t) = \sqrt{\gamma}a(t) + \beta_{u_1}(t). \tag{108}$$

We choose the controller to be an optical amplifier or attenuator with gain g, and the beamsplitter parameters (ϵ, δ), such that $\delta g < 1$. Therefore, using the feedback loop of Fig. 14 and the steady-state model for an amplifier in Sec. III B 3 (used here for simplicity) implies that

$$\beta_{u_1} = \delta g\ \beta_{y_1}. \tag{109}$$

Combining (108) and (109), we find that

$$\beta_{u_1} = \frac{\delta g}{1-\delta g}\sqrt{\gamma}a = \sqrt{\gamma}Ga, \quad (110)$$

where $G = \delta g/(1-\delta g) > 0$. Thus, using convention (7),

$$\beta_{u_1,r} = \sqrt{\gamma}Gq \quad \text{and} \quad \beta_{u_1,i} = \sqrt{\gamma}Gp. \quad (111)$$

To see the effect of this feedback, we include the nonzero β_{u_1} terms to recalculate [cf. Eq. (105)]

$$d\langle q^2(t)+p^2(t)\rangle = \langle -\gamma[q^2(t)+p^2(t)] - 2\sqrt{\kappa}[\beta_{u_2,i}(t)p(t) + p(t)\beta_{u_2,i}(t)] + \lambda_2 - \sqrt{\gamma}[\beta_{u_1,r}(t)q(t) + q(t)\beta_{u_1,r}(t) + \beta_{u_1,i}(t)p(t) + p(t)\beta_{u_1,i}(t)]\rangle dt, \quad (112)$$

where λ_2 is a suitable constant. Substituting in the feedback terms from (111) gives

$$d\langle q^2(t)+p^2(t)\rangle = \langle -\gamma(1+2G)[q^2(t)+p^2(t)] - 2\sqrt{\kappa}[\beta_{u_2,i}p(t) + p(t)\beta_{u_2,i}(t)] + \lambda_2\rangle dt. \quad (113)$$

The gain from u_2 to y_2 with the controller in place is estimated from (102) and (113) to be less than

$$g_{u_2\to y_2,(\mathrm{fb})} \leqslant \frac{\kappa}{\gamma(1+2G)} + 1 < \frac{\kappa}{\gamma} + 1. \quad (114)$$

The feedback increases the effective coupling rate γ, so that this gain is smaller than the gain without feedback (106), and therefore improves the robustness of the oscillator against environmental influence.

The design parameters δ and g can be chosen in some appropriate manner. Also, a good design will also need to take into account the effects of added noise, which are reflected in the constant λ_2. Indeed, one could use a degenerate parametric amplifier ([13], Sec. 7.2.9) in place of the amplifier used previously, to avoid additional amplifier noise by carefully selecting the appropriate quadrature gains.

The design procedure used here is a Lyapunov technique that exploits a *passivity* property of the open and damped oscillator. Such passivity-based control design techniques are well known in control engineering, e.g., [31,32], and have been used recently with success in quantum feedback control, e.g., [11].

We note that it is also possible to consider the use of a classical controller for robust stabilization, although we do not here.

VI. CONCLUSIONS

In this paper we have demonstrated that the small gain theorem is applicable to the stability analysis of quantum feedback networks. These networks may include classical components. While we have focused on specific examples involving quantum optical elements, the general principles should be apparent. We have also applied these principles to problems of robust stability analysis and design. We expect the small gain theorem and other stability methods will be useful for the design of quantum technologies. Future work will include the further development and application of stability methods to quantum networks.

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[33] The gain g is given in [13], Eq. (7.2.13) derived using a steady-state approximation. It also equals the H^{∞} norm [17] of the amplifier frequency response $G(i\omega)=(\gamma+\kappa-2i\omega)/(\gamma-\kappa-2i\omega)$ ([13], Eq. (7.2.17)).
[34] $\check{\beta}_r(t)$ is the *conditional expectation* of $\beta_r(t)$ ([29], Sec. 3.4).