Matter-wave solitons supported by dissipation

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We show how long-lived self-localized matter waves can exist in Bose-Einstein condensates with a nonlinear dissipative mechanism. The ingredients leading to such structures are a spatial phase generating a flux of atoms toward the condensate center and the dissipative mechanism provided by the inelastic three-body collisions in atomic Bose-Einstein condensates. The outcome is a striking example of nonlinear structure supported by dissipation.

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I. INTRODUCTION

The realization of Bose-Einstein condensation with ultracold atomic gases has opened the door for many spectacular realizations of matter waves. In particular, elastic two-body collisions between condensed atoms provide an effective nonlinear interaction within the atomic cloud. These nonlinear interactions have been used to obtain different types of self-localized matter waves [1–3] denoted as matter-wave solitons.

Most of the proposed and all of the experimentally realized solitons in ultracold atomic gases lead to self-localization only along one spatial dimension. In the remaining directions these structures must be externally confined by either magnetic or optical means [4,5]. Truly multidimensional solitons cannot be supported by the nonlinear interactions arising from elastic two-body collisions alone since nonlinearity is not able to robustly balance dispersion in these scenarios [6,7].

It has been theoretically predicted that in Bose-Einstein condensates (BECs) in which Feschbach resonance management is used to modulate periodically the scattering length certain stable two-dimensional solitons may exist, the so-called *stabilized solitons* [8–10]. However, the idea does not extend trivially to three spatial dimensions since these structures become less stable [11,12], or require the addition of extra (external) confining potentials [5,10].

Another physical effect coming from the interaction between atoms in the condensate is dissipation. Three-body collisions usually lead to expulsion of atoms from the condensate leading to an effective dissipation which is thought to be responsible for its finite lifetime.

Dissipation usually acts against self-localization since it tends to take the system closer to the linear situation where no stable nonlinear structures exist. However, in some contexts dissipation has been shown to play a stabilizing role. In Ref. [13] the addition of a phenomenological Landau damping to the Feschbach-resonance-managed model has been shown to enhance stability. Another example is nonspreading (linear) wave packets with external imaginary potential [14].

In this paper we present nonlinear structures in Bose-Einstein condensates self-trapped by the effect of nonlinear *dissipative terms*. The physical idea behind this paper is that imprinting an appropriate spatially dependent phase on a BEC leads to a flux of particles from its periphery to the center, which compensates the particles lost by three-body inelastic collisions. The outcome is a long-lived nonlinear structure supported by dissipation.

The structure of the paper is as follows. In Sec. II we write our model equations (Sec. II A) and obtain the equations for stationary solutions as well as their asymptotic solutions (Sec. II B) and also compute numerically the profiles of these solutions (Sec. II C). In Sec. III we discuss how stationary solutions may exist in a system with dissipation (and without *gain*) and how to construct physically interesting objects from these mathematical solutions. In Sec. IV we propose several ideas which could be useful in order to generate these multidimensional solitary waves. Finally in Sec. V we discuss our results and summarize our conclusions.

II. MODEL EQUATIONS AND STATIONARY SOLUTIONS

A. Mean-field model

We will work in the mean-field approximation in which a BEC is modeled by the Gross-Pitaevskii equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + V\Psi + \frac{4\pi\hbar^2 a_s}{m}|\Psi|^2\Psi + i\Gamma_3|\Psi|^4\Psi \quad (1)$$

where V accounts for any external trapping potential, a_s is the s-wave scattering length for elastic two-body collisions, and $\Gamma_3=-\hbar K_3/12$, K_3 being the thresholdless three-body recombination rate [15]. Equation (1) has been used as a model to study collapse in BECs with attractive interactions [16–23]. However, the nonlinear structures to be studied in this paper are not related to collapse phenomena, a fact that we will stress by working out examples with $a_s=0$ or even $a_s>0$. Models similar to Eq. (1) arise in the propagation of optical beams in Kerr media with multiphoton absorption processes [24,25].

First, we change to the dimensionless variables $\boldsymbol{r}\equiv\boldsymbol{x}/a_0$, $\tau\equiv t/T$ $(\nu_0=1/T)$, and $\psi\equiv\sqrt{a_0^3}\Psi$, where $\nu_0=\hbar/a_0^2 m$, and a_0

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is a characteristic size of the BEC so that Eq. (1) becomes (from now on we set $V=0$)

$$i\frac{\partial\psi}{\partial\tau} = -\frac{1}{2}\Delta\psi + g|\psi|^2\psi + i\gamma|\psi|^4\psi, \tag{2}$$

with $g=4\pi a_s/a_0$ and $\gamma=-K_3/(12\nu_0 a_0^6)$.

B. Equations for stationary solutions and asymptotic behavior

Stationary nontrivial solutions of Eq. (2) with $\gamma=0$ have been discussed extensively in the literature since they correspond to the simplest model of classical superfluids (see, e.g., recent work in Ref. [28]). In the context of trapped BECs there have also been many studies of stationary solutions of these equations with $\gamma=0$ beyond the simplest ground-state and vortex solutions [26,27,29,30].

To obtain stationary self-trapped solutions of Eq. (2) we first write its solutions in the modulus-phase representation,

$$\psi(\boldsymbol{r},\tau) = A(\boldsymbol{r},\tau)\exp[i\phi(\boldsymbol{r},\tau)], \tag{3}$$

with $A(\boldsymbol{r},\tau)>0$. Then Eqs. (2) become

$$\frac{\partial(A^2)}{\partial\tau} = -\boldsymbol{\nabla}\cdot(A^2\,\boldsymbol{\nabla}\,\phi) + 2\gamma A^6, \tag{4a}$$

$$\frac{\partial\phi}{\partial\tau} = \frac{1}{2}\left(\frac{1}{A}\Delta A - (\boldsymbol{\nabla}\phi)^2\right) - gA^2. \tag{4b}$$

Stationary solutions of Eq. (2) satisfy

$$\partial_\tau A^2 = 0. \tag{5}$$

Then, from Eq. (4b) we get

$$\phi(\boldsymbol{r},\tau) = \varphi(\boldsymbol{r}) - \delta\tau, \tag{6}$$

with $\delta>0$ being obviously the chemical potential in adimensional units. Rescaling the spatial variables with $\boldsymbol{\rho}\equiv\sqrt{2\delta}\boldsymbol{r}$ we find

$$-2(\boldsymbol{\nabla}_{\boldsymbol{\rho}}A)(\boldsymbol{\nabla}_{\boldsymbol{\rho}}\phi) - A\Delta_{\boldsymbol{\rho}}\phi + \frac{\gamma}{\delta}A^5 = 0, \tag{7a}$$

$$\Delta_{\boldsymbol{\rho}}A - (\boldsymbol{\nabla}_{\boldsymbol{\rho}}\phi)^2 A - \frac{g}{\delta}A^3 + A = 0. \tag{7b}$$

In this paper we will focus on solutions with radially symmetric amplitude $A(\boldsymbol{\rho})=R(\rho)$. In quasi-two-dimensional situations we will study solutions with phase given by $\varphi(\boldsymbol{\rho})=\Phi(\rho)+m\theta$, where $\theta=\arctan(\rho_2/\rho_1)$ and $\boldsymbol{\rho}=(\rho_1,\rho_2)$, i.e., including a possible vorticity. In three spatial dimensions we will restrict our attention to spherically symmetric stationary solutions of the form $\varphi(\boldsymbol{\rho})=\Phi(\rho)$. Then, Eq. (7a) can be integrated by using the divergence theorem and we get

$$\Phi'(\rho) = \frac{\gamma}{R^2\rho^{d-1}\delta}\int_0^\rho r^{d-1}R(r)^6\,dr. \tag{8}$$

Equation (8) allows us to obtain the asymptotic behavior of the phase,

$$\Phi'(\rho) \approx \begin{cases} \dfrac{\gamma}{2\delta}R^4(\rho)\rho & \text{when } \rho \ll 1 \quad (9a) \\ Q/[\rho^{d-1}R^2(\rho)] & \text{when } \rho \gg 1, \quad (9b) \end{cases}$$

where we assume that the quantity Q defined as

$$Q = \frac{\gamma}{\delta}\int_0^\infty r^{d-1}R^6\,dr \tag{10}$$

is a finite quantity, an assumption that will be justified later.

The amplitude equation (7b) becomes

$$R'' + \frac{d-1}{\rho}R' - \left((\Phi')^2 + \frac{m^2}{\rho^2}\right)R - \frac{g}{\delta}R^3 + R = 0, \tag{11}$$

plus $R(\infty)=0$ and $R(0)=R_0$ $(m=0)$ or $R^{(|m|)}(0)>0$ $(m\neq 0)$, where $R^{(|m|)}(\rho)$ is the $|m|$th derivate of $R(\rho)$. In the latter case, it is easy to obtain $R(\rho)\sim R^{|m|}(0)\rho^{|m|}$ for small ρ. For large ρ we get

$$R'' + \frac{d-1}{\rho}R' + qR \simeq 0, \tag{12}$$

which gives

$$R(\rho) \sim 1/\rho^{(d-1)/2} \tag{13}$$

asymptotically.

To obtain the following order in the approximation of $R(\rho)$ for large ρ we define $h(\rho)=R(\rho)\rho^{(d-1)/2}$, which satisfies the Ermakov-Pinney equation [31]

$$h'' + h = Q^2/h^3 \tag{14}$$

whose solution can be found in the explicit form

$$R(\rho) \sim \frac{\sqrt{(Q^2/C_1^2)\cos^2\rho + (C_1\cos\rho + C_2\sin\rho)^2}}{\rho^{(d-1)/2}}, \quad \rho \gg 1. \tag{15}$$

When $d=2$ this result is similar to that of Ref. [25] with a different definition of Q. Equation (15) guarantees the finiteness of Q, which was assumed previously.

C. Numerical study of the stationary solutions

We have solved numerically Eqs. (7a) and (11) by using a standard shooting algorithm to find specific stationary configurations. Some results for $g=0$ with $m=0$ [Figs. 1(a) and 2] and $m=1$ [Fig. 1(b)] are shown as a function of the shooting parameter $\alpha=\delta/\gamma R^4(0)$. Solutions are found above a certain threshold value $\alpha>\alpha_{th}=0.54, 2.93, 0.36$ corresponding to the two-dimensional (2D) solution without and with vorticity ($m=1$), and three-dimensional (3D) solution, respectively. When the effect of the nonlinearity is included (see Fig. 3) we obtain a compression of the solutions for $g<0$ [Fig. 3(a)] and an expansion leading to slower decay for $g>0$ [Fig. 3(b)]. The threshold α_{th} is lowered when $g<0$ and increased when $g>0$ there being a maximum positive value $g_*=\delta/R^2(0)$ above which stationary solutions do not exist. It is remarkable that stationary solutions supported by dissipation can exist even when the scattering length is positive. In

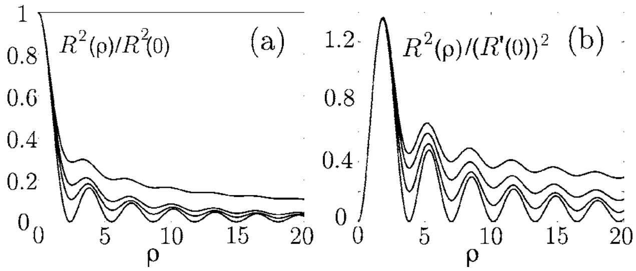

FIG. 1. (a) Radial profile of stationary solutions of Eq. (11) with $g=0$ and $\alpha=\delta/[\gamma R^4(0)]=0.55,0.65,0.8,\infty$ (from higher to lower ones). (b) Stationary radial profile of vortex solutions with $m=\pm 1$ for $\delta/\gamma[R'(0)]^4=2.9,3.2,4.1,\infty$.

three spatial dimensions we find a similar behavior with a faster decay of the amplitude (Fig. 4). In all cases the oscillatory behavior of the solutions and their decay are well described by Eq. (15).

III. PHYSICAL INTERPRETATION, REALISTIC IMPLEMENTATION, AND STABILITY

How can stationary solutions exist in a system with dissipation? From the previous asymptotic analysis $R(\rho) \sim 1/\rho^{(d-1)/2}$; thus we can estimate the number of particles as

$$N \sim \int_0^\infty r^{d-1} R^2 dr = \infty. \tag{16}$$

Moreover,

$$\frac{d}{d\tau} N = 2\gamma \int_{\mathbb{R}^d} |\psi|^6 d\mathbf{r} < \infty. \tag{17}$$

Intuitively speaking, it is the fact that the solution has an infinite number of particles together with a *finite* particle loss given by the right-hand side of Eq. (17) which allows for the existence of stationary solutions even in the presence of three-body recombination losses.

A coment is in order. In dissipationless systems and in the absence of trapping one would expect the far field to be defined by the chemical potential. However, in our case the only posible value of the far field is $R(\infty)=0$ since otherwise the right-hand side of Eq. (17) would be unbounded and thus such stationary solutions could probably not exist.

However, realistic solutions must have a finite number of particles. A simple way to construct realistic quasistationary

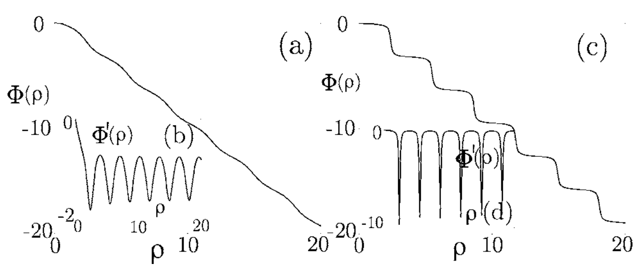

FIG. 2. Phase and its derivative for solutions of Eq. (11) with $g=0$ and (a), (b) $\alpha=\delta/[\gamma R^4(0)]=0.65$; (c), (d) 5.

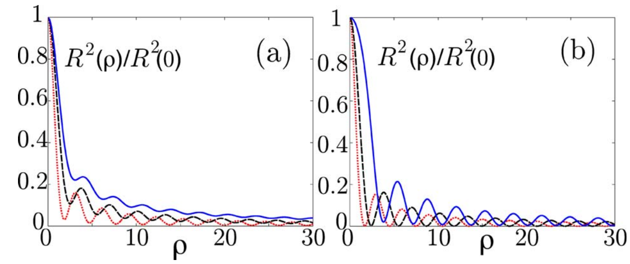

FIG. 3. (Color online) Radial profile of solutions of Eq. (11) with (a) $\delta/[\gamma R^4(0)]=0.8$ and $gR^2(0)/\delta=-1.3$ (dotted line), 0 (dashed line), and 0.35 (solid line); (b) $\delta/[\gamma R^4(0)]=5$, $gR^2(0)/\delta=-2$ (dotted line), 0 (dashed line), and 0.85 (solid line).

solutions is to cut the stationary solutions at a specific point far enough from the origin.

To study the effect of this cutoff on stationary solutions and also to get an idea of their stability we will make numerical simulations of our model equations (2). In all the simulations of time evolution to be shown in this paper we have used a standard second order in time and spectral in space (Fourier polynomials), split-step integrator implementing absorbing boundaries to get rid of the outgoing radiation [10,38,39]. Note that in all figures in this paper showing time evolution of the different quantities we use the physical time t instead of the rescaled one τ in order to give a better idea of the lifetime of these structures.

In Fig. 5 we show simulations of the evolution of one of those modified stationary solutions. It can be seen how the central soliton survives for very long times with quasistationary amplitude as shown in Fig. 5(d). This is a consequence of the phase structure of stationary solutions (see Fig. 2), which leads to a particle flux toward the condensate center and thus induces a refilling mechanism of the central soliton which is manifest in Figs. 5(a)–5(c). The rings surrounding the central peak play the role of a reservoir of atoms and constantly feed the central peak and dissppear progressively as time proceeds. From the practical point of view, and since the amplitude of the rings is small, what one would observe is a very long-lived soliton lasting for times of the order of the condensate lifetime.

The number of atoms used in our simulations is about an order of magnitude above those available in present experiments. However, what it is really important is the shooting

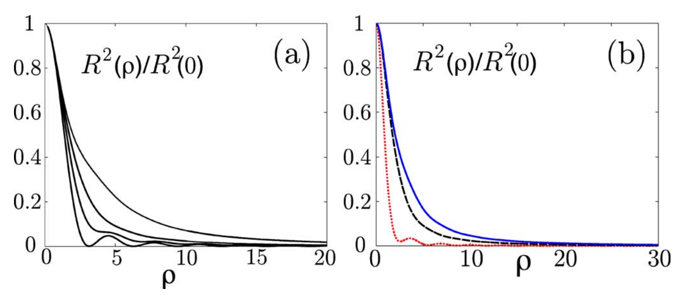

FIG. 4. (Color online) Radial profile of solutions of Eq. (11) for $d=3$ with (a) $\delta/[\gamma R^4(0)]=0.36,0.4,0.5,\infty$ and $g=0$; (b) $\delta/[\gamma R^4(0)]=0.4$ and $gR^2(0)/\delta=-2$ (dotted line), 0 (dashed line), and 0.1 (solid line).

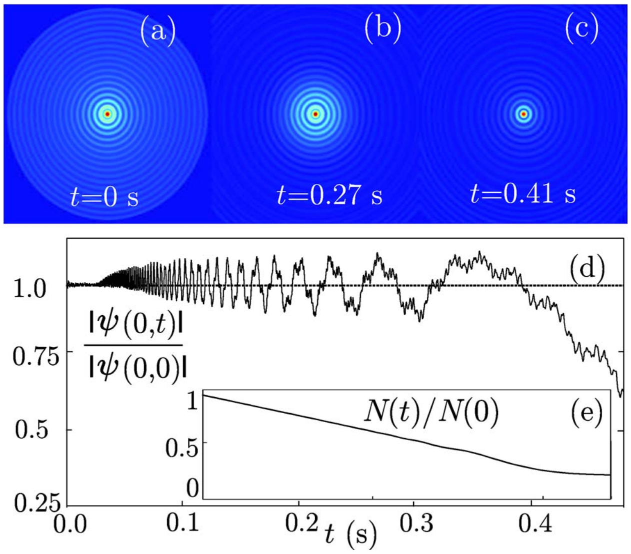

FIG. 5. (Color online) Time evolution of a stationary solution of Eq. (2) for $\gamma=-2\times10^{-10}$ (from $K_3=2\times10^{-26}$ cm^6/s for ^{7}Li), $g=0$, $\delta=0.008$, $R_0=77$. The solution is solved in a square region of 1.2 mm size and the stationary solution is set to zero outside a disk of radius 450 μm which leaves about $N_0=5.7\times10^7$ atoms inside. (a)–(c) Pseudocolor plot showing $|\psi(x,y,t)|$ for different times. (d), (e) Evolution of the (d) amplitude of the wave function and (e) number of particles in the condensate. Time is shown in physical units (not in adimensional ones) to provide an estimate of the real lifetime of these structures.

parameter $\delta/[\gamma R^4(0)]$. By choosing smaller values for $R(0)$ one may get smaller particle numbers but at the price of decreasing δ, thus changing the solution and widening the spatial scales. These are options which must be taken on the basis of particular experimental scenarios.

We have checked that the same mechanism works for solutions of vortex type as shown in Fig. 6.

The scenario described above is also valid for nonzero (positive and negative) values of g as shown in Fig. 7. This means that the phenomenon described here is not merely a result of elastic two-body collisions. Moreover, it seems that the phenomenon generates a very long-lived central soliton even in condensates with positive scattering length, which is a very surprising result.

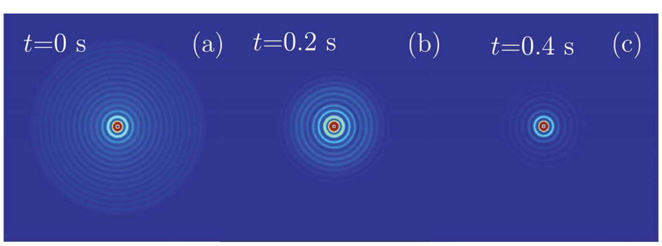

FIG. 6. (Color online) (a)–(c) Pseudocolor plots showing the evolution of $|\psi(x,y,t)|^2$ for different time values of a vortex-type, $m=1$, solution computed with $g=0$, $\delta=10^{-2}$, $R_{max}=57$, on a square of side 900 μm, and setting the profile to zero outside a disk of radius 350 μm. The total number of atoms is $N_0=7.6\times10^7$. Time is shown in physical units (not in adimensional ones) to provide an estimate of the real lifetime of these structures.

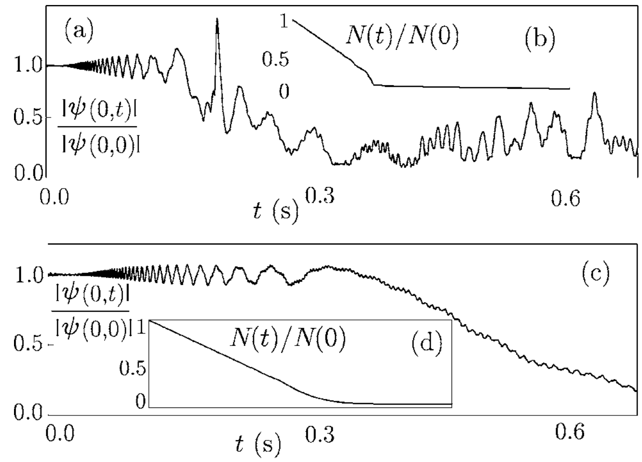

FIG. 7. Evolution of a stationary solution of Eq. (2) computed with $\delta=10^{-2}$, $\gamma=-2\times10^{-10}$. Shown are the evolutions of the amplitude (a), (c) and number of particles (b), (d) in the condensate for (a), (b) $g=-5.7\times10^{-6}$, $R_0=132$, on a square of side 800 μm, with the solution cut at 300 μm, leading to $N_0=7.6\times10^7$ atoms and (c), (d) $g=7.4\times10^{-7}$, $R_0=82$, on a square physical domain of side 1.2 mm with the profile set to zero outside a disk of radius 450 μm which implies $N_0=1.2\times10^8$ atoms. Time is shown in physical units (not in adimensional ones) to provide an estimate of the real lifetime of these structures.

We have not found any signs of the azimuthal instabilities predicted in Ref. [25] for systems with higher-order dissipation. Understanding this fact would require further research. A posible reason would be a suppression of this instability in our situation. It could be also just a consequence of the finite lifetime of our structure, which could be smaller than the typical time for the instability to set in.

IV. PROPOSALS FOR EXPERIMENTAL GENERATION

To generate these structures in real experiments it would be very important to obtain appropriate initial data with amplitude and, more importantly, phase close to those corresponding to stationary solutions. In quasi-2D condensates phase imprinting methods [32–35] can be used. In fact, imprinting radially symmetric phases is simpler to do using absorption plates than originally proposed for vortices [32]. As to the amplitude profile it could be achieved by using a Bessel beam instead of a Gaussian beam for trapping the condensate before releasing it. A similar idea has been proposed in Ref. [28] to generate the much simpler stationary structures which appear when $\gamma=0$.

In fully three-dimensional Bose-Einstein condensates phase imprinting methods would be more difficult to apply. However, three-dimensional systems with attractive scattering length and three-body recombination in the collapsing regime develop the phenomenon known as superstrong collapse [7,36,37,39] which has been ignored in the physical literature studying collapse in Bose-Einstein condensates. This means that a collapsing condensate spontaneously develops a structure of shells in both the amplitude and phase similar to those shown in Fig. 1, constantly feeding atoms to the region of higher density of atoms. In fact, those structures have been seen in numerical simulations of collapse in Bose-Einstein condensates [16,17,19,23].

The phenomenon of spontaneous phase self-modulation in superstrong collapse provides a way for generating the matter-wave solitons presented in this paper. If during the initial stage of a blow-up event the scattering length is changed to a subcritical value the violent compression and high losses associated with collapse would be suppressed but the superimposed phase structure could evolve to one of the attractors of the system, i.e., the stable stationary states. This procedure could be a way to generate the matter-wave soliton supported by dissipation proposed in this paper.

V. CONCLUSIONS AND DISCUSSION

In this paper we have shown how unusual types of long-lived self-localized matter waves can be constructed with Bose-Einstein condensates. The structures presented in this paper differ essentially from those previously discussed in BECs which are always supported by an interplay of dispersion and nonlinearity (with maybe the help of an external potential to confine along one or several dimensions). It is not posible to connect our structures directly to dissipative optical solitons, which are usually supported by an equilibrium between gain and losses, and they can be better seen as an equilibrium between the matter flux traveling to the condensate center supported by their specific phase structure and the loss terms which eliminate those particles from the condensate. The best analogy for this structure found in other fields would be nondiffracting Bessel beams and their nonlinear extensions which have been recently constructed in media with multiphoton absorption [24,25].

The ingredients leading to such structures are a spatial phase generating a flux of atoms toward the condensate center and the dissipative mechanism provided by the inelastic three-body collisions in atomic Bose-Einstein condensates. The outcome is a striking example of a nonlinear structure supported by dissipation.

We have also discussed in our paper how it would be posible to generate these types of matter waves in real experiments.

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