Dark solitons as quasiparticles in trapped condensates

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We present a theory of dark soliton dynamics in trapped quasi-one-dimensional Bose-Einstein condensates, which is based on the local-density approximation. The approach is applicable for arbitrary polynomial nonlinearities of the mean-field equation governing the system as well as to arbitrary polynomial traps. In particular, we derive a general formula for the frequency of the soliton oscillations in confining potentials. A special attention is dedicated to the study of the soliton dynamics in adiabatically varying traps. It is shown that the dependence of the amplitude of oscillations vs the trap frequency (strength) is given by the scaling law $X_0 \propto \omega^{-\gamma}$ where the exponent γ depends on the type of the two-body interactions, on the exponent of the polynomial confining potential, on the density of the condensate, and on the initial soliton velocity. Analytical results obtained within the framework of the local-density approximation are compared with the direct numerical simulations of the dynamics, showing a remarkable match. Various limiting cases are addressed. In particular for the slow solitons we computed a general formula for the effective mass and for the frequency of oscillations.

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I. INTRODUCTION

One of the main properties of solitons, making them to be of special interest for physical applications, is preserving their localized shapes during evolution and mutual interactions [1]. Due to this robustness solitons can be regarded as quasiparticles and systems possessing large number of such excitations can be described in terms of the distribution function governed by the kinetic equation [2].

In the mean-field theory [3] description of the quasi-one-dimension homogeneous Bose gas is reduced to the exactly integrable nonlinear Shrödinger (NLS) [or one-dimensional (1D) Gross-Pitaevskii (GP)] equation, and therefore solitons are expected to play a prominent role in the dynamical and statistical properties of low-dimensional condensates. When interatomic interactions are repulsive, the GP equation possesses dark (or grey) soliton solutions [3–5]. Existence of the dark solitons was confirmed by a number of recent experiments with Bose-Einstein condensates (BEC's) confined by elongated traps [6].

In practice, condensates appear to be never homogeneous, and therefore effect of external potentials on the dark-soliton dynamics is a subject of special interest (see, e.g., Refs. [7–11], and references therein). An inhomogeneity of a system by itself does not invalid possibilities of description of solitons as quasiparticles (in some approximation, of course). In particular, one can explore the Hamiltonian approach to an effective particle with one degree of freedom, instead of dealing with the original equation for the macroscopic wave function, which is a system with infinite degrees of freedom. Moreover, one can extend the respective description on the gas of solitons, which now will be described by a distribution function governed either by the Fokker-Planck equation (for the case where a soliton bearing system interacts with a thermal bath, see, e.g., Ref. [13]) or by a kinetic equation with respective collision integral, as this is shown in Ref. [7] for the case of interaction of solitons with a noncondensed atoms.

A quasiparticle description of dark solitons can be obtained from the perturbation theory in adiabatic approximation [9] (sometimes called the collective variable approach). At the same time, as was shown in Ref. [10], a concept of a quasiparticle naturally emerges from the Landau theory of superfluidity and can be justified on the basis of the mean-field theory within the framework of the *local-density approximation*. It turns out that a dark soliton moves in an external potential without deformation of its density profile as a particle of mass 2 m. The local-density approximation is rather general, allowing direct extension to other nonlinear equations, related to the BEC dynamics, as well as to various (nonparabolic) types of the trap potential. Building up such a generalized theory is the main goal of the present paper.

In real experimental conditions the external trap potential can depend not only on the coordinate, but also on time. That is why the second aim of the present work is the description of the effect of adiabatic time dependence of the external parameters on the dark soliton motion.

The paper is organized as follows. We start with the dynamics of a dark soliton in an adiabatically changing parabolic trap (Sec. II). In Sec. III, we develop our Hamiltonian theory for solitons described by generalized polynomial NLS equations and show how such an approach is related to the

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mean-field approximation. In Sec. IV we consider in detail examples of dark soliton dynamics, which include the cases of nonparabolic trap and models with higher nonlinearity. The consideration is provided within the framework of the local-density approximation and is verified by direct numerical simulations of the dark soliton dynamics. In this section we also show how one can modify the perturbation theory for dark solitons to take into account adiabatic change of the trap frequency (Sec. V B) and make comments on the dynamics of small amplitude dark solitons (Sec. V C). The outcomes are summarized in the Conclusion and technical details of some calculations are given in the Appendixes.

II. DARK SOLITON IN A TIME-DEPENDED PARABOLIC TRAP

Let us start with the dynamics of a dark soliton described by the GP equation

$$i\hbar\,\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx} + \frac{1}{2}m\omega^2x^2\Psi + g|\Psi|^2\Psi - \mu\Psi. \qquad (1)$$

Here $g=2\hbar^2 a_s/(ma_\perp^2)$, a_s is the s-wave scattering length and $a_\perp$ is the transverse linear oscillator length, which describes the BEC in an elongated trap at low densities [3] (see also Ref. [9] for the details of derivation by means of the multiple scale expansion method).

It has been shown in Ref. [10] (see also the details below Sec. III) that the dark soliton dynamics in a parabolic trap can be successfully described within the framework of the local-density approximation. This means that, in spite of the presence of the trap, one starts with the solution of the 1D homogeneous (i.e., when $\omega=0$) GP equation [5] (see also Ref. [3], Chap. 5.5):

$$\Psi(x,t) = \sqrt{n_0}\left(i\frac{v}{c} + \frac{\sqrt{c^2-v^2}}{c}\tanh\left[\frac{x-X(t)}{\ell}\right]\right), \qquad (2)$$

where $X(t)=vt$, v is the velocity of the soliton, n_0 is the unperturbed linear density, $c=\sqrt{gn_0/m}$ is the speed of sound, and $\ell=\hbar/(m\sqrt{c^2-v^2})$ is the width of the soliton. Then the influence of the trap is accounted by considering a general function $X(t)$ which dependence on time is to be obtained.

The energy of the system can be defined as

$$E = \int\left[\frac{\hbar^2}{2m}|\Psi_x|^2 + \frac{g}{2}(|\Psi|^2-n_0)^2\right]dx = \frac{4}{3}\hbar c n_0\left(1-\frac{v^2}{c^2}\right)^{3/2} \qquad (3)$$

and for the dark soliton solution (2) can be rewritten in a form of the conservation law

$$c^2(X) - v^2 = (\mathcal{G}E)^{2/3}, \qquad (4)$$

where $\mathcal{G}=3g/(4\hbar m)$. The introduced dependence $c=c(X)$ is the key point of the local-density approximation: the sound velocity is substituted by its local value computed in the point where the center of the soliton is located. In the Thomas-Fermi (TF) approximation, when the atomic density is given by $n(x)=\frac{1}{g}\left(\mu-\frac{m\omega^2x^2}{2}\right)$, [21] one has

$$c^2(X) = \frac{g}{m}n(X) = c_0^2 - \frac{1}{2}\omega^2X^2 \qquad (5)$$

with $c_0=\sqrt{\mu/m}$. Substituting $v=dX/dt$ in Eq. (4), the energy conservation can be rewritten as follows [10]:

$$\frac{m_s}{2}\left(\frac{dX}{dt}\right)^2 + \frac{m_s\omega_s^2}{2}X^2 = E_*. \qquad (6)$$

Here we introduced the effective mass of the soliton considered as a quasiparticle

$$m_s = 2m, \qquad (7)$$

the frequency of the soliton oscillations $\omega_s=\omega/\sqrt{2}$ [8–12] and the effective soliton energy

$$E_* = \frac{m_s c_*^2}{2}, \quad c_*^2 = c_0^2 - (\mathcal{G}E)^{2/3} \qquad (8)$$

which, altogether with E, is a constant of motion. The amplitude of oscillations governed by Eq. (6) is

$$X_0 = \sqrt{2E_*/m_s\omega_s^2}. \qquad (9)$$

One of the characteristic features of the introduced quasiparticles is that their dynamics is determined not only by their local properties (velocity and amplitude) but also by the environment, i.e., by the unperturbed density. As a result any change of the trap characteristics (say, trap frequency or geometry) will affect solitons not only by changing the domain of their motion but also through the change of the density. It turns out that the local-density approximation is a suitable framework for description of mentioned phenomena in the case when time variation of the parameters of the system is slow enough.

According to a general law of the Hamiltonian mechanics, the adiabatic invariant

$$I(E) = \frac{1}{2\pi}\oint p\,dX \qquad (10)$$

stays constant [14]. Time dependence of the amplitude of oscillation can be defined from this condition. The canonical momentum, which enters in Eq. (10), can be computed explicitly using the formula

$$p = \int_0^v \frac{\partial E}{\partial v}\frac{dv}{v} \qquad (11)$$

which gives

$$p = -2n\hbar\left[\frac{v}{c}\sqrt{1-\frac{v^2}{c^2}} + \arcsin\left(\frac{v}{c}\right)\right]. \qquad (12)$$

It turns out, however, for calculation of the adiabatic invariant it is more convenient to use the general equation between I and the frequency of oscillations:

$$\frac{dI}{dE} = \frac{1}{2\pi}\oint\frac{dX}{v} = (\omega_s)^{-1}. \qquad (13)$$

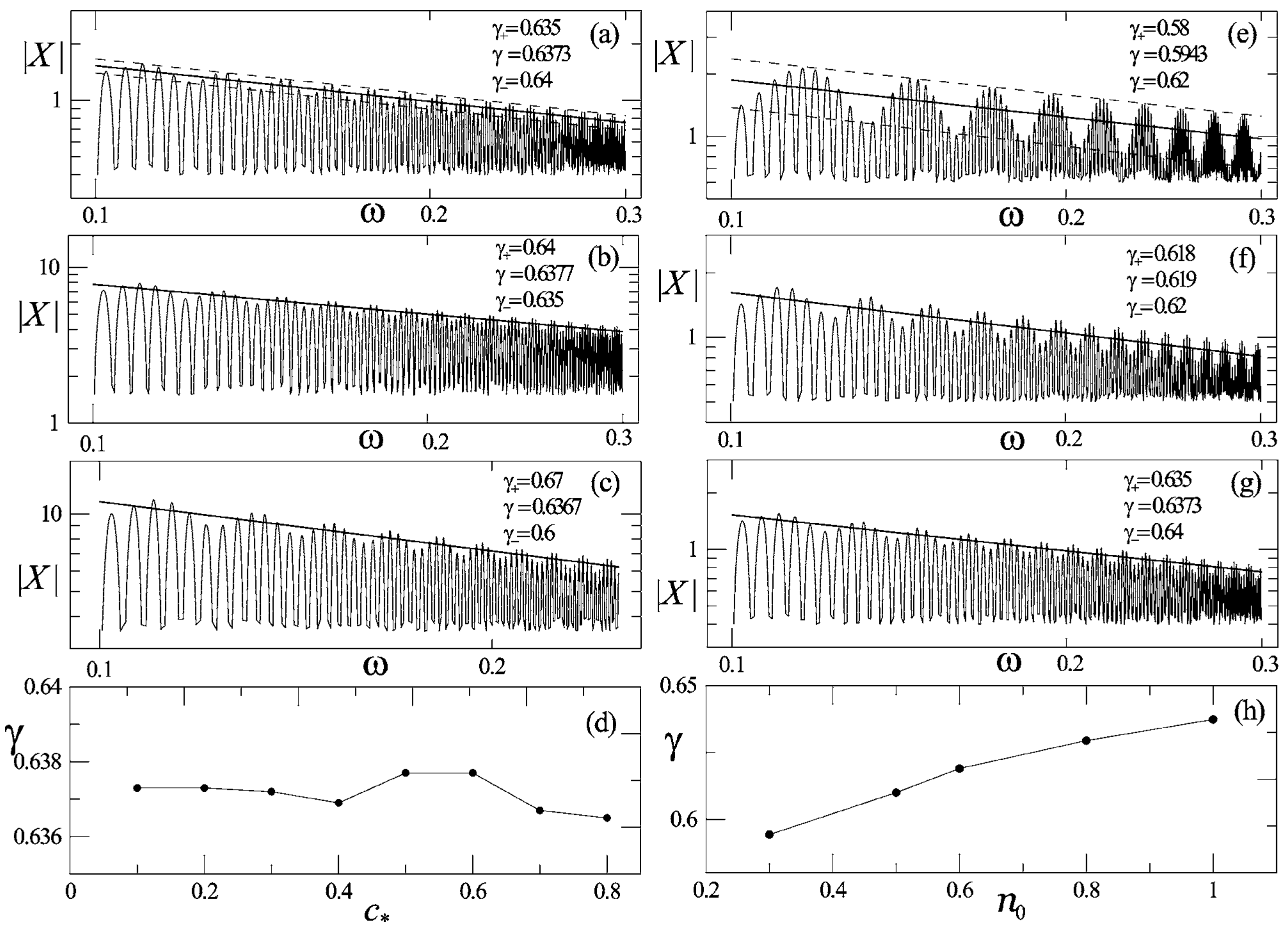

FIG. 1. Time dependence of the soliton coordinate on the adiabatically changing frequency, modeled by the function $\omega(t)=(1+0.001t)\times 0.1$, in logarithmic coordinates: panels (a)–(c) and (e)–(g). Straight solid lines visualize the law $X_0=\widetilde{X}_0\omega^{-\gamma}$. Dashed lines in panels (a) and (e) show the exponents $\gamma_\pm$ [see Eq. (17)] while in the rest of the panels only the average γ is shown. In panels (a)–(c) the parameters are $n_0=1$ and $v=0.1$, 0.5, and 0.8, correspondingly. The respective matching parameter is $\widetilde{X}_0=0.385$, 1.93, 2.6. In panels (e)–(g) the parameters are $v=0.1$, $n_0=0.3;0.6$, and 1, correspondingly. The respective matching parameter is $\widetilde{X}_0=0.622$, 0.445, 0.385. In panel (d) we plot dependence of γ on the initial soliton velocity v for the case $n_0=1$. In panel (h) we depict the dependence of γ on the initial density n_0 corresponding to the case of the initial velocity $v=0.1$. In the numerical calculations we take $\hbar=1$, $m=1$, and $g=1$.

Taking into account that $\omega_s=\omega/\sqrt{2}$ does not depend on E and using an obvious boundary condition $I=0$ at $v\to 0$, we easily find a simple equation,

$$I=\frac{\sqrt{2}}{\omega}\left(E-\frac{4\hbar m}{3g}c_0^3\right). \tag{14}$$

It is not difficult to show [see, for example, Eq. (17.10) in Ref. [3]], that in the TF approximation one has

$$c_0^2=gn(0)/m\propto\omega^{2/3},\quad \text{i.e.,}\quad c_0^3\propto\omega, \tag{15}$$

so the second term on the right-hand side of Eq. (14) is constant.

Thus preserving the adiabatic integral in an adiabatic process implies preserving E/ω, which in the case of slowly varying frequency implies $E\propto\omega$. Taking again into account that according to Eq. (15) in the TF approximation $c_0\propto\omega^{1/3}$, one deduces from Eq. (8) that $E_*\propto\omega^{2/3}$. Finally, the scaling law for the amplitude of oscillations, defined by Eq. (9), reads

$$X_0\propto\omega^{-2/3}. \tag{16}$$

It is worth emphasizing that this law is different than one for a conventional harmonic oscillator, where $X_0\propto\omega^{-1/2}$, even though the motion of the soliton is purely harmonic. The point is that in our case the ratio E/ω, but not E_*/ω, is preserved.

An important feature of the soliton dynamics is that in the case at hand the soliton frequency does not depend on the energy. Hence the frequency of the soliton oscillations does not depend on the amplitude of the soliton, which corroborates with the analysis of the oscillations of the small-amplitude solitons [see Eq. (93) below and subsequent discussion] as well with the earlier studies [10,11].

We have checked the obtained predictions, made on the basis of the local-density approximation, numerically. The typical results are presented in Fig. 1.

The local-density approximation essentially uses that the background of the condensate is static, i.e., that the dark soliton motion does not excite the motion of the whole condensate. In practice, due to finiteness of the system, such a supposition strictly speaking does not hold, and the whole condensate also undergoes oscillations with the frequency of the condensate ω, which follows directly from the Ehrenfest theorem. The difference of the frequencies of the condensate and of the dark soliton, i.e., between ω and ω_s, results in the beating of the dark soliton [9], which are clearly observable in Fig. 1. Respectively, one can identify the two slopes corresponding to the maxima and to the minima of the soliton amplitudes. We will use the subindexes "+" and "–" for the respective quantities. In other words, each of the results presented in panels (a)–(c) and (e)–(g) is characterized by the two scaling laws: $X_{0,\pm}=\widetilde{X}_\pm\omega^{-\gamma_\pm}$ shown explicitly in Figs. 1(a) and 1(e). The exponents $\gamma_\pm$ are different (although their difference is relatively small), which requires a definition of some averaged exponent γ which could be compared with the theoretical predictions. We obtain such exponent numerically from the dynamics of the averaged amplitude, i.e., using the formula

$$\frac{\widetilde{X}_0}{\omega^\gamma}=\frac{1}{2}\left(\frac{\widetilde{X}_+}{\omega^{\gamma_+}}+\frac{\widetilde{X}_-}{\omega^{\gamma_-}}\right). \qquad (17)$$

A summary of the results for the averaged exponent γ are presented in panels (d) and (h). As one can see from the figures the law of the change of the amplitude of soliton oscillations stays close to the predicted law $\gamma=2/3$ for relatively slow solitons and relatively large densities. Meantime deviations are clearly seen in Figs. 1(e) and 1(f). In the last case the exponent γ is essentially less than that predicted in our analytical consideration. It turns out, however, that the mentioned deviation from $2/3$ law is observed for small densities. This is natural from the point of view of the theory. Indeed, our consideration was based on the TF approximation for the atomic density, when $n_0\propto\omega^{2/3}$. This approximation fails at low densities, and must be substituted by the Gaussian distribution, where $n_0\propto\omega^{1/2}$. Then by repeating the above arguments for the Gaussian distribution, instead if the TF one, one finds

$$X_0\propto\omega^{-1/2}, \qquad (18)$$

i.e., the law of the dependence of the amplitude of oscillation of the conventional linear oscillator on the frequency, which corroborates with the numerical findings.

III. GENERAL APPROACH

A. Generalized equation

The theory developed in the previous section can be generalized for NLS equation with arbitrary power-law nonlinearity and nonparabolic potential. More specifically, in the present section we consider the equation

$$i\hbar\,\Psi_t=-\frac{\hbar^2}{2m}\Psi_{xx}+U(x)\Psi+g|\Psi|^{2\alpha}\Psi-\mu\Psi, \qquad (19)$$

where α is a positive integer and $g>0$, which describes interacting particles of mass m in an external potential $U(x)$. The exponent α characterizes the effective interparticle interactions. In particular when $\alpha=1$ and $g=2\hbar^2a_s/(ma_\perp^2)$ one recovers the GP equation (1) considered in the previous section.

The chemical potential μ introduced in Eq. (19) is determined by the link valid for a homogeneous condensate: $\mu=gn^\alpha$. Thus the sound speed c connected to the chemical potential by the relation $mc^2=nd\mu/dn$ can be expressed as follows:

$$c^2=\frac{\alpha g}{m}n^\alpha. \qquad (20)$$

There are several reasons to consider more the general Eq. (19). First of all, Eq. (1), being completely integrable, possesses very specific soliton properties. It is interesting to investigate the soliton dynamics in a more general situation. The case of $\alpha=2$ is particularly important, because a corresponding equation can be used in different physical problems. Such a situation can take place near the Feshbach resonance. In this case the s-wave scattering length depends on the magnetic field as $a_s=a_g+\Delta/(B-B_0)$ where a_g is the background value of the scattering length, and B_0 and Δ are the location and width of the resonance. If the magnetic field is equal to $B_c=B_0-\Delta/a_g$, the scattering length turns to zero and the dominant interaction among atoms is due to three-body effects.

Indeed, in the higher approximations of the Bogoliubov theory expansion of the chemical potential of a uniform gas with respect to density n has form

$$\mu=a_sn\left(b_1+b_2(na_s^3)^{1/2}+b_3(na_s^3)\ln\frac{1}{na_s^3}\right)+g_2n^2, \qquad (21)$$

where $b_1=4\pi\hbar^2/m$ and other coefficients b can be calculated (see Ref. [3], Chap. 4.2). Coefficient g_2 depends on three-body interactions and cannot be calculated explicitly. However, it stays finite for $B=B_c$, while three first terms disappear, giving $\mu=g_2n^2$ [15]. Correspondingly, the nonlinear term in the mean-field equation has the form $g_2|\Psi|^4\Psi$. The sign of g_2 cannot be defined from general considerations. We assume that $g_2>0$. After averaging with respect to the transverse motion we obtain Eq. (19) with $\alpha=2$ and $g=g_2/(3\pi^2a_\perp^4)$.

Another physical system where the equation of the state with $\alpha=2$ is valid is a 1D Bose gas in the so-called Tonks-Girardeau (TG) limit of inpenetrable particles. This limit can be achieved for a gas of small density. It has been shown by Girardeau [16] that there exists an exact mapping between states of this system and an *ideal* 1D Fermi gas. In particular in this case one has $\mu=gn^2$ with $g=\hbar^2\pi^2/(2m)$. It has also been rigorously shown that one can find density distribution of such a gas in a 1D trap by minimization of the energy functional [17],

$$E=\int\left(\frac{\hbar^2}{2m}[(\sqrt{n})_x]^2+\frac{g}{3}n^3+U(x)n\right)dx. \qquad (22)$$

On the basis of these considerations the authors of Ref. [18] suggested to use Eq. (19) for dynamics of the TG gas. However, the hydrodynamiclike equation (19) can not give a satisfactory description of dynamics of an ideal Fermi gas. Nevertheless, it can be useful for a Bose gas near the TG limit, where equation of state approximately follows the $\alpha=2$ law, but dynamic is still not an ideal gas type.

The case $\alpha=2$ is often referred to also as a quintic nonlinear Schrödinger (QNLS) equation. For the sake of brevity in what follows we use this terminology. We mention that other polynomial models are also considered in the literature [19].

B. Soliton in the generalized equation

Let us consider now a condensate in the absence of external field, $U(x)=0$. Equation (19) takes the form

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx} + g|\Psi|^{2\alpha}\Psi - \mu\Psi \tag{23}$$

and is subject to the finite density boundary conditions

$$\lim_{x\to\pm\infty}\Psi(x,t) = \sqrt{n_0}e^{\pm i\theta}, \tag{24}$$

where the constant θ can be considered without restriction of generality in the interval $[0,\pi/2]$: $\theta\in[0,\pi/2]$. Then dark solitons, $\Psi_s(x,t)$, will be associated with traveling-wave solutions, characterized by the following dependence of the density on the spatial coordinate and time:

$$|\Psi_s(x,t)|^2 \equiv \eta^2(x-vt), \tag{25}$$

where v is the soliton velocity. Below such solutions will also be referred to as unperturbed.

The energy of the soliton solution can be defined as

$$E = \int \mathcal{E}(x)dx \tag{26}$$

where the energy density $\mathcal{E}(x)$ is given by

$$\mathcal{E}(x) = \frac{\hbar^2}{2m}|\Psi_x|^2 + \frac{g}{\alpha+1}(|\Psi|^{2\alpha+2} - n_0^{\alpha+1}) - gn_0^{\alpha}(|\Psi|^2 - n_0). \tag{27}$$

The energy is an integral of motion. Hence taking into account that the dark soliton depends on the two parameters (n_0,v) and connecting the mean density with the speed of sound by Eq. (20), one concludes that the energy of the dark soliton is a function of c and v:

$$E = E(c,v). \tag{28}$$

C. Local-density approximation

Consider now propagation of a dark soliton in a condensate with the density, varying due to the external trap potential: $n=n(x)$ with $n(0)=n_0$ [for the sake of definiteness the trap potential will be assumed having minimum at $x=0$: $U(0)=0$]. In particular, in the TF approximation the function $n(x)$ is given by

$$n(x) = n_{TF}(x) \equiv g^{-1/\alpha}[\mu - U(x)]^{1/\alpha}. \tag{29}$$

This formula determines the dependence of the sound velocity on the spatial coordinate [cf. Eq. (5)]:

$$c^2(x) = c_0^2 - \frac{\alpha}{m}U(x), \tag{30}$$

where c_0 is expressed through n_0 by the link (20).

Now we define the *local-density approximation* as an assumption that the conservation law (28) is valid for a soliton in the inhomogeneous condensate, i.e., that c can be changed to its local value $c(X)$, where X is the position of the center of the soliton, computed using the unperturbed soliton wave function $\Psi_s(x,t)$. Respectively, X and v are considered as functions of time related by the equation $dX/dt=v(t)$.

Thus in the local-density approximation the equation of motion of the soliton is determined from Eq. (28):

$$E(c(X),v) - E = 0. \tag{31}$$

Here E is the constant energy of the soliton.

Equation (31) can be viewed as an equation of motion of a quasiparticle, which can be associated to the dark soliton. Then $E(c(X),v)$ must be associated with the Hamiltonian of the quasiparticle after expressing the velocity v through the canonical momentum p according to the formula (11). After inverting this formula, one obtains the Hamiltonian of the quasiparticle:

$$H(p,X) \equiv E(c(X),v(p,X)). \tag{32}$$

Finally, the adiabatic invariant and the frequency are computed according to formulas (10) and (13), which are obviously valid in the general case.

D. Justification of the local-density approximation

In the present subsection we show that equation for the energy of a soliton, obtained for a uniform condensate, is actually valid also for a trapped condensate in the local-density approximation. Thus the trapping potential does not enter explicitly in the expression for the energy of a soliton.

To this end we define a real-valued wave function of the background $F(x)$ such that $n_0(x)=n_0F^2(x)$ is the density of the condensate in the absence of the soliton and $F(x)$ solves the equation [9]

$$-\frac{\hbar^2}{2m}F_{xx} + gn_0^{\alpha}F^{2\alpha+1} + [U(x) - \mu]F = 0, \tag{33}$$

where $n_0=n_0(0)$ subject to the normalization conditions $F(0)=1$ and $F_x(0)=0$.

The density of the "grand canonical energy" of the inhomogeneous condensate can be written down as follows:

$$\mathcal{E}'(x) = \frac{\hbar^2}{2m}|\Psi_x|^2 + \frac{g}{1+\alpha}n^{\alpha+1}(x) + [U(x) - \mu]n(x). \tag{34}$$

Here $n(x)=|\Psi|^2$. Let the soliton center be at $x=X$, ℓ be a soliton width, and L_0 be the spatial extension of the condensate. Then we introduce δ such that $L_0 \gg \delta \gg \ell$ and separate the integration on two domains,

$$E' = \int_{|x-X|>\delta} \mathcal{E}' dx + \int_{|x-X|<\delta} \mathcal{E}' dx. \quad (35)$$

Next, we add to the first term an integral $\int_{|x-X|<\delta}\mathcal{E}_0' dx$ where $\mathcal{E}_0'$ is the energy density of the background and, correspondingly, deduct it from the second term in E'.

For the case of a dark soliton solution, which is exponentially localized around $x=X$, the first integral can be approximated (with the exponential accuracy) as follows

$$\int_{|x-X|>\delta} \mathcal{E}' dx + \int_{|x-X|<\delta} \mathcal{E}_0' dx \approx \int_{-\infty}^{\infty} \mathcal{E}_0' dx = E_0, \quad (36)$$

where E_0 is the energy of the unperturbed condensate.

In order to compute the other two integrals we represent

$$\mathcal{E}'(x) - \mathcal{E}_0'(x) = \frac{\hbar^2}{2m}|\Psi_x|^2 + \frac{g}{1+\alpha}[n^{\alpha+1}(x) - n_0^{\alpha+1}(x)] - g n_0^{\alpha}(x) \times [n(x) - n_0(x)] + \frac{\hbar^2}{2m}\frac{F_{xx}}{F} n(x) - \frac{\hbar^2 n_0}{4m}(F^2)_{xx}. \quad (37)$$

As it is shown in Appendix A the last two terms can be made as small as necessary by choosing the potential large enough, while in the rest of the terms related to the background x can be securely substituted by X (due to their smoothness in the region of the soliton motion). This leads us to the final expression for the energy of the soliton:

$$E_s = \int_{|x-X|<\delta} (\mathcal{E}' - \mathcal{E}_0') dx \approx \int_{|x-X|<\delta} \left\{ \frac{\hbar^2}{2m}|\Psi_x|^2 + \frac{g}{\alpha+1}[|\Psi|^{2\alpha+2} - n_0^{\alpha+1}(X)] - g n_0^{\alpha}(X)[|\Psi|^2 - n_0(X)] \right\} dx. \quad (38)$$

The obtained integral does not depend (in the leading order) on the particular choice of the parameter δ. Then comparing the expression (38) with Eqs. (26) and (27) one can verify that they lead to the same expression for the soliton energy, where the only substitution n_0 by $n_0(X)$ must be made.

IV. EXAMPLES OF LANDAU DYNAMICS OF DARK SOLITONS

In the present section we consider two examples relevant in different ways to the BEC dynamics in low dimensions.

A. Dark soliton of the GP equation in a polynomial trap

1. General approach

Let us now turn to the case where the "polynomial" trap

$$U(x) = \frac{m}{2}\omega^{2r} x^{2r} \quad (39)$$

with r being a positive integer, $r=1,2,\ldots$, and ω being a function slowly depending on time: $\omega=\omega(t)$. If $r=1$, then $U(x)$ is transformed in the conventional parabolic trap considered in Sec. II. Then ω is the trap frequency. For this reason and for the sake of brevity of notations in what follows ω is referred to as a frequency independent on the value of r.

The question we are interested in is the dependence of the amplitude of the soliton oscillations on the frequency, subject to the adiabatic change of the last one. The explicit form of p, given by Eq. (12), allows one to solve the problem analytically in a general case, i.e., for the arbitrary integer r.

Now the link between the velocity and the coordinate (4) reads

$$v^2 + \frac{1}{2}\omega^{2r} X^{2r} = c_*^2 \quad (40)$$

[c_* was defined in Eq. (8)] and the expression for the amplitude of the oscillations of the soliton, X_0 is given by

$$X_0 = \frac{2^{1/(2r)} c_*^{1/r}}{\omega}. \quad (41)$$

Next one can compute the following quantities:

(i) The normalization condition

$$N = \int_{-x_{TF}}^{x_{TF}} n(x) dx = \frac{2r(2n_0)^{1+1/(2r)}}{2r+1}\left(\frac{g}{m}\right)^{1/(2r)} \frac{1}{\omega}, \quad (42)$$

where N is the total number of atoms and we introduced the TF radius

$$x_{TF} = \left(\frac{2gn_0}{m}\right)^{1/(2r)} \frac{1}{\omega}. \quad (43)$$

(ii) The adiabatic invariant

$$I = \frac{\hbar m^{1-1/(2r)} c_*^{2+1/r}}{g\omega} G_r, \quad (44)$$

where the constant G_r is defined in Eq. (C2) and the details of calculations are presented in Appendix C.

(iii) The frequency of the soliton [using Eq. (13)]

$$\omega_s = R_r c_*^{1-1/r} \omega, \quad (45)$$

where

$$R_r = \frac{\pi}{2^{1/(2r)}}\left(\int_{-1}^{1} \frac{dx}{\sqrt{1-x^{2r}}}\right)^{-1}. \quad (46)$$

The obtained relations as the well as the constancy of the total number of particles N and of the adiabatic invariant I subject to slow change of the frequency readily allow one to get the scaling relations [they follow from Eqs. (42) and (44), respectively]:

$$n_0 \propto \omega^{2r/(1+2r)} \quad \text{and} \quad c_* \propto \omega^{r/(1+2r)}. \quad (47)$$

Finally, taking into account the link (41) we arrive at the general scaling relation determining the dependence of the amplitude of the soliton oscillations on the frequency

$$X_0 \propto \omega^{-\gamma}, \quad \gamma = \frac{2r}{1+2r} \quad (48)$$

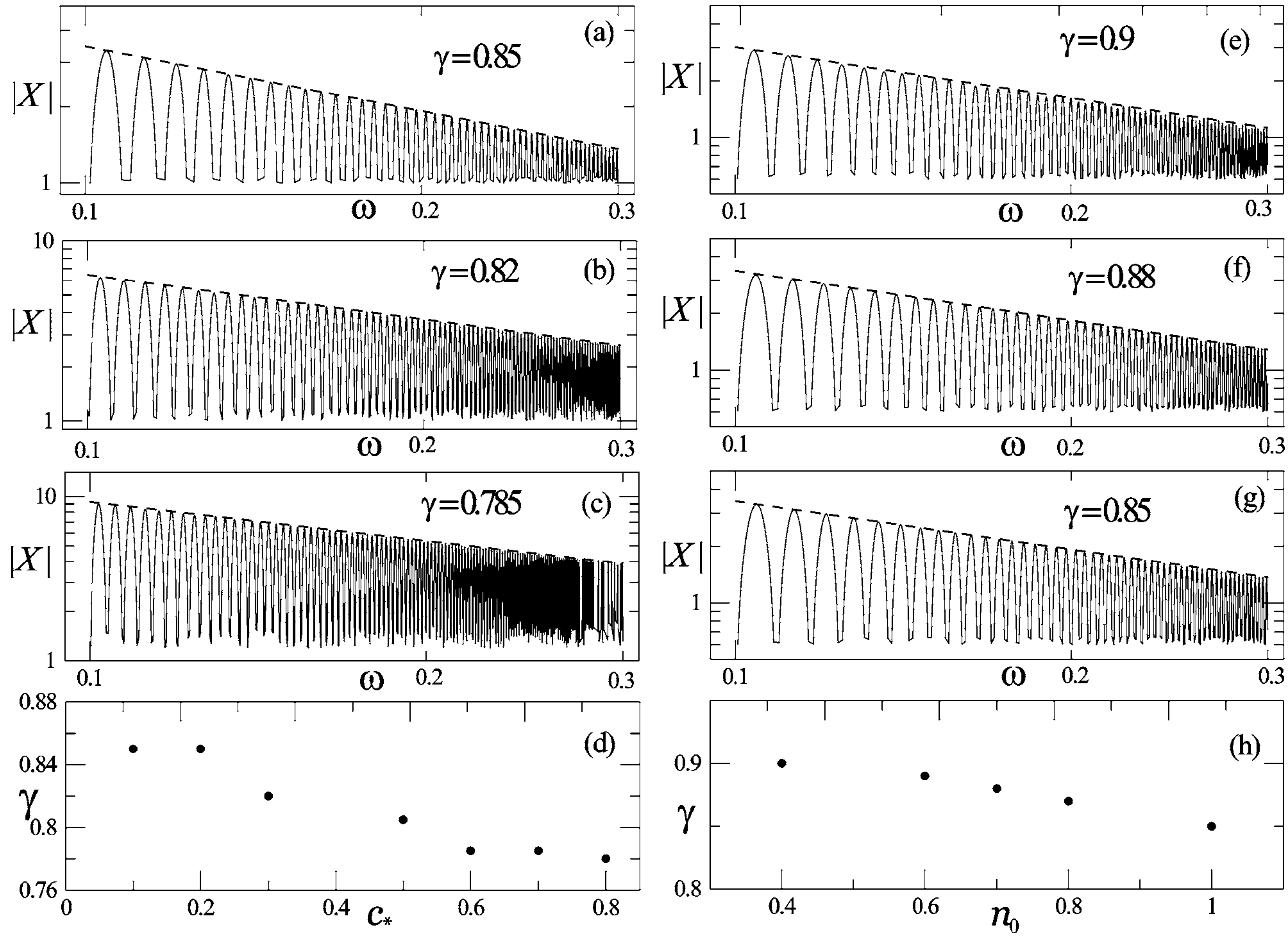

FIG. 2. Dependence of the soliton coordinate on the frequency in the logarithmic scale. Adiabatic change of the frequency is modeled by the law $\omega(t)=(1+0.001t)\times 0.1$. The dashed lines visualize the law $X_0=\widetilde{X}_0\omega^{-\gamma}$. In panels (a)–(c) the parameters are $n_0=1$ and $v=0.1$, 0.3, and 0.6, respectively. The matching parameter is $\widetilde{X}_0=0.49$, 0.98, 1.52. In panels (e)–(g) the parameters are $v=0.1$, and $n=0.4$, 0.7, and 1, respectively. The matching parameter is $\widetilde{X}_0=0.38$, 0.445, 0.49. In panel (d) we show the dependences of the exponent γ vs soliton velocity v at the density $n_0=1$. In panel (h) we show the dependences of the exponent γ vs density n_0 at $v=0.1$. In the numerical calculations we used $\hbar=1$, $m=1$, and $g=1$.

2. GP dark soliton in an x^4 trap

Let us consider in more details the dynamics of a soliton in a nonparabolic trap with the potential energy

$$U(x)=\frac{m}{2}\omega^4x^4 \tag{49}$$

(i.e., the case $r=2$). Now $R_2=\pi 2^{1/4}K(1/\sqrt{2})\approx 0.847$, $K(\cdot)$ being the complete elliptic integral of the first kind, and the frequency of soliton oscillations depends on the energy of the condensate [see Eqs. (40) and (45)]. The exponent defined by Eq. (48) is $\gamma=0.8$.

The numerical study of the soliton dynamics in a quartic trap are presented in Fig. 2. While the predicted exponential law $\omega^{-0.8}$ is now also obtained with reasonable accuracy, there are several features which distinguish the present case from the case shown in Fig. 1. First, one does not observe beatings (while they are well pronounced in the case of a parabolic trap). This fact can be explained by the absence of the unique frequency of the background oscillations: in the case at hand the Ehrenfest theorem does not result in a coupled equation for the averaged coordinate of the center of mass of the condensate. Second the dependencies of the exponent γ of the soliton velocity and on the density appear to be decreasing functions, as it is shown in Figs. 2(d) and 2(h).

B. Dark soliton in the QNLS limit

As the next example we consider the equation

$$i\hbar\Psi_t=-\frac{\hbar^2}{2m}\Psi_{xx}+\frac{1}{2}m\omega^2x^2\Psi+g|\Psi|^4\Psi-\mu\Psi. \tag{50}$$

Now $\alpha=2$ and $r=1$. Although the general approach, similar to the one developed in the preceding section, is also available in the case at hand, it becomes rather involved and cumbersome. That is why here we consider the physically relevant case of the parabolic potential which reveals the main physical features of highly nonlinear models.

The dark soliton solution has the following form [18]:

$$\Psi_s(x,t)=\sqrt{n_s(x,t)}e^{i\theta_s(x,t)}, \tag{51}$$

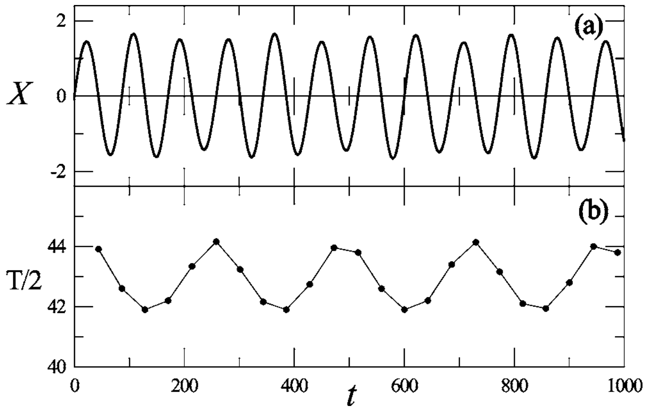

FIG. 3. (a) Dependence of the position of the center of the QNLS dark soliton on time for $\omega=0.1$, $n_0=1$, and $v=0.1$ (b) Time dependence of the half period $T/2$ substracted from (a). As before we take $\hbar=1$, $m=1$, and $g=1$.

$$n_s(x,t) = \sqrt{g}n_0 - \frac{12\sqrt{g}n_0(c^2-v^2)e^{[x-X(t)]/\ell}}{c^2(4+e^{[x-X(t)]/\ell})^2 - 12(c^2-v^2)}, \quad (52)$$

$$\theta_s(x,t) = -\arctan\left(\frac{c^2e^{[x-X(t)]/\ell} - 2c^2 + 6v^2}{6v\sqrt{c^2-v^2}}\right), \quad (53)$$

where $X(t)=vt+x_0$, x_0 is a constant, and $\ell=\hbar/(2m\sqrt{c^2-v^2})$.

The TF distribution now acquires the form

$$n_{TF}(x) = \frac{1}{\sqrt{g}}\sqrt{\mu - \frac{1}{2}m\omega^2x^2} \quad (54)$$

and the normalization conditions defines the chemical potential $\mu=\sqrt{2mg}\omega N/\pi$.

The energy is computed from Eqs. (26) and (27) to be

$$E = \hbar\sqrt{\frac{m}{g}}\frac{\sqrt{3}}{4\sqrt{2}}(c^2-v^2)\ln\frac{2c+\sqrt{3}u\sqrt{c^2-v^2}}{2c-\sqrt{3}u\sqrt{c^2-v^2}}. \quad (55)$$

Taking into account that due to Eq. (29) now

$$c^2 = c_0^2 - \omega^2x^2 \quad (56)$$

and introducing the notation

$$\mathcal{E}_0 = \hbar n_0 c_0 \quad (57)$$

we obtain

$$E = \mathcal{E}_0\frac{\sqrt{3}}{4}\left(1 - \frac{\omega^2x^2}{c_0^2} - \frac{v^2}{c_0^2}\right) \times \ln\frac{2\sqrt{c_0^2-\omega^2x^2} + \sqrt{3}\sqrt{c_0^2-\omega^2x^2-v^2}}{2\sqrt{c_0^2-\omega^2x^2} - \sqrt{3}\sqrt{c_0^2-\omega^2x^2-v^2}}. \quad (58)$$

Respectively, the energy of the zero velocity dark soliton is $E_0=E(v=0)\approx 1.14\mathcal{E}_0$.

In Fig. 3(a) we present a typical trajectory of the QNLS dark soliton in a constant trap. One of the main features observed is that the dynamics is not strictly periodic, but undergoes slow modulations (see Fig. 3). The averaged frequency of the dynamics shown is approximately 0.07 (this corresponds to the relation $\omega_s\approx 0.7\omega$) while the frequency of the large oscillations of the period is approximately five times less. It is worth pointing out that the theoretical prediction for the frequency of the large amplitude (slow) dark solitons in the QNLS model gives $\omega_s\approx 0.6572\omega$ (see Table I below) while small amplitude solitons should oscillate with the frequency close to $\omega_s\approx\omega$ (see the discussion in Sec. V C).

For the next step we studied the adiabatic dynamics of the QNLS dark soliton in a slowly varying trap. The respective results are shown in Fig. 4.

Like in the case of the NLS dark soliton in a parabolic trap one can observe the beating of the solution. From the left column [panels (a)–(c)] one detects increase of the frequency with increase of the initial value of the velocity, which is expected in view of the above discussion. The right column [panels (d)–(f)] show that the frequency of soliton oscillations decay when the background density increases. This last fact is also explained in view of the above discussion, by the fact that increase of the local density subject to constant velocity v results in increase of the relativistic factor c^2-v^2, and thus in bigger difference between the speed of sound and soliton velocity. In all the cases, however, one observes well pronounced scaling law with the exponent $\gamma=0.5$ for a relatively large range of the parameters.

V. LIMITING CASES

A. Small-velocity solitons

1. General relations

As we have seen above, increase of the power of the nonlinearity (i.e., of the exponent α) makes the problem of the computing the frequency and dependence of the frequency on the amplitude of oscillations rather complicated, not allowing one to obtain a general formula linking X_0 and ω for arbitrary α. It turns out, however, that the problem can be solved in the limit of small velocities: $v\ll c$. To this end we take into account that the static dark soliton for any $\alpha>0$ has zero amplitude in its center, and hence the limit of small velocities corresponds to the limit of small X. Then, expanding Eq. (31) with respect to v^2 and X^2, one obtains in the leading orders

$$E = E_0 + \frac{\partial E_0}{\partial v^2}v^2 + \frac{\partial E_0}{\partial X^2}X^2, \quad (59)$$

where the subindex "0" stands to indicate that the respective quantities are computed in the point $X=0$ and $v=0$. This formula must be viewed as a standard expression

$$E_s = \frac{m_s}{2}(v^2 + \omega_s^2x^2) \quad (60)$$

for the energy of a harmonic oscillator having mass m_s and frequency ω_s. Comparison of Eq. (59) with Eq. (60) gives the expressions for the effective mass,

$$m_s = 2\frac{\partial E_0}{\partial v^2}, \quad (61)$$

and for the frequency of oscillations

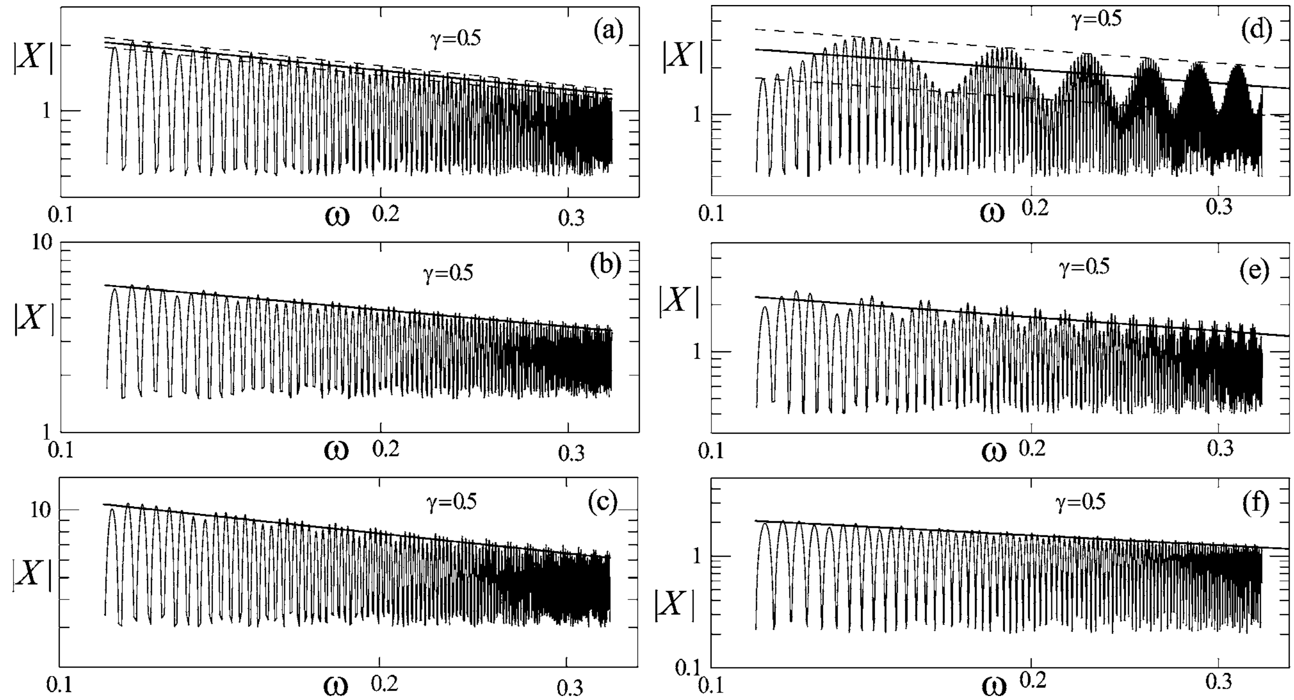

FIG. 4. Dependence of the soliton coordinate on the frequency adiabatically varying according to the law $\omega(t)=(1+0.001t)\times 0.11$. Straight lines show the law $X_0 \propto \omega^{-\gamma}$. In panels (a) and (d) we show also the laws $X_{0,\pm} \propto \omega^{-\gamma_\pm}$ [see Eq. (17)] by dashed lines. In (a)–(c) parameters are $n_0=1$, and $v=0.14$, 0.42, and 0.85, correspondingly. In (d)–(f) parameters are $v=0.14$, and $n_0=0.4$, 0.6, and 1, correspondingly. In the numerical calculations we take $\hbar=1$, $m=1$, and $g=1$.

$$\omega_s = \left(\frac{\partial E_0}{\partial X^2} \bigg/ \frac{\partial E_0}{\partial v^2} \right)^{1/2} \tag{62}$$

of a small amplitude dark soliton.

Thus to compute frequency dependence of the amplitude of the soliton oscillations from Eq. (23) we have to expand the energy $E(c(X),v)$ for small X and v. It is convenient to do this in dimensionless variables which we define as follows:

$$\psi = n_0^{-1/2}\Psi, \quad \zeta = \frac{mc_0}{\hbar}\sqrt{\frac{2}{\alpha}}x, \quad \tau = \frac{c_0^2 m}{\hbar\alpha}t, \tag{63}$$

allowing one to rewrite Eq. (23) in the dimensionless form

$$i\psi_\tau = -\psi_{\zeta\zeta} + (|\psi|^{2\alpha} - 1)\psi \tag{64}$$

[here we used the relations (20)]. Also we will use the notation $V=\sqrt{\frac{\alpha}{2}}\frac{v}{c_0}$. Then looking for the dark soliton solution, i.e., one having the form (25) and thus depending only on the running variable $x-vt$ (which in dimensionless variables means dependence on $\zeta-2V\tau$), and representing $\psi=\eta\exp(i\theta)$ one obtains (see Appendix B) the link

$$\theta_\zeta = -V\frac{1-\eta^2}{\eta^2} \tag{65}$$

and the equation for η [notice that according to Eq. (24) the boundary conditions now are $\eta \to 1$ and $\theta_\zeta \to 0$ as $\zeta \to \pm\infty$],

$$\eta_{\zeta\zeta} + (1-\eta^{2\alpha})\eta - V^2\frac{1-\eta^4}{\eta^3} = 0. \tag{66}$$

The last equation can be integrated with respect to ζ which gives

$$P^2 = \frac{1}{\alpha+1}(\eta^{2\alpha+2} - 1) + 1 - \eta^2 - V^2\frac{(1-\eta^2)^2}{\eta^2}, \tag{67}$$

where we designated $\eta_\zeta = P \equiv P(\eta)$.

Now the energy of the soliton can be rewritten in the form (see Appendix D)

$$E = \mathcal{E}_0 \frac{2\sqrt{2}}{\sqrt{\alpha}} \int_{\eta_m}^1 \left[P^2(\eta) + \frac{(1-\eta^2)^2}{\eta^2}V^2 \right] \frac{d\eta}{P(\eta)}, \tag{68}$$

where $\mathcal{E}_0$ was introduced in Eq. (57) and η_m determines the soliton amplitude in its center and solves the equation

$$P(\eta_m) = 0. \tag{69}$$

For a particular case of the zero-velocity dark soliton one has

$$E_0 = \mathcal{E}_0 Q_\alpha, \tag{70}$$

where

$$Q_\alpha = \frac{2\sqrt{2}}{\sqrt{\alpha(\alpha+1)}} \int_0^1 \sqrt{\eta^{2\alpha+2} - (\alpha+1)\eta^2 + \alpha}\, d\eta.$$

[Notice that in this case $\eta_m=0$, but $P(\eta=0)=\sqrt{\alpha/(\alpha+1)} \neq 0$.] Particular values of the energy for some relevant models are presented in Table I.

2. Effective mass of a dark soliton

Let us consider now a soliton moving with a small velocity, $V \ll 1$. To execute the expansion of the energy we first notice that from Eqs. (69) and (67) it follows that in the leading order

$$\eta_m \approx \sqrt{\frac{\alpha+1}{\alpha}} V. \tag{71}$$

Next, we introduce a constant η_0 which satisfies the condition $\eta_m \ll \eta_0 \ll 1$ and split the integral in Eq. (68) in two: $E = E_1 + E_2$ where

$$E_1 = \mathcal{E}_0 \frac{2\sqrt{2}}{\sqrt{\alpha}} \int_{\eta_0}^{1} [\cdots] \frac{d\eta}{P(\eta)}, \quad E_2 = \mathcal{E}_0 \frac{2\sqrt{2}}{\sqrt{\alpha}} \int_{\eta_m}^{\eta_0} [\cdots] \frac{d\eta}{P(\eta)}.$$

As it follows from Eq. (67),

$$\frac{\partial P}{\partial V^2} = -\frac{1}{2P} \frac{(1-\eta^2)^2}{\eta^2}$$

and thus in the limit $V \to 0$

$$\begin{aligned} \frac{dE_1}{dV^2} &= \mathcal{E}_0 \frac{2\sqrt{2}}{\sqrt{\alpha}} \int_{\eta_0}^{1} \left[\frac{1}{\alpha+1}\left(\eta^{2\alpha+2}-1\right) + 1 - \eta^2 \right] \frac{\partial}{\partial V^2} \frac{1}{P} d\eta \\ &\approx \mathcal{E}_0 \frac{\sqrt{2}}{\sqrt{\alpha}} \int_{\eta_0}^{1} \frac{(1-\eta^2)^2}{P_0(\eta)} \frac{d\eta}{\eta^2} \\ &\approx \mathcal{E}_0 \frac{\sqrt{2}}{\sqrt{\alpha}} \int_{0}^{1} \frac{d}{d\eta}\left(\frac{(1-\eta^2)^2}{P_0(\eta)} \right) \frac{d\eta}{\eta} + \mathcal{E}_0 \frac{\sqrt{2(1+\alpha)}}{\alpha \eta_0} \end{aligned} \tag{72}$$

(to obtain the last line where P_0 is P at $V=0$ we substituted the lower limit by zero, due to fast convergence of the integral, and integrated by parts).

To calculate the derivative of E_2 we take into account that η is small over the whole range of integration. Thus

$$E_2 \approx \mathcal{E}_0 \frac{2\sqrt{2}}{\sqrt{\alpha+1}} \int_{\eta_m}^{\eta_0} \frac{\eta}{\sqrt{\eta^2 - \eta_m^2}} d\eta$$

$$\approx \mathcal{E}_0 \frac{2\sqrt{2}}{\sqrt{\alpha+1}} \eta_0 - \mathcal{E}_0 \sqrt{\frac{2}{\alpha+1}} \frac{\eta_m^2}{\eta_0}$$

and in the limit $V \to 0$ [due to Eq. (71)],

$$\frac{dE_2}{dV^2} = -\mathcal{E}_0 \frac{\sqrt{2(\alpha+1)}}{\alpha \eta_0}. \tag{73}$$

The sum of Eqs. (72) and (73) gives us the derivative that we are looking for:

$$\frac{dE_0}{dV^2} = \mathcal{E}_0 F_\alpha, \quad F_\alpha = \sqrt{\frac{2}{\alpha}} \int_0^1 \frac{1}{\eta} \frac{d}{d\eta} \frac{(1-\eta^2)^2}{P_0(\eta)} d\eta. \tag{74}$$

(We recall that the subindex "0" on the left-hand side stands for $v=0$ and $X=0$.) Finally, the definition of the effective mass (61), the explicit expression for the momentum P_0 (12), and the last formula (74) yields the general expression for the mass of the dark soliton:

TABLE I. Characteristics (the zero-velocity energy E_0, the number of particles N, the effective mass m_*, and the frequency of oscillations in the parabolic trap ω_s) of dark solitons with small velocities for different powers of the nonlinearity α.

α	E_0	$\lvert N\rvert$	m_*	ω_s
2	$2\sqrt{3}\ln\frac{1+\sqrt{3}}{\sqrt{2}}\mathcal{E}_0$	$4\sqrt{3}\ln\frac{1+\sqrt{3}}{\sqrt{2}}\frac{\mathcal{E}_0}{mc_0^2}$	$\frac{\sqrt{3}+2\ln[(1+\sqrt{3})/\sqrt{2}]}{2\ln[(1+\sqrt{3})/\sqrt{2}]}m$	0.6572ω
1	$\frac{4\sqrt{2}}{3}\mathcal{E}_0$	$2\sqrt{2}\frac{\mathcal{E}_0}{mc_0^2}$	$2m$	$\frac{\omega}{\sqrt{2}}$

$$m_s = \alpha F_\alpha \frac{\mathcal{E}_0}{c_0^2}. \tag{75}$$

In order to relate the mass of the soliton to the atomic mass m, we recall that $-dE/d\mu = N$ where N is the negative "total number of particles" associated with the soliton. Thus $m_* = -m_s/N$ can be considered as the effective mass of a "solitonic" particle. In the limit $V \to 0$ the above quantities can be easily calculated to give

$$N = \frac{\mathcal{E}_0}{mc_0^2} N_\alpha, \quad N_\alpha = \sqrt{2\alpha} \int_0^1 (1-\eta^2) \frac{d\eta}{P_0(\eta)} \tag{76}$$

and

$$m_* = \frac{\alpha F_\alpha}{N_\alpha} m. \tag{77}$$

In Table I we present examples of the effective mass for two relevant cases. From the provided values one can see that the effective mass of the soliton particle is bigger than the mass of a free particle: $m_* > m$.

3. Frequency of oscillations of a dark soliton

To conclude this subsection we compute the frequencies of oscillations of dark solitons in a trap, which can be done using the relation (62). To this end we notice that including the trap potential into the scheme developed in the preceding subsection can be done by simply changing the chemical potential μ by $\mu - U(X)$. Thus for the parabolic trap we have

$$\frac{\partial E_0}{\partial X^2} = -\frac{\omega^2}{2} \frac{\partial E_0}{\partial \mu} = \frac{\omega^2 N}{2}. \tag{78}$$

This leads us to the formula

$$\omega_s = \frac{N_\alpha}{\alpha F_\alpha} \omega. \tag{79}$$

B. Analysis based on the perturbation theory

In Ref. [10] it has been argued that the phenomenological approach formulated above can be justified from the viewpoint of the original GP equation with the help of the perturbation theory for dark solitons [9,20] when motion occurs in a constant parabolic trap. The proof was based on a possibility of effective factorization of the solution on the constant background and the dark soliton solution moving against it. In the case of a time-dependent trap, the background cannot be considered as a constant, and the theory requires revision. The goal of the present subsection is to develop the modification of the perturbation theory and to obtain from it the exponent γ which describes change of the amplitude of oscillations of the GP dark soliton in a parabolic trap.

To this end we start with the dimensionless form of the (1D) GP equation [the variable are introduced in Eq. (63), see also Eq. (64)],

$$i\psi_\tau + \psi_{\zeta\zeta} - \frac{1}{2}\nu^2\zeta^2\psi - |\psi|^2\psi = 0, \tag{80}$$

where $\nu \equiv \nu(\tau) = \hbar/(2^{1/2}c_0^2 m)\omega(t)$. We assume that $\omega(t)$ is a slow function of time, which is expressed by the adiabaticity condition $\frac{1}{\omega^2}\left|\frac{d\omega}{dt}\right| \ll 1$. Accordingly, $\nu(\tau)$ is also a slow function. Notice that $\nu(0)=\nu_0 \ll 1$ as a condition for the local-density approximation. We look for a solution of Eq. (80) in a form of the ansatz (analogous of the well-known lens transformation)

$$\psi(\zeta,\tau) = e^{-if(\tau)\zeta^2}\frac{1}{\sqrt{L(\tau)}}\phi(\xi,\tilde{\tau}(\tau)), \tag{81}$$

where ξ is a function of time and of spatial coordinate given by $\xi = \zeta/\sqrt{L(\tau)}$, while $\tilde{\tau}$ is a new temporal variable related to the old one by the equation $\tilde{\tau}_\tau = 1/L$. The functions $L(\tau)$ and $f(\tau)$ are to be determined below. Substitution of Eq. (81) into Eq. (80) yields

$$i\phi_\tau + \phi_{\xi\xi} - |\phi|^2\phi - \left(f_\tau + 4f^2 + \frac{1}{2}\nu^2\right)L^2\xi^2\phi - \frac{i}{2}(L_\tau - 4fL)\phi - \frac{i}{2}(L_\tau - 8fL)\xi\phi_\xi = 0. \tag{82}$$

Let us now require the trap frequency of the new equation (i.e., the term proportional to $\xi^2\phi$) to be constant, say $1/2\nu_0^2$, and dissipative terms, i.e., linear with respect to ϕ, to vanish. This gives us two equations:

$$\left(f_\tau + 4f^2 + \frac{1}{2}\nu^2\right)L^2 = \frac{1}{2}\nu_0^2 \tag{83}$$

and

$$L_\tau = 4fL. \tag{84}$$

The obtained equations will be supplied by the natural initial conditions $f(0)=0$ and $L(0)=1$. Then Eq. (82) takes the form

$$i\phi_\tau + \phi_{\xi\xi} - |\phi|^2\phi - \frac{1}{2}\nu_0^2\xi^2\phi = -2ifL\xi\phi_\xi. \tag{85}$$

We emphasize that the last equation is *exact* with no approximation made, so far.

Before the analysis of Eq. (85), let us consider Eqs. (83) and (84) in more detail. They can be reduced to a single equation for L:

$$\frac{L}{2}L_{\tau\tau} + \nu^2L^2 - \nu_0^2 = 0. \tag{86}$$

Due to adiabaticity the first term in Eq. (86) is small in comparison with the other ones. Neglecting that term, we find in the leading order

$$L = \frac{1}{\tilde{\tau}_\tau} = \frac{\nu_0}{\nu(\tau)} \quad \text{and} \quad f = -\frac{\nu_\tau}{4\nu}. \tag{87}$$

Then, simple estimates give $L_{\tau\tau} \sim \nu^2\left(\frac{1}{\omega^2}\left|\frac{d\omega}{dt}\right|\right)^2 \ll \nu^2$ and $L \approx \nu_0/\nu \sim 1$, which justifies the approximation made.

Next we introduce the notation R for the right-hand side of Eq. (85): $R \equiv -2ifL\xi\phi_\xi$. Since f is small (because of the adiabaticity of the change of ν) this term gives us a perturbation, which is complementary to the perturbation introduced by a constant parabolic trap, provided ν_0 is small, the case considered in detail in Ref. [9]. Due to their smallness, the effect of different perturbations on the dynamics of the soliton center is additive, allowing one to compute only the contribution of R to the dynamical equation of the soliton center and add it to the equation describing soliton in a stationary potential obtained in Ref. [9] [see Eq. (32) there]. We skip the description of tedious but straightforward calculations [25] and present only the final result: the equation for the soliton coordinate, in terms of the rescaled by L variables, is given by

$$\frac{dX}{d\tau} = V - \frac{1}{2}\nu_0^2\int_0^\tau X(\tau')d\tau' - \frac{1}{4}\left(\frac{\nu_0}{\nu}\right)_\tau VX. \tag{88}$$

Next we differentiate the last equation with respect to τ and eliminate the "dissipative" term by means of the substitution

$$X = Y(\tilde{\tau})e^{\vartheta(\tilde{\tau})}, \tag{89}$$

where

$$\vartheta = -\ln\left(\frac{\nu}{\nu_0}\right)^\delta, \quad \delta = -\frac{V}{8}. \tag{90}$$

Having done this and restoring the original variables we arrive at the final formula

$$X_0 \propto \nu^{-\gamma}, \quad \gamma = \frac{1}{2} + \delta. \tag{91}$$

Comparing this result with Eqs. (16) and (18) one can see a remarkable agreement. The perturbation theory, valid for relatively low densities of the condensate, and thus to the Gaussian background, corrects the law (18) based on the phenomenological approach, by means of small shift (recall that $V \ll 1$ and thus $\delta \ll 1$) toward the larger exponent which in the TF limit is given by Eq. (16). Moreover, the perturbation theory introduces an explicit dependence of the exponent on the velocity (at this point it is relevant to recall also that the frequency itself does not depend on the soliton velocity). Finally we mention that the obtained result corroborates the numerical results on the dependence of the exponent γ on the soliton velocity and on the density of the background depicted in Figs. 1(d) and 1(h).

C. A comment on small amplitude solitons

Small amplitude dark solitons of the NLS equation with a polynomial nonlinearity of any power of the nonlinearity are described by the Korteweg-de Vries (KdV) equation [22] and they move with the sound velocity (or more precisely with a velocity slightly deviating from the sound velocity). While self-consistent reduction of the 3D GP equation to the 1D KdV equation seems to be not possible for realistic condensates (as it is explained in Ref. [23]), the KdV being academic rather than practical allows one to predict some features of the underline GP equation.

Generally speaking one can easily argue that the small amplitude limit of a dark soltion in a parabolic trap is not available. Indeed, existence of a soliton in a trap implies smallness of the soliton width ℓ compared to the trap width $\sqrt{\hbar/(m\omega)}$. Using the expression for the width of a dark soliton, which is given by Eq. (2), one immediately obtains the limitation $\hbar\omega/m \lesssim c^2 - v^2$. Thus the existence of a trap does not allow the true small amplitude limit, which would correspond to $v \to c$.

Let us now *formally* compute the half period of oscillations of a small amplitude soliton in a parabolic trap. Under the half period we understand the time necessary for a soliton to pass the distance between two turning points. To this end associate the velocity

$$c(x) = \omega\sqrt{\frac{\alpha}{2}(x_0^2 - x^2)}, \tag{92}$$

where $x_0 = 2\mu/(m\omega^2)$, with the velocity of the soliton. Then direct computation gives

$$\omega_{sol} = \frac{\pi}{\displaystyle\int_{-x_0}^{x_0} [dx/c(x)]} = \sqrt{\frac{\alpha}{2}}\omega. \tag{93}$$

Thus for the small amplitude GP dark soliton in a parabolic trap we obtain $\omega/\sqrt{2}$, which coincides with the results known for relatively large velocity for the soliton. Equation (93) also gives $\omega_s = \omega$ for $\alpha = 2$, the result recently reported in Ref. [24].

We emphasize, however, that presently there are no available results confirming the validity of the law (93) for small amplitude NLS solitons. The main physical reason for this, mentioned in Ref. [9], is that in the vicinity of the turning points the density becomes small enough making the problem linear and thus not allowing solitonic propagation due to dominating dispersion. Mathematically, the problem occurs due to divergence (see, e.g., the second of equations (11) in Ref. [24]) of the small amplitude expansion near the points where the condensate density, and thus the speed of sound, in the TF approximation becomes zero (see also the discussion of the failure of the small amplitude limit in Ref. [11]).

VI. CONCLUSION

In this paper we presented development of the theory suggested in the earlier publication [10], providing a detailed description of the one-dimensional dynamics of a dark soliton in a Bose-Einstein condensate confined by an external potential. The theory is based on the local-density approximation and allows one to interpret the dark soliton as a Hamiltonian particle. We addressed various generalizations of the theory including the nonlinearity of a general polynomial type as well as nonparabolic potential. We have obtained that the dependence of the amplitude of the soliton oscillations in an external trap depends on the adiabatically changing frequency through the scaling law $X_0 \propto \omega^{-\gamma}$ where the exponent γ depends on the type of the nonlinearity and on the type of the confining potential. It turns out also that the frequency dependence of the amplitude of the oscillations depends also on the density of the condensate and on the initial velocity, even in the cases when the frequency itself is independent on the above quantities as in the case of the standard nonlinear Schrödinger dark solitons. Also the obtained scaling law in a general case appears to be very different from the corresponding law for the linear oscillator.

We dedicated special attention to the cases of dark solitons within the framework of the Gross-Pitaevskii and quintic nonlinear Schrödinger models. We also have shown that in the limiting case of slow, and thus large amplitude, solitons one can obtain the general explicit expressions for the effective mass of the dark soliton, considered as a quasiparticle, and for the frequency of its oscillations in the external confining trap.

The results have been verified numerically, showing good agreement with theory, and were shown to be in agreement with outcomes of the direct perturbation theory for solitons.

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APPENDIX A: ESTIMATES FOR THE BACKGROUND

In the present appendix we provide estimates for the last two terms in Eq. (37). For the sake of simplicity the consideration will be restricted to the case of a polynomial parabolic trap (39).

Let us consider the behavior of the function $F(x)$ in the vicinity of the point $x \ll L_0$ (we recall that L_0 is an effective trap length). The background is obviously an even function of the trap which allows us to look for its solution in a form of the expansion

$$F = 1 + \sum_{k=1}^{\infty} F_k \zeta^k, \quad \zeta = x^2. \tag{A1}$$

More specifically we are looking for the coefficients F_k, all of which become zero in the homogeneous case when $\omega = 0$. It follows directly from Eq. (33) that

$$\mu = gn_0^\alpha - \frac{\hbar^2}{m}F_1. \tag{A2}$$

In the homogeneous condensate the chemical potential is given by $\mu_0 = gn_0^\alpha$ and thus it should be verified that $F_1 \ll mgn_0^\alpha/\hbar^2$ for ω small enough.

Next from Eq. (33) one can obtain the recurrent formulas

$$\frac{\hbar^2}{2m}(2k+2)(2k+1)F_{k+1} = \frac{gn_0^\alpha}{k!}\left(\frac{d^k F^{2\alpha+1}}{d\zeta^k}\right)_{\zeta=0} - \mu F_k \quad \text{for } k<r \tag{A3}$$

$$\frac{\hbar^2}{2m}(2r+2)(2r+1)F_{r+1} = \frac{gn_0^\alpha}{r!}\left(\frac{d^r F^{2\alpha+1}}{d\zeta^r}\right)_{\zeta=0} - \mu F_r + \frac{m}{2}\omega^{2r}. \tag{A4}$$

In order to satisfy the constraint $F_k=0$ at $\omega=0$, we require $F_{r+1} \ll F_r$. From Eqs. (A3) and (A4) we obtain the following asymptotic relations:

$$F_k = O(F_r) = O(\omega^{2r}), \quad k \leqslant r,$$

$$F_{r+1} = o(\omega^{2r})$$

which in their turn guarantee the smallness of the integrals

$$\int_{|x-X|<\delta} \frac{F_{xx}n(x)}{F}dx \propto \omega^{2r} \quad \text{and} \quad \int_{|x-X|<\delta} \frac{d^2 n_0(x)}{dx^2}dx \propto \omega^{2r}$$

when $\omega \to 0$.

APPENDIX B: ON THE LINK AMONG FORMULAS (64)–(66)

In terms of the amplitude η and the phase θ, both depending on $\zeta - 2V\tau$, Eq. (64) can be rewritten in the form of a system

$$-2V\theta_\zeta = \frac{\eta_{\zeta\zeta}}{\eta} - \theta_\zeta^2 + 1 - \eta^{2\alpha}, \tag{B1}$$

$$2V\eta_\zeta = 2\eta_\zeta\theta_\zeta + \eta\theta_{\zeta\zeta}. \tag{B2}$$

Multiplying Eq. (B2) by η, integrating with respect to θ and using the boundary conditions $\eta \to 1$ and $\theta \to$const as $\zeta \to \pm\infty$, one obtains the link (65).

In order to obtain Eq. (66) it is enough to substitute θ_ζ expressed in terms of η through the relation (65) in Eq. (B1) and multiply the result by η.

APPENDIX C: ADIABATIC INTEGRAL FOR THE GP SOLITON IN A POLYNOMIAL TRAP

The adiabatic integral for the GP dark soliton is computed, using Eq. (10) and links (40) and (20) for $\alpha=1$, as follows:

$$\begin{aligned} I &= -4\hbar \int_0^{X_0} n\left[\frac{v}{c}\sqrt{1-\frac{v^2}{c^2}} + \arcsin\left(\frac{v}{c}\right)\right]dx \\ &= \frac{2^{2+1/(2r)}m^{1-1/(2r)}\hbar}{rg\omega}\int_0^{c_*} \frac{v(v^2+u^2)}{(u^2-v^2)^{(2r-1)/(2r)}}\left[\frac{vu}{v^2+u^2} + \arcsin\left(\frac{v}{\sqrt{v^2+u^2}}\right)\right]dv = \frac{\hbar m^{1-1/(2r)}u^{2+1/r}}{g\omega}G_r, \end{aligned} \tag{C1}$$

where the constant G_r is given by

$$G_r = \frac{2^{2+1/(2r)}}{r}\int_0^1 \frac{y(1+y^2)}{(1-y^2)^{1-1/(2r)}}\left[\frac{y}{1+y^2} + \arcsin\left(\frac{y}{\sqrt{1+y^2}}\right)\right]dy. \tag{C2}$$

APPENDIX D: CALCULATION OF THE ENERGY (68)

Starting with the definition (26) and (27) written as

$$E = E_0\int_{-\infty}^{\infty}\left(P^2 + \frac{1}{\alpha+1}(\eta^{2\alpha+2}-1) + 1 - \eta^2 + \frac{(1-\eta^2)^2}{\eta^2}V^2\right)d\zeta, \tag{D1}$$

and excluding $P(\eta)$ with the help of Eq. (12) one obtains

$$\begin{aligned} E &= 2E_0\int_{-\infty}^{\infty}\left(\frac{1}{\alpha+1}(\eta^{2\alpha+2}-1) + 1 - \eta^2\right)d\zeta \\ &= 4E_0\int_{\eta_m}^{1}\left(\frac{1}{\alpha+1}(\eta^{2\alpha+2}-1) + 1 - \eta^2\right)\frac{d\eta}{P(\eta)}. \end{aligned}$$

Formula (68) follows from the last equality.

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