Schmidt number for quantum operations

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To understand how entangled states behave under local quantum operations is an open problem in quantum-information theory. The Jamiolkowski isomorphism provides a natural way to study this problem in terms of quantum states. We introduce the Schmidt number for quantum operations by this duality and clarify how the Schmidt number of a quantum state changes under a local quantum operation. Some characterizations of quantum operations with Schmidt number k are also provided.

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Entanglement is one of the central concepts in quantum-information theory. It makes possible many nonintuitive applications, such as quantum parallelism [1], quantum cryptography [2], quantum teleportation [3], and quantum dense coding [4]. Such applications proceed more effectively if quantum states are more entangled. One is then interested in how much a quantum state is entangled. Many entanglement measures have been suggested, most notably the entanglement of distillation [5] and of formation [5], and the relative entropy of entanglement [6,7]. A basic idea regarding entanglement measures is that the entanglement measures cannot increase under local quantum operations with classical communications (LOCC). The theory of entanglement measurements is then connected with the behavior of the quantum states under quantum operations. To understand how entanglement behaves when only part of an entangled state is manipulated becomes a challenging open problem in quantum-information theory.

By quantum operations we mean general quantum state manipulations including unitary transformations, positive-operator-valued measurements, and postselections. In general, the trace of a density matrix may not be preserved under quantum operations. Let S denote the set of positive matrices with trace less than or equal to 1. In this paper the elements of S are generally called quantum states. Mathematically, quantum operations are linear, completely positive, and trace-nonincreasing mappings Φ from S into itself. A natural way of describing quantum operations Φ is given by the Jamiolkowski isomorphism $J(\Phi)=\rho$ between operations Φ and states ρ, which encodes the dynamical properties of operations Φ with the static properties of states ρ [8,9].

Suppose that $\mathcal{H}$ is an n-dimensional Hilbert space and $\Phi:\mathcal{B}(\mathcal{H})\rightarrow\mathcal{B}(\mathcal{H})$ is a linear bounded mapping from $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})$. To each Φ we associate a matrix $\rho\in\mathcal{B}(\mathcal{H}\otimes\mathcal{H})$, according to

$$\rho = I\otimes\Phi(P_+) \tag{1}$$

where $P_+=(1/n)\Sigma_{i,j=1}^{n}|ii\rangle\langle jj|$ is the normalized maximally entangled state. Here P_+ is so chosen that ρ is a quantum state if Φ is a quantum operation. The mapping $\rho=J(\Phi)$ is called Jamiolkowski isomorphism. This duality between Φ and ρ has been discussed in recent works [10–12] and employed for various purposes [13–15]. The inverse mapping is given by

$$\Phi(X) = n\mathrm{Tr}_1\rho(X^T\otimes I) \tag{2}$$

for all $A\in\mathcal{B}(\mathcal{H})$ where A^T is the transpose of A with respect to the fixed basis $|i\rangle$ and Tr_1 means the partial trace over the first Hilbert space. Two important relations between Φ and ρ are that (i) Φ is Hermitian (preserving) if and only if ρ is Hermitian and Φ is completely positive if and only if ρ is positive; (ii) Φ is trace preserving if and only if

$$\mathrm{Tr}_1\rho = \frac{1}{n}I. \tag{3}$$

Hence the associated quantum states of quantum operations are positive matrices whose partial traces are smaller than or equal to the maximally entangled state.

The physical interpretation of ρ in Eq. (1) corresponding to the quantum operation Φ is then straightforward if we identify the first Hilbert space with the system held by Bob and the second by Alice. Suppose that Alice and Bob share the maximally entangled normalized state P_+. Alice performs the quantum operation Φ on her own subsystem and tells Bob her outcome. Then ρ is the (not necessarily normalized) quantum state shared by Alice and Bob after Alice's local operation. It has been shown that the most general strategy of entanglement manipulations of a pure bipartite is equivalent to a strategy involving only a single quantum operation by Alice followed by one-way communication of the result from Alice to Bob (and finally locally unitary transformations by Bob and Alice) [16]. Thus, if ρ represents the result of LOCC on P_+ done by Alice and Bob together, Φ is the total effect of these LOCC on P_+ done only by Alice. In this way we see that ρ reflects the nonlocal effect of Φ, though Φ is manipulated locally.

Since ρ contains all the dynamical information of Φ, the entanglement properties of ρ may reflect how the entanglement of quantum states behaves under Φ. For example, if Φ is a quantum channel, i.e., Φ is a linear, completely positive, and trace-preserving mapping and the associated quantum state ρ is separable, Φ is called an entanglement-breaking (EB) channel, which maps every quantum state to a separable quantum state [17,18]. The entanglement of quantum

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states is totally lost after the local operation Φ. The EB channels were introduced by Holevo [19] and can be written in the following standard form:

$$\Phi(\rho) = \sum_k \sigma_k \operatorname{Tr} A_k \rho \tag{4}$$

where σ_k is a density matrix and $\{A_k\}$ forms a positive-operator-valued measurement. $I \otimes \Phi$ can be done by Bob and Alice together as follows. Suppose that Alice and Bob share a quantum state γ together. Alice makes a measurement $\{A_k\}$ on her own part, and sends the classical outcome k to Bob through a classical channel. Alice prepares the agreed upon state σ_k. Then Bob and Alice will share the product state $\gamma_k \otimes \sigma_k$ with probability $\alpha_k = \operatorname{Tr}(I \otimes A_k)\gamma$ where $\gamma_k = (1/\alpha_k)\operatorname{Tr}_2[(I \otimes \sqrt{A_k})\gamma(I \otimes \sqrt{A_k})]$. Two important EB channels are classical-quantum channels if each A_k is a one-dimensional projection, and quantum-classical channels if σ_k is a one-dimensional projection.

Another example is given by the binding entanglement (BE) channels if the associated quantum state ρ is bounded entangled [20]. An entangled state ρ is called bounded entangled if ρ can not be distilled into maximally entangled states by means of LOCC [21]. The corresponding BE channel has the property of producing only bound entanglement states. BE channels have all capacities zero and have connection with the conjecture of nonpositive partial transpose bounded entanglement [20,22].

In this paper we introduce a classification of quantum operations Φ with respect to the Schmidt number of the associated quantum states $\rho = J(\Phi)$. Recall that a pure state $|\phi\rangle \in \mathcal{H} \otimes \mathcal{H}$ has Schmidt number r if it has Schmidt decomposition $|\phi\rangle = \Sigma_{i=1}^r \sqrt{a_i}|e_i\rangle|f_i\rangle$, where $\langle e_i|e_j\rangle = \langle f_i|f_j\rangle = \delta_{ij}$, $a_i > 0$, and $\Sigma_{i=1}^r a_i = 1$. It is easy to see that the Schmidt number r of a pure state $|\phi\rangle$ is the rank of the reduced density matrix, $r = \operatorname{rank}(\operatorname{Tr}_1|\phi\rangle\langle\phi|) = \operatorname{rank}(\operatorname{Tr}_2|\phi\rangle\langle\phi|)$.The Schmidt number of the pure states can also be characterized by the minimal number of nonzero coefficients λ_i if $|\phi\rangle$ is expanded in terms of product pure states $|\phi\rangle = \Sigma\lambda_i|\phi_{1i}\rangle|\phi_{2i}\rangle$. The nonzero coefficients a_i provide a measure of entanglement. Terhal and Horodecki generalized the Schmidt number to the case of mixed states [23].

Definition 1. Let ρ be a density matrix and $\{p_i, |\phi_i\rangle\}$ a decomposition of ρ, i.e., $\rho = \Sigma_i p_i |\phi_i\rangle\langle\phi_i|$. Let $r_{\max} = \max\{r_i\}$ be the maximum Schmidt number within the decomposition $\{p_i, |\phi_i\rangle\}$ where r_i is the Schmidt number of $|\phi_i\rangle$. Then ρ is said to have Schmidt number $k = \min\{r_{\max}\}$ where the minimum is taken over all possible decompositions $\{p_i, |\phi_i\rangle\}$ of ρ.

Let S_k denote the set of the elements in S with Schmidt number less than or equal to k. S_k is a convex compact subset of S and $S_k \subset S$. Terhal and Horodecki showed that Schmidt number never increases under LOCC and the quantum state of Schmidt number k can be witnessed by k-positive mappings. It is easy to see that Schmidt number is independent of the trace of the positive matrices. Hence we can also define Schmidt number for any positive matrix ρ that ρ has Schmidt number k if $(1/\operatorname{Tr}\rho)\rho$ is of Schmidt number k. The Schmidt number for the quantum operator Φ can be defined as follows.

Definition 2. A completely positive map Φ has Schmidt number k if the associated quantum state ρ has Schmidt number k.

In the following, some examples of quantum operations with Schmidt number k are given. The unitary transformations have the largest Schmidt number n since they preserve the entanglement of the maximally entangled state P_+. The associated quantum states ρ corresponding to entanglement breaking quantum channels are separable and therefore the EB channels have the smallest Schmidt number 1. Another example is given by the state ρ_λ [23],

$$\rho_\lambda = \frac{1-\lambda}{n^2-1}(I - P_+) + \lambda P_+,$$

which has Schmidt number k if and only if

$$\frac{k-1}{n} < \lambda \leq \frac{k}{n}.$$

Therefore the quantum channel Φ_λ corresponding to ρ_λ,

$$\Phi(A) = n \operatorname{Tr}_1(\rho_\lambda(A^T \otimes I)) = \frac{1-\lambda}{n^2-1}(n^2 A^T - A) + \lambda A,$$

is of Schmidt number k if and only if $(k-1)/n < \lambda \leq k/n$.

Let $\mathcal{O}$ denote the set of quantum operations and $\mathcal{O}_k$ the set of quantum operations of Schmidt number k or less. It is easy to check that $\mathcal{O}_k$ is a convex compact subset of $\mathcal{O}$ and $\mathcal{O}_k$ is invariant under unitary transformations. Moreover, the image $J(\mathcal{O}_k)$ of $\mathcal{O}_k$ under the Jamiolkowski isomorphism is a proper subset of S_k.

We can transfer properties of the Schmidt number for quantum states to cases of quantum operations. Since the Schmidt number for mixed states is not additive [23], the Schmidt number for quantum operations is also not additive. That is, if $\mathcal{N}(\Phi) = \ln k$ for a quantum operation with Schmidt number k, then $\mathcal{N}(\Phi^{\otimes n})$ is not necessarily equal to $n \ln k$. Moreover, we may say that Φ_1 is simpler than Φ_2, in symbol $\Phi_1 \prec \Phi_2$, if ρ_2 can be obtained from ρ_1 by LOCC. The physical picture is that the effect of $I \otimes \Phi_2$ on the maximal entangled states is equivalent to the composition of $I \otimes \Phi_1$ and other LOCC. Suppose that Φ_i has the Schmidt number k_i for $i = 1, 2$. If $\Phi_1 \prec \Phi_2$, we have $k_1 \geq k_2$. If $k_1 = k_2$, we consider the Schmidt number of Φ^n. If the Schmidt number of Φ_1^n is larger than the Schmidt number of Φ_2^n for some n, then we can conclude that $\Phi_2 \nprec \Phi_1$. Moreover, the classification of quantum states with respect to LOCC [24] can be translated to the set of quantum operators.

Our first result concers the behavior of quantum states under local quantum operations $\Phi \in \mathcal{O}_k$.

Theorem 1. Let $\Phi \in \mathcal{O}_{k_1}$ and $\sigma \in S_{k_2}$. Then $I \otimes \Phi(\sigma) \in S_k$ with k at most $\min\{k_1, k_2\}$.

Proof. Our proof begins with the Kraus representation of completely positive linear mappings. Recall that every completely positive linear mapping Φ can be written in the form

$$\Phi(X) = \sum_{\mu} V_\mu X V_\mu^* \tag{5}$$

where the operators $V_\mu \in \mathcal{B}(\mathcal{H})$ are called Kraus operators. Φ may have different Kraus representations. From the definition (1) of the associated quantum state ρ we obtain the following relation between the associated quantum state ρ and the Kraus operators V_μ of a quantum operation Φ:

$$\rho = \frac{1}{n}\sum_{\mu}\left((I\otimes V_\mu)\sum_i |ii\rangle\right)\left(\sum_j \langle jj|(I\otimes V_\mu)^*\right). \tag{6}$$

The summands in Eq. (6) are pure states. In general, $\mathrm{Tr}\,\rho \neq 1$. We still may call the set $\{(1/\sqrt{n})(I\otimes V_\mu)\Sigma_i|ii\rangle\}$ a ρ ensemble [25]. Specially, if Φ has only one Kraus operator V, then the associated quantum state ρ is a pure state, $\rho = |\phi\rangle\langle\phi|$ where $|\phi\rangle$ is uniquely determined up to a phase.

On the other hand, if the associated quantum state ρ of the quantum operation Φ is pure, $\rho=|\phi\rangle\langle\phi|$ with $|\phi\rangle = \Sigma_{ij}c_{ij}|i,j\rangle$, it follows from the equivalence of two representations of Φ, Eqs. (2) and (5), that there exists exactly one Kraus operator V up to a phase such that

$$n\,\mathrm{Tr}_1(|\phi\rangle\langle\phi|X^T\otimes I) = VXV^*, \tag{7}$$

where $V=e^{i\theta}\sqrt{n}\Sigma_{ij}c_{ij}|j\rangle\langle i|$ for some θ. Suppose that ρ is a mixed state and $\{|\phi_1\rangle,\ldots,|\phi_r\rangle\}$ a ρ ensemble. Chose V_μ for $|\phi_\mu\rangle$ determined by Eq. (7) with some phase for each μ. It follows that $\{V_\mu\}$ constitute a Kraus representation of Φ. Let $\{|\psi_1\rangle,\ldots,|\psi_s\rangle\}$ be another ρ ensemble. Generalizing the results in [25] to the caseTr $\rho \neq 1$, there exists an $s\times r$ matrix M with $\Sigma_{\nu=1}^{s} M_{\nu\mu}^* M_{\nu\lambda} = \delta_{\mu\lambda}$ for $\mu,\lambda = 1,\ldots,r$ such that

$$|\psi_\nu\rangle = \sum_{\mu=1}^{r} M_{\nu\mu}|\phi_\mu\rangle \tag{8}$$

for $\nu=1,\ldots,s$. One can check that the operators

$$W_\nu = \sum_{\mu=1}^{r} M_{\nu\mu}V_\mu \tag{9}$$

are also Kraus operators for Φ. The situation is similar for Kraus operators. Two sets of Kraus operators, $\{V_\mu\}$ and $\{W_\nu\}$ are connected by a isometric matrix [9]. If we know some ρ ensemble corresponding to $\{V_\mu\}$, we can use this matrix to get another ρ ensemble corresponding to $\{W_\nu\}$.

Furthermore, for the pure states ρ there is an important correspondence between Schmidt number for ρ and the rank of V. It follows from (7) that

$$\mathrm{Tr}_1\rho = VV^*. \tag{10}$$

Therefore, a pure state ρ is of Schmidt number k, i.e., the operator $\mathrm{Tr}_1\rho$ is of rank k, if and only if the corresponding Kraus operator V in Eq. (7) is of rank k.

To check the Schmidt number of $(I\otimes\Phi)\sigma$, we first consider the case that both Φ and σ are pure. Suppose Φ is a pure quantum operation with Schmidt number k_1 and hence the Kraus operator V is of rank k_1. Let $\sigma=|\psi\rangle\langle\psi|$ be a pure state with Schmidt number k_2. $|\psi\rangle$ can be written in the form $|\psi\rangle=\Sigma_{i=1}^{n}|i,b_i\rangle$ where $|b_i\rangle$ are vectors in the second Hilbert space and span a k_2-dimensional subspace. Applying $I\otimes\Phi$ to σ, we obtain a quantum state σ',

$$\sigma' = I\otimes\Phi|\psi\rangle\langle\psi| = \left(I\otimes V\sum_i |i,b_i\rangle\right)\left(\sum_j \langle j,b_j|(I\otimes V)^*\right). \tag{11}$$

σ' is also a pure quantum state. The rank of $\mathrm{Tr}_1\sigma'$ is the dimension of the subspace spanned by the vector $\{V|b_i\rangle\}$ which is at most $\min\{k_1,k_2\}$. Hence σ' has the Schmidt number of at most $\min\{k_1,k_2\}$.

Suppose that the associated quantum state ρ of Φ is mixed and has the decomposition $\rho=\Sigma_\mu|\phi_\mu\rangle\langle\phi_\mu|$. Let $|\phi_\mu\rangle$ be of Schmidt number r_μ and hence the corresponding Kraus operator V_μ is of rank r_μ. It follows that

$$\sigma' = (I\otimes\Phi)|\psi\rangle\langle\psi| = \sum_\mu (I\otimes V_\mu)|\psi\rangle\langle\psi|(I\otimes V_\mu)^*.$$

In general, σ' is a mixed state and $\{(I\otimes V_\mu)|\psi\rangle\}$ is a σ' ensemble. If $\{|\phi'_\nu\rangle\}$ is another σ ensemble, then it can be connected with $\{I\otimes V_\mu|\psi\rangle\}$ by an isometric matrix [see Eq. (8)]. Hence it is of the form $\{(I\otimes W_\nu)|\psi\rangle\}$ for some Kraus operators $\{W_\nu\}$ of Φ, which in turn corresponds to another ρ ensemble. Since $I\otimes V_\mu|\psi\rangle$ is of Schmidt number at most $\min\{r_\mu,k_2\}$, it follows that σ' has Schmidt number at most $\min\{k_1,k_2\}$.

Similar arguments can be applied to the remaining cases. The theorem holds.

This theorem gives the Schmidt number for quantum operation Φ a dynamical feature. We know that the Schmidt number for quantum state σ cannot increase under the local quantum operation $I\otimes\Phi$. The theorem tells more than that. The Schmidt number of the quantum operation gives another constraint on the behavior of quantum states. After the local quantum operation the Schmidt number of the quantum state cannot be higher than those of Φ and σ. Moreover, it follows that the Schmidt number of the composition of two quantum operations Φ_1 and Φ_2 with Schmidt number k_1 and k_2 is at most $\min\{k_1,k_2\}$.

Due to the correspondence between the Kraus representations of Φ and the decompositions of ρ, the extreme points ρ of $J(\mathcal{O}_k)$ correspond to the extreme points of $\mathcal{O}_k$. From the proof of the theorem it follows that the extreme points Φ of $\mathcal{O}_k$ are of the form $\Phi(A)=VAV^*$ such that V has at most k nonzero singular values which are less than or equal to 1 and at least one of the singular values is 1.

Corollary 1. $\Phi\in\mathcal{O}_k$ if and only if $I\otimes\Phi$ maps every quantum state to a quantum state in S_k, i.e., $I\otimes\Phi(\sigma)\in S_k$ for all $\sigma\in S$.

Proof. By Theorem 1 we know that $I\otimes\Phi(S)\subset S_k$. We have to show that $I\otimes\Phi(S)$ cannot be a subset of S_{k-1}. But it is obvious since $I\otimes\Phi(P_+)=\rho$ is of the Schmidt number k. Hence Φ is of the Schmidt number k if and only if Φ sends quantum states into S_k.

The Schmidt number of a quantum operation is characterized that it is the highest Schmidt number of the image of S under $I\otimes\Phi$. Another characterization of $\mathcal{O}_k$ is related to the characterization of the Schmidt number of the quantum states by the k-positive mappings. A linear Hermiticity-preserving map Ψ is k positive if and only if

$$I \otimes \Psi(|\phi\rangle\langle\phi|) \geq 0$$

for all $|\phi\rangle\langle\phi|$ with the Schmidt number k. It has been shown [23] that a mixed state ρ has Schmidt number at least $k+1$ if and only if there exists a k-positive linear map Ψ

$$I \otimes \Psi(\rho) \not\geq 0.$$

By this fact and Corollary 1 we conclude the equivalence of conditions 1 and 2 of the following Corollary 2.

Corollary 2. The following statements are equivalent: (1) Φ has Schmidt number k. (2) $\Psi \circ \Phi$ is completely positive for every k-positive linear mapping Ψ. (3) $\Phi \circ \Psi$ is completely positive for every k-positive linear mapping Ψ.

Proof. To show the equivalence of conditions 2 and 3 we consider the adjoint map $\Lambda \rightarrow \Lambda^{\dagger}$ defined by $\mathrm{Tr}\,\Lambda(\rho)\sigma = \mathrm{Tr}\,\rho\Lambda^{\dagger}(\sigma)$ for all ρ, σ in $\mathcal{B}(\mathcal{H}\otimes\mathcal{H})$. Λ is positive if and only if $\Lambda^{\dagger}$ is. Moreover, $(\Lambda_1 \circ \Lambda_2)^{\dagger} = \Lambda_2^{\dagger} \circ \Lambda_1^{\dagger}$. The equivalence of conditions 2 and 3 follows.

Thus the linear completely positive mappings can be classified by linear positive mappings. Our results are also a generalization of Theorem 4 in [17].

In summary, we have demonstrated the idea that the behavior of quantum states under local quantum operations can be understood in terms of the properties of the associated quantum states of quantum operations. In particular, it means that to know the behavior of all states under $I \otimes \Phi$, we need to consider only the entanglement properties of $I \otimes \Phi(P_+)$ of the normalized maximally entangled quantum state P_+ under $I \otimes \Phi$.

We showed that the Schmidt number for the associated quantum states is a good index for describing the nonlocal effect of local quantum operations. Moreover, a connection between linear completely positive mappings with Schmidt number k and linear k-positive mappings is found. The duality between quantum states and quantum operations also provides a way to study the properties of quantum states in terms of quantum operations [14,26]. Thus the duality is important in analyzing the properties of both quantum states and quantum operations. We hope our results attract more attention to this point.

Note added. We acknowledge that the concept of the Schmidt number for quantum operations is also introduced in [27] and our Theorem 1 can be obtained from the Theorem 1 in [27].

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