Simultaneous histories, path sums, and measurements for noncommuting variables

D. Sokolovski
School of Mathematics and Physics, Queen's University of Belfast, Belfast, BT7 1NN, United Kingdom

R. Sala Mayato
Departamento de Fisica Fundamental II, Universidad de La Laguna, La Laguna (S/C de Tenerife), Spain

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We analyze von Neumann-like quantum measurements in terms of simultaneous virtual paths constructed for two noncommuting variables. The approach is applied to measurements of operator functions of conjugate variables and to the joint measurements of such variables. The limits of applicability of the restricted phase space path integral are studied. We demonstrate that, for a simple joint measurement, using entangled meter states allows one to manipulate the order in which the measurements are conducted. The effects of "weakening" a measurement by choosing unsharp meter states are also discussed.

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I. INTRODUCTION

There have been several attempts to use the Feynman path integral [1–3] as a basis for a general approach to the quantum measurement theory (see, for example, [4–7], and references therein). In particular, in [7], the path integral was used to propose a solution to the tunneling time problem [8], which is essentially the one of defining the duration τ a quantum particle spends in a specified region of space Ω, e.g., inside the barrier. One difficulty one encounters in analyzing the problem by conventional methods is that τ is related to a time interval T during which a particle moves across the region, rather than to one particular instant. It was shown in [7] that, although such a quantity is not represented in a conventional way by a Hermitian operator, the probability amplitude to have a value τ can be obtained by restricting the Feynman path integral to the paths spending this duration in Ω. Having defined the "traversal time wave function" in such a way, one also obtains a recipe for measuring this quantity. One of the corresponding measuring devices (the Larmor clock) is a version of a conventional von Neumann meter [9], with the distinction that while a von Neumann meter acts over a short period of time, the Larmor clock monitors the particle throughout the time interval T and does not project the particle's initial state on any orthogonal basis [7]. The desire to see whether a more general approach, similar to the one just described, can be applied to describe the measurements of quantities other than the traversal time was the motivation for Ref. [10] and also the present work. In [10] we have shown that, for a given variable A, one can define paths representing the virtual records of A in the same way a Feynman path represents a possible virtual record of the particle's position. As with the Feynman trajectories, an unrestricted sum over such paths yields the propagator, i.e., the transition amplitude between the initial and final states of the system. More importantly for the present discussion of quantum measurements, it was also shown that a path sum restricted to the paths giving a particular value f to a functional $\int \mathcal{F}(A)dt'$ yields the probability amplitude to obtain the pointer reading f for a system coupled to a von Neumann-like meter similar to the Larmor clock [7]. Thus, the path summation approach was shown to be helpful in both identifying the quantity measured in a finite time measurement [11] and providing a simple mathematical tool for analyzing various aspects of such a measurement, such as its accuracy and strength [12].

With the calculation of measurements probabilities reduced to summing individual amplitudes over classes of virtual paths, one might be tempted to formulate a measurement theory without mentioning either the operators or the Hilbert space. This, however, is not possible. One aspect not discussed in Ref. [10] is the existence of physical quantities whose operators do not commute with that of A, which is the main subject of the present paper. The noncommutation of $\hat{A}$ and $\hat{B}$ results in that the paths constructed for the quantity A are not convenient elements for building the measurement amplitudes involving B, and vice versa. For example, the Feynman paths are well suited to analyze the traversal time, but not the momentum measurements [13]. Although the measurement amplitudes for B can always be expressed in terms of the paths constructed specifically for this quantity, this does not provide a solution for the case when both $\hat{A}$ and $\hat{B}$ are measured at the same time [14,15]. For such a joint measurement it is natural to attempt obtaining simultaneous virtual records for both A and B. The best known example of such histories are the phase space paths [3], which simultaneously trace the virtual values of the particle's position and momentum. An unrestricted sum over these paths gives the well-known phase space path integral for the Feynman propagator, and a particular type of restriction was used, for example, in [16] to describe continuous quantum measurements. In this paper, we analyze the way to construct simultaneous paths for a general pair of noncommuting variables $\hat{A}$ and $\hat{B}$ and the possibility (or the lack of it) to use such paths to construct the probability amplitudes for simultaneous measurements of $\hat{A}$ and $\hat{B}$ as well as for various functions of these quantities. The rest of paper is organized as follows. In Sec. II, we briefly discuss the path decomposition of the measurement amplitude for single variable A. In Sec. III, we extend the approach to noncommuting variables A and B, demonstrate nonuniqueness of the simultaneous path decomposition, and relate it to the operator ordering problem. In Sec. IV, we give path analysis of a simultaneous measurement of the position x and momentum p, similar to that analyzed by Arthurs and Kelly [14] and discuss the case of two meters prepared initially in an entangled state. Section V contains a phase space path analysis of measurements of a linear combination of x and p. Our conclusions are presented in Sec. VI.

II. PATH DECOMPOSITION FOR A SINGLE VARIABLE A

Consider a quantum system in a Hilbert space of an arbitrary dimension, prepared in an initial state $|\Psi_0\rangle$, which at $t=0$ is coupled to a von Neumann-like meter with pointer position f ($\hbar=1$)

$$i\partial_t|\Psi(t|f)\rangle = [\hat{H} - i\beta(t)\hat{A}\partial_f]|\Psi(t|f)\rangle, \tag{1}$$

$$|\Psi(t=0|f)\rangle = G(f)|\Psi_0\rangle. \tag{2}$$

The duration and strength of the coupling, which involves the product of the pointer's momentum with the measured variable A, is controlled by the *switching function* $\beta(t)$. At some time t the meter, whose initial wave function in the position representation is $G(f)$, is read, i.e., its position is accurately determined and used to extract the information about the observed system. In the following, we will refer to $|\Psi(t|f)\rangle$ as the *measurement state*. If a system is postselected [12] in a known state $|\Psi_1\rangle$ the *measurement amplitude*, i.e., the conditional probability amplitude to find the meter reading f, is given by

$$A(f,\Psi_0,\Psi_1) = \langle\Psi_1|\Psi(t|f)\rangle \tag{3}$$

and distribution of the meter's reading is given by the standard formula

$$\rho(f,\Psi_0,\Psi_1) \equiv \frac{|A(f,\Psi_0,\Psi_1)|^2}{\int |A(f',\Psi_0,\Psi_1)|^2 df'}. \tag{4}$$

If there is no control of the system's state after the measurement, the probability to find the pointer at f is proportional to the norm of the measurement state

$$\rho(f,\Psi_0) \equiv \frac{\langle\Psi(t|f)|\Psi(t|f)\rangle}{\int |G(f)|^2 df}. \tag{5}$$

A well-known (and, indeed, defining) property of the von Neumann meter [9] is that when a strong coupling is applied over a short period of time, during which it overrides the system's own Hamiltonian $\hat{H}$, e.g.,

$$\beta(t) = \lim_{T\to 0} T^{-1}\theta_{0T}(t), \tag{6}$$

$$\theta_{ab}(z) \equiv 1 \quad \text{for} \quad a \leqslant z \leqslant b, \quad \text{and} \quad 0 \quad \text{otherwise}, \tag{7}$$

the resultant measurement state is

$$|\Psi(t|f)\rangle = \sum_k \langle a_k|\Psi_0\rangle G(f-a_k)|a_k\rangle, \tag{8}$$

where a_k and $|a_k\rangle$ are the eigenvalues and eigenfunctions of the measured operator $\hat{A}$, respectively,

$$\hat{A}|a_k\rangle = a_k|a_k\rangle. \tag{9}$$

For a sufficiently narrow $G(f)$, the final position of the pointer coincides with one of the eigenvalues of $\hat{A}$ and Eq. (8) is often used to identify the scalar product $\langle a_k|\Psi_0\rangle$ with the probability amplitude that the variable A has a value a_k a given moment in time.

It is easy to show that while a von Neumann meter "measures" the eigenvalues of an operator $\hat{A}$, the more general procedure (1) measures the value of a particular functional on the system's virtual paths [10]. We note first that any solution of Eq. (1) can be written as a convolution [10]

$$|\Psi(t|f)\rangle = \int G(f-f')|\Phi(t|f')\rangle df', \tag{10}$$

where $|\Phi(t|f)\rangle$ is the measurement state for to the case when the initial pointer position is known exactly, i.e., $|\Phi(0|f)\rangle = \delta(f)|\Psi_0\rangle$. Using Eqs. (1) and (10), we can write the Fourier transform $|\Psi(t|\lambda)\rangle$ [to simplify the notations we will use the same letter to denote the function and its Fourier transform with respect to f, e.g., $G(\lambda)$ stands for the Fourier transform of $G(f)$],

$$|\Psi(t|f)\rangle = (2\pi)^{-1}\int d\lambda \exp(i\lambda f)|\Psi(t|\lambda)\rangle, \tag{11}$$

as

$$|\Psi(t|\lambda)\rangle = G(\lambda)\exp\left\{-i\int_0^t [H + \lambda\beta(t')\hat{A}]dt'\right\}|\Psi_0\rangle. \tag{12}$$

Next we divide the interval $[0,t]$ into N subintervals,

$$j\epsilon < t < (j+1)\epsilon, \quad \epsilon \equiv \frac{t}{N}$$

and assume that the initial meter's wave function $G(f)$ is peaked around $f=0$ with a half width Δf, so that the factor $G(\lambda)$ effectively limits the range of λ in Eq. (11) to $-1/\Delta f \leqslant \lambda \leqslant 1/\Delta f$. Thus, in the limit $N\to\infty$ the exponential in Eq. (12) can be written as

$$\prod_{j=1}^{N}\sum_{k_j} \exp[-i\hat{H}(j\epsilon)\epsilon]|a_{k_j}\rangle\exp[-i\lambda\beta(j\epsilon)a_{k_j}]\langle a_{k_j}|, \tag{13}$$

where we have used the well-known result [2]

$$\frac{\exp(\hat{A}+\hat{B})}{N} = \exp\left(\frac{\hat{A}}{N}\right)\exp\left(\frac{\hat{B}}{N}\right) + O\left(\frac{1}{N^2}\right) \tag{14}$$

to factorize the exponential containing $\hat{H}$ and $\hat{A}$ and used the spectral decomposition for the latter. Inserting Eq. (13) into Eq. (11) and carrying out integration over λ yields

$$|\Psi(t|f)\rangle = \int df' G(f-f')\sum_{[a]} \delta \times\left[f - \int_0^t \beta(t')a(t')dt'\right]|\Phi(t|[a])\rangle, \quad (15)$$

where we have introduced a path $a(t')$, which at each $0 \leq t' \leq t$ takes its value from the spectrum of the operator $\hat{A}$, as a limit of the set of discrete values $\{a_{k_1}, a_{k_2}, \ldots a_{k_N}\}$ for $N \to \infty$. A functional $Z[a]$ on such a path is defined by the limit of its discretised form $Z(a_{k_1}, a_{k_2}, \ldots a_{k_N})$ as $N \to \infty$. In particular, in Eq. (15) the *measurement substate* $|\Phi(t|[a])\rangle$ is given by

$$|\Psi(t|[a])\rangle \equiv \lim_{N\to\infty}\left\{\prod_{j=1}^{N} \exp(-i\hat{H}\epsilon)|a_{k_j}\rangle\langle a_{k_j}|\right\}|\Psi_0\rangle \quad (16)$$

and the integral in the argument of the δ function is understood as the limit of its Riemann sum,

$$\int_0^t \beta(t')a(t')dt \equiv \lim_{N\to\infty}\sum_{j=1}^{N} \epsilon\beta(j\epsilon)a_{k_j}. \quad (17)$$

Finally, for a functional $Z[a]$ the sum over paths is defined as

$$\sum_{[a]} Z[a] \equiv \lim_{N\to\infty}\sum_{k_1,k_2\ldots k_N} Z(a_{k_1}, a_{k_2}, \ldots, a_{k_N}). \quad (18)$$

Equation (15) illustrates the nature of the von Neumann-like measurement (1). Instead of a (virtual) value of a variable A at a given time t', one is encouraged to think of a virtual record (path) traced by A throughout a time interval $[0,t]$ with $|\Phi(t|[a(t')])\rangle$ yielding the result of evolution of the system's initial state $|\Psi_0\rangle$ along this particular path. A von Neumann-like meter with a switching function $\beta(t)$ measures the value of the functional $F[a] \equiv \int_0^t \beta(t')a(t')dt'$ by destroying coherence between the classes of paths corresponding to different values of F. Introducing a quantum uncertainty Δf in the initial pointer position results in separating the paths into "fuzzy" classes, such that in the class labeled by the value f the actual value of the functional F lies within the range $f-\Delta f < F[a] < f+\Delta f$.

III. PROPERTIES OF THE MEASUREMENT SUBSTATES: FEYNMAN PATHS AND SOME EXAMPLES

Certain properties of the measurement substates $|\Phi(t|[a(t')])\rangle$ should be mentioned here. Two substates corresponding to two different values of f are, in general, not orthogonal. This is seen from the way they were constructed, as the number of substates usually exceeds that of the eigenstates of the measured operator $\hat{A}$.

The paths are such that at each time their values must coincide with one of the eigenvalues of the operator $\hat{A}$, so that an impulsive von Neumann measurement can be seen as distinguishing between the paths "passing" through different eigenvalues of $\hat{A}$ at a given time. An unrestricted coherent sum of the substates over all paths yields Schroedinger state of the system at the time t,

$$\sum_{[a]} |\Phi(t|[a])\rangle = \exp(-i\hat{H}t)|\Psi_0\rangle, \quad (19)$$

The number of eigenvalues may vary from two, for a two-level system, to an infinite continuum for a particle whose coordinate varies between $-\infty$ and ∞. For the latter, the choice, $\hat{A}=\hat{x}$ makes the path sum in (19) the Feynman path integral [1]. In particular, the Feynman propagator [1] between the space-time points $(x_0,0)$ and (x_1,t) is obtained by projecting the measurement substate onto the final state $|x_1\rangle$ and summing over all coordinate histories (Feynman paths) $\Sigma_{[x]}\langle x_1|\Phi(t|[x])\rangle$ for a particle preselected in the state $|\Psi_0\rangle = |x_0\rangle$.

The "eigenpaths" $\{a(t')\}$ can be immersed into a larger set $\{\varphi(t')\}$ of all functions taking, at all times, arbitrary real values by introducing the δ functional, $\delta[\varphi] \equiv \lim_{N\to\infty}\Pi_{j=1}^{N}\delta(\varphi(j\epsilon))$, to write

$$|\Psi(t|[\varphi])\rangle = \sum_{[a]} \delta[\varphi - a]|\Psi(t|[a])\rangle.$$

It can then be shown that the substates $\Psi(t|[\varphi])\rangle$ satisfy the functional "recorder" equation

$$i\partial_t|\Psi(t|[\varphi])\rangle = \left[\frac{\hat{H} - i\hat{A}\,\delta}{\delta\varphi(t)}\right]|\Psi(t|[\varphi])\rangle. \quad (20)$$

A detailed analysis of this equation and its relation to von Neumann-like measurements can be found in [10].

The substates $|\Psi(t|[a])\rangle$ can also be used to construct measurement amplitudes for other (possibly time-dependent) variables whose operators commute with $\hat{A}$ at all times. The general expression for such an operator is given by the spectral representation

$$\mathcal{F}(\hat{A},t) = \sum_k |a_k\rangle\mathcal{F}(a_k,t)\langle a_k|, \quad (21)$$

where $\mathcal{F}(z,t)$ is some real function. Replacing in Eq. (1) $-i\beta(t)\hat{A}$ by $-i\mathcal{F}(\hat{A},t)$ and repeating the steps leading to Eq. (15), we obtain

$$|\Psi(t|f)\rangle = \int df' G(f-f')\sum_{[a]} \delta\left[f - \int_0^t \mathcal{F}(a,t')dt'\right]|\Phi(t|[a])\rangle, \quad (22)$$

which shows that a path sum restricted to give a definite value to a functional

$$F[a] = \int_0^t \mathcal{F}(a,t')dt' \quad (23)$$

corresponds to a measurement of the variable represented by an operator function $\mathcal{F}(\hat{A},t)$. Finally, Eqs. (1) and (15) are easily generalized to the case where several (M) meters conduct simultaneous measurements of different commuting functions of $\hat{A}$. In this case the measurement amplitude for the coupling $-i\Sigma_{i=1}^{M}\mathcal{F}_m(\hat{A},t')dt'$ is obtained as a restricted path sum over classes of paths giving definite values to the functionals $\int_0^t \mathcal{F}_m(a,t')dt'$, $m=1,2,\ldots,M$ [10].

Previous work, which motivated development of the general approach presented in Sec. II and III, is worth a brief discussion. Its initial objective was to propose a solution to the tunneling or the traversal time problem (for a comprehensive review of the effort and some references, see [7]). The problem is, essentially, that of determining the duration τ a quantum particle spends in a given region of space Ω. By its very nature, τ is defined for a given time interval $[0,t]$ during which the particle moves. Classically, τ can be measured by equipping the particle with a Larmor clock, a magnetic moment that precesses when inside a constant magnetic field contained within Ω. Quantally, such a clock performs not an instantaneous von Neumann measurement, but rather a more general von Neumann-like measurement described in the beginning of Sec. II. Thus, a traversal time measurement is not associated with a conventional orthogonal expansion of the Schroedinger state of the observed particle, which explains much of the controversy surrounding the subject. It is, however, readily analyzed by the method of Sec. II. Indeed, the particle's operator, which occurs in the coupling between the magnetic moment and the field, is the projector onto Ω, $\hat{\theta}_\Omega(x)$, with eigenvalues 1 and 0 for the states localized inside and outside Ω, respectively. The projector commutes with the particle's position $\hat{x}$, and, with the help of Eq. (22), one can analyze the problems using the eigenpaths of the latter, i.e., the Feynman paths [1].The functional in Eq. (23) becomes the "stopwatch" expression for the net time a given Feynman path spends inside the region,

$$F[x] = \int_0^t \theta_\Omega(x)dt',$$

and the quantum Larmor clock is readily seen as a natural generalization of the classical one. Where classical mechanics provides a unique duration τ, quantum mechanics offers a range of durations, each assigned a probability amplitude equal to the net Feynman weight $\exp\{iS[x(t)]\}$ for all paths spending exactly τ in the region. Accordingly, there is a range of precession angles for the magnetic moment, which acts as a filter destroying the coherence between the classes of paths. In practice, the magnetic moment, or a spin, may have only a finite number of components [17] j and a discrete set of rotation angles. The path integral analysis still applies and the clock becomes a von Neumann-like meter of Sec. II in the limit $j\to\infty$ (see, for example, Eq. (15) of Ref. [17]).

A similar Feynman path approach can be applied to any variable that commutes with the position $\hat{x}$, $\mathcal{F}(\hat{x})$. The value of a functional $\int_0^t \mathcal{F}(x)dt'$ can be measured by a Larmor clock interacting with the magnetic field whose strength B varies with x as $\mathcal{F}(x)$ [cf. $\mathcal{F}(x)=\theta_\Omega(x)$ for the traversal time just discussed]. With the choice

$$\mathcal{F} = T^{-1}\int_0^t A(x)dt,$$

the precession angle of the clock (which runs until $t=T$) yields the time average of the variable $A(x)$. For example, a measurement of the particle's position, $A(x)=x$, requires a field $B\sim x$. Classically, if the instantaneous position is of interest, one may choose $T\to 0$ and apply a very large field over a short time, so that the particle will not move while the impulsive measurement continues. Then the angle, by which the clock is rotated (positive if $x>0$, negative if $x<0$ and zero for $x=0$) would indicate the current distance from the origin $x=0$. In all other cases, the clock would register the position averaged over the duration of the measurement. A path integral analysis of a quantum measurement of particle's position is given in Ref. [18]. Its outcome crucially depends on the initial state in which the meter is prepared. If this state is chosen to be Gaussian, an impulsive measurement makes the particle pass through a Gaussian slit [18], first discussed in [1]. If a measurement takes a finite time, the meter coupled to a linearly changing magnetic field separates Feynman paths into noninterfering classes labeled by the value of the mean position. In the case of position, a meter with a finite number of spin components can be analyzed just as in the case of the traversal time, as was done in Sec. VI of Ref. [18].

On should also bear in mind that time extended measurements of the traversal time or the mean position share one basic property with the impulsive von Neumann measurement, which, after all, is their special case. Namely, an accurate measurement perturbs the particle's motion. In the impulsive case, this perturbation takes the form of a sudden conversion of the particle's pure state into a mixture and its eventual collapse (the origin and nature of this collapse is beyond the scope of the present work). In a finite time measurement, the meter acts on the particle until the measurement is completed and the particle ends up in a mixture of states obtained by the evolution restricted to a particular class of Feynman paths. (For an attempt to minimize this perturbation by using a "kicked," rather than continuous Larmor clock see [19].) An accurate meter can destroy a subtle interference phenomenon, such as tunneling, by making the particle "go over the barrier." If so, does one really measure the tunneling time? On one hand, tunneling occurs when all Feynman paths and, therefore, all τ's interfere destructively to produce an exponentially small transition probability. Thus, all information about the amount of time the particle spends in the barrier is lost to interference, just as when the information about the slit chosen by an electron is lost if one chooses to observe an interference pattern in a two-slit experiment [1]. On the other hand, to compute the probabilities for obtaining a particular value of τ one uses the set of paths and probability amplitudes defined for a particle *in the absence* of a meter. However, in assigning the probabilities, one already assumes that the coherence between the classes of paths is destroyed. A suitable physical agent for effecting this decoherence is a Larmor clock whose interaction with the particle and tunneling. The situation is now similar to the two-slit experiment in which the passage of electron via a particular slit is observed at the cost of destroying the interference pattern. An attempt to minimize the perturbation incurred by the meter leads to the so-called weak measurements [12], which are known to produce anomalous results and would not satisfy anyone expecting a well-defined classical-like duration of tunneling [7].

The fact that converting interfering amplitudes into exclusive ones is synonymous with subjecting the system to an external interaction is the basic feature of all quantum measurements and cannot be bypassed without departing from the basic principles of the conventional quantum mechanics. Feynman [20] pointed out that the two-slit experiment contains, in a nutshell, the essence of the quantum theory and warned against attempts to rationalize electron's behavior. We would simply note that the path integral approach allows one to extend the notion of a quantum measurement beyond expanding the state of the measured system in some orthogonal basis. Such an extension retains, although presented in a slightly different form, all conceptual problems of a standard impulsive von Neumann measurement [9]. Our discussion of the relation between restricted path sums and von Neumann-like measurements would end here were it not for the existence of variables that do not commute with the chosen operator $\hat{A}$ (e.g., the momentum $\hat{p}$, which does not commute with the position $\hat{x}$ [13]). Each such variable, $\hat{B}$, requires a different set of paths to represent its measurement amplitude. This presents a problem if one wishes to discuss simultaneous measurements of such quantities in the language of path sums. In Sec. IV we will analyze the possibility of constructing simultaneous histories for two noncommuting variables and their relation to von Neumann-like measurements.

IV. SIMULTANEOUS HISTORIES AND MEASUREMENTS FOR NONCOMMUTING VARIABLES

Consider next two variables A and B whose operators do not commute, $[\hat{A},\hat{B}]\neq 0$. The presence of two, quantities rather than just one, makes the path analysis of measurements involving both $\hat{A}$ and $\hat{B}$ more complicated. Following the considerations of Sec. III, we begin with a simultaneous von Neumann-like measurement of two operators [the operator functions are understood as in Eq. (21)]

$$\hat{C}_i = \mathcal{F}_{i1}(\hat{A},t) + \mathcal{F}_{i2}(\hat{B},t), \quad i=1,2 \tag{24}$$

so that the Schroedinger equation describing the system coupled to two meters becomes $[\underline{f}\equiv(f_1,f_2)]$

$$i\partial_t|\Psi(t|\underline{f})\rangle = [\hat{H} - i\hat{C}_1\partial_{f_1} - i\hat{C}_2\partial_{f_2}]|\Psi(t|\underline{f})\rangle, \tag{25}$$

$$|\Psi(t=0|\underline{f})\rangle = G(\underline{f})|\Psi_0\rangle. \tag{26}$$

As in Eqs. (11) and (12), the wave function can be obtained as a Fourier transform $[\underline{\lambda}=(\lambda_1,\lambda_2),\ \underline{\lambda f}=\lambda_1 f_1+\lambda_2 f_2]$,

$$|\Psi(t|\underline{f})\rangle = (2\pi)^{-1}\int d\underline{\lambda}G(\underline{\lambda})\exp(i\underline{\lambda f}) \times\exp\left\{-i\int_0^t[H+\lambda_1\hat{C}_1+\lambda_2\hat{C}_2]dt'\right\}|\Psi_0\rangle. \tag{27}$$

Slicing again the time interval $[0,t]$ into N segments of length ϵ, we encounter exponential terms of the form

$$\exp\{-i\hat{H}(j\epsilon)+\lambda_1[\mathcal{F}_{11}(\hat{A},j\epsilon)+\mathcal{F}_{12}(\hat{B},j\epsilon)] + \lambda_2[\mathcal{F}_{21}(\hat{A},j\epsilon)+\mathcal{F}_{22}(\hat{B},j\epsilon)]\epsilon\}$$

which, for $\epsilon\to 0$, may be factorized by applying the relation (14) twice

$$\exp\{-i\hat{H}(j\epsilon)\epsilon\}\exp\{-i\epsilon[\lambda_1\mathcal{F}_{11}(\hat{A},j\epsilon)+\lambda_2\mathcal{F}_{21}(\hat{A},j\epsilon)]\} \times\exp\{-i\epsilon[\lambda_1\mathcal{F}_{12}(\hat{B},j\epsilon)+\lambda_2\mathcal{F}_{22}(\hat{B},j\epsilon)]\}. \tag{28}$$

Note that the choice of the order of the terms in Eq. (28) is not uniquely prescribed as we could, for example, interchange any of the two exponentials. We will proceed for now with the ordering defined by Eq. (28) because it leads, for example, to the well-known phase space path integral [2,3] when A and B are the particle's coordinate an momentum. We will discuss other possibilities at the end of this section. With the order chosen, inserting the spectral representation of the last two exponentials into Eq. (28) and performing integration over $d\underline{\lambda}$ in Eq. (27) yields a restricted path sum

$$\langle\Psi_1|\Psi(t|\underline{f})\rangle = \int d\underline{f'}G(\underline{f}-\underline{f'}) \times\sum_{[ab]}\delta\left\{f_1-\int_0^t[\mathcal{F}_{11}(a,t')+\mathcal{F}_{12}(b,t')]dt'\right\} \times\delta\left\{f_2-\int_0^t[\mathcal{F}_{21}(a,t')+\mathcal{F}_{22}(b,t')]dt'\right\} \times\langle\Psi_1|\Phi(t|[a,b])\rangle, \tag{29}$$

where the substates $|\Phi(t|[a,b])\rangle$ are given by

$$|\Psi(t|[a,b])\rangle \equiv \lim_{N\to\infty}\left\{\prod_{j=1}^{N}\exp[-i\epsilon\hat{H}(j\epsilon)]|a_{k_j}\rangle\langle a_{k_j}|b_{n_j}\rangle\langle b_{n_j}|\right\} \times|\Psi_0\rangle, \tag{30}$$

and the path sum $\Sigma_{[a,b]}$ denotes, as before, the limit $\lim_{N\to\infty}\Sigma_{k_1,\ldots,k_N,n_1,\ldots,n_N}$.

It is instructive to compare the quantum result (29) and (30) with its classical counterpart. Consider for this purpose a classical system coupled to two classical meters with the positions (f_1,f_2) and the momenta (λ_1,λ_2) respectively, so that the total Hamiltonian is

$$H(p,x)+\lambda_1 C_1(p,x)+\lambda_2 C_2(p,x),$$

where $C_i(p,x)$ are the classical variables corresponding to the operators (24). It is readily seen that the momenta of both meters are conserved. Setting $\lambda_1=\lambda_2=0$ as well as $f_1(0)=f_2(0)=0$ and integrating the pointer's equations of motion yields

$$f_1(t)=\int_0^t[\mathcal{F}_{11}(A,t')+\mathcal{F}_{12}(B,t')]dt',$$

$$f_2(t)=\int_0^t[\mathcal{F}_{21}(A,t')+\mathcal{F}_{22}(B,t')]dt',$$

$$\dot{A}=\{A,H\},$$

$$\dot{B}=\{B,H\},$$

where $\{Z,H\}$ denotes the Poisson's bracket [21]. Thus, in the classical case, there is only one simultaneous path

$\{A(t),B(t)\}$ and the only possible pointer positions are those coinciding with the values of the functionals $\int_0^t[\mathcal{F}_{11}(A,t')+\mathcal{F}_{12}(B,t')]dt'$ and $\int_0^t[\mathcal{F}_{21}(A,t')+\mathcal{F}_{22}(B,t')]dt'$. In the quantum case, there is a variety of paths giving various values to the two functionals. As a result, the probability amplitude to find the meter readings f_1 and f_2 is found as the sum of the individual path amplitudes $\langle\Psi_1|\Phi(t|[a,b])\rangle$ over the paths that give these values to these functionals. This simple analysis of the work of von Neumann-like meters is one of the central results of this paper.

The path decomposition (29) is similar to that in Eq. (15) with the difference that it involves two-component paths $\{a(t'),b(t')\}$ whose first and second components take the values from the spectra of the operators $\hat{A}$ and $\hat{B}$, respectively. A coherent sum over the values of one variable reduces (29) to the single variable path decomposition (16) for the other, e.g.,

$$\sum_{[b]}|\Psi(t|[a,b])\rangle=|\Psi(t|[a])\rangle. \tag{31}$$

Choosing A and B to be the coordinate p and momentum x, respectively, of a non relativistic particle with the classical Hamiltonian

$$H(p,x)=p^2/2m+V(x),$$

we obtain the phase space path integral [2,3]. Indeed, using

$$\langle p|x\rangle=\langle x|p\rangle^*=(2\pi)^{-1/2}\exp(ipx)$$

and noting that $\lim_{\epsilon\to0}\langle p|\exp\{-i[\hat{p}^2/2m+V(\hat{x})]\epsilon\}|x\rangle\approx\exp\{-i[p^2/2m+V(x)]\epsilon\}\langle p|x\rangle$, we rewrite Eq. (30) as

$$|\Psi(t|[x,p])\rangle=\lim_{N\to\infty}(2\pi)^{1/2-N}\exp(-\hat{H}\epsilon)|p_N\rangle\langle x_1|\Psi_0\rangle \times\exp\left\{i\sum_{j=1}^{N-1}\left[\frac{p_jx_{j+1}-p_j^2}{2m-V(x_{j+1})}\right]\epsilon-i\sum_{j=1}^{N}p_jx_j\epsilon\right\}. \tag{32}$$

For a particle starting in a state $|x'\rangle$ and then postselected in some $|x''\rangle$, the amplitude for a path becomes

$$A([p,x],x'',x')\equiv\langle x''|\Psi(t|[x,p])\rangle =\exp\left\{-i\int_0^t[p\dot{x}-H(p,x)]dt'\right\} \tag{33}$$

and summing Eq. (33) over all paths connecting x' and x'', yields the standard phase space path integral for the propagator [2,3]

$$K(x'',x',t)=\int DpDx\exp\left\{-i\int_0^t[p\dot{x}-H(p,x)]dt'\right\}.$$

Note that the fact that the commutator of the position and momentum operators is a number rather than an operator, allowed us to write the substate (30) in a more compact form (32). We will further explore this property in the following sections.

The convenience of Eqs. (29) and (30) is that they can be used in at least two different ways. (i) The choice $\mathcal{F}_{12}=\mathcal{F}_{21}=0$ corresponds to two von Neumann-like measurements of the variables $\mathcal{F}_{11}(\hat{A},t)$ and $\mathcal{F}_{22}(\hat{B},t)$ for which the accurate measurement amplitude can now be obtained as a sum over of paths giving definite values to the functionals

$$F_1[a]=\int_0^t\mathcal{F}_{11}(a,t')dt'$$

and

$$F_2[b]=\int_0^t\mathcal{F}_{22}(b,t')dt'.$$

Depending on the choice of $\mathcal{F}_{11}$ and $\mathcal{F}_{22}$, the two measurements can occur consecutively, partially overlap in time, or coincide. The latter case was studied in the famous work by Arthurs and Kelly [14] and we will return to the joint measurements in Sec. V. (ii) The choice $\mathcal{F}_{21}=\mathcal{F}_{22}=0$ corresponds to a single von Neumann-like measurement of the sum of the variables $\mathcal{F}_{11}(\hat{A},t)$ and $\mathcal{F}_{12}(\hat{B},t)$. One can, of course, treat this sum as a new variable, calculate its spectrum and eigenpaths, and construct the measurement amplitude as described in Sec. II. We are, however, interested in using for this purpose the simultaneous path for the operators $\hat{A}$ and $\hat{B}$ defined in Eq. (30). With the help of Eq. (30), the measurement amplitude can now be obtained by summing over the classes of path giving a definite value to the functional

$$F[a,b]=\int_0^t[\mathcal{F}_{11}(a,t')+\mathcal{F}_{22}(b,t')]dt',$$

which involves both components of a path

$$|\Psi(t|f)\rangle=\sum_{[ab]}\delta\left\{f-\int_0^t[\mathcal{F}_{11}(a,t') +\mathcal{F}_{22}(b,t')]dt'\right\}|\Phi(t|[a,b]). \tag{34}$$

It is important to note here that the simultaneous path expansion (29) cannot be extended to quantities that involve products, rather than sums of the variables A and B. The difficulty arises because the same classical product, e.g., A^2B^2, may have different Hermitian operator realizations in the quantum case, for example, $[\hat{A}^2\hat{B}^2+\hat{B}^2\hat{A}^2]/2$, $\hat{A}\hat{B}^2\hat{A}$, etc. Now, the right-hand side of Eq. (29) involves the substates $|\Phi(t|[a,b])\rangle$, which were defined previously without any reference to a particular ordering of the operator product to be measured. Neither can this information be contained in the δ-function restriction term. Thus, it is not clear to which of the two realizations, $[\hat{p}^2\hat{x}^2+\hat{x}^2\hat{p}^2]/2$ or $\hat{x}\hat{p}^2\hat{x}$, would correspond, for example, the restricted phase space path integral

$$\langle x_1|\Phi'(t|f)\rangle=\int DpDx\,\delta\left(f-\int_0^tp^2x^2dt'\right) \times\exp\left\{\int_0^t[p\dot{x}-H(p,x)]dt'\right\}. \tag{35}$$

A close inspection (see Appendix A) shows that it corresponds to neither. In fact, the evolution of the substate $|\Phi'(t|f)\rangle$ is nonunitary and cannot, therefore, be related to any measurement procedure. Without such a relation the probability $|\langle x_1|\Phi'(t|f)\rangle|^2$ has no obvious meaning and the restricted path integral remains a purely mathematical construct. Note that a problem, similar to the one just described, arises whenever path integrals are used to impose operator ordering [2]. The form of the measurement subsets employed in Eq. (35) was fixed when by our arbitrary choice of the order of the three terms in Eq. (28). As shown in Appendix A, valid path decompositions for various realizations for the operators containing products of powers of the particle's coordinate and momentum can be obtained by using alternative definitions of the substates in $|\Phi(t|[a,b])\rangle$ in Eq. (30) and the paths themselves, at the cost of losing the simplicity and elegance of the conventional phase space path integral (33). These realizations correspond to different resolutions of the exponential operator $\exp(-i\hat{H}\epsilon - i\lambda\hat{p}\epsilon - i\lambda\hat{x}\epsilon)$, which are possible because the factorization formula (14) only states that for $\epsilon\to 0$ the exponential $\exp[(\hat{A}+\hat{B}+\hat{C}+\ldots)\epsilon]$ can be factorized but does not prescribe a unique way for doing so. The equation of motion obeyed by a restricted path sum of a particular type usually depends on the choice of such factorization. As a result, it is always possible to choose a suitable path sum to describe the measurement of a particular operator realization of a quantity involving products of the two variables. One does not, however, find a unique recipe to suit all cases and caution should be exercised when using path integrals similar to (35). Restricted phase space path integrals were used, for example, in [16], but we are not aware of any detailed analysis of the limits of their applicability. In Sec. V, we provide examples of our approach by considering in further detail measurements of the particle's position and momentum.

V. SIMULTANEOUS MEASUREMENTS OF TWO CONJUGATE VARIABLES: CHANGING THE ORDER BY ENTANGLEMENT

We continue by considering a system with a zero Hamiltonian

$$\hat{H} \equiv 0. \tag{36}$$

The approximation (36), which is often used [12,14], assumes that the measurement(s) are sufficiently fast for the effects of the Hamiltonian to be neglected while the system interacts with the meter(s). In fact, our analysis can be applied to any two operators whose commutator is a purely imaginary C number

$$[\hat{A},\hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} = C = i|C|. \tag{37}$$

Applying the Baker-Campbell-Hausdorf formula, we have the exact result

$$\exp(a\hat{A} + b\hat{B}) = \exp(a\hat{A})\exp(b\hat{B})\exp(-abC/2), \tag{38}$$

which holds for arbitrary constants a and b and which we will use to verify the derivation of Sec. IV. Consider next two von Neumann-like meters, measuring $\hat{A}$ and $\hat{B}$, respectively, whose switching functions $\beta_j(t)$, $j=1,2$, may or may not overlap in time. As has been shown in Sec. II, such a measurement makes distinguishable the classes of simultaneous paths labelled by the values f_1 and f_2 of the functionals $\int_0^t \beta_1(t')a(t')dt'$ and $\int_0^t \beta_2(t')b(t')dt'$. The corresponding measurement state can be written as the Fourier integral

$$\begin{aligned}|\Psi(t|\underline{f})\rangle = (2\pi)^{-2}\int d\underline{\lambda}\exp(i\underline{\lambda}\underline{f})G(\underline{\lambda})\\ \times\exp\left[\epsilon^2\lambda_1\lambda_2 C\sum_{j=1}^{N}\beta_1(j\epsilon)\beta_2(j\epsilon)\right]\\ \times\left\{\prod_{j=1}^{N}\exp[-i\lambda_1\beta_1(j\epsilon)\epsilon\hat{A}]\right.\\ \left.\times\exp[-i\lambda_2\beta_2(j\epsilon)\epsilon\hat{B}]\right\}|\Psi_0\rangle,\end{aligned} \tag{39}$$

where, as before, G is the initial state of the two meters, and we have used Eq. (38) to factorize the exponential $\exp\{-i\epsilon[\lambda_1\beta_1(j\epsilon)\hat{A}+\lambda_2\beta_2(j\epsilon)\hat{B}]\}$ and separate the terms that contain the commutator C. It is readily seen that the contribution of these terms can be neglected, as assumed while deriving Eq. (29), since the first exponent in Eq. (39) tends to $\epsilon\lambda_1\lambda_2\int_0^t\beta_1(t')\beta_2(t')dt'$ and vanishes as $\epsilon\to 0$. Thus, we obtain Eq. (29) with the substates $|\Phi(t|[a,b])\rangle = \prod_{j=1}^{N}|a_{jk}\rangle\langle a_{jk}|b_{jl}\rangle\langle b_{jl}|$, which have a particularly simple form since $\hat{H}\equiv 0$.

If $\hat{A}$ is measured before $\hat{B}$, i.e., if the first meter is switched on and off before the second meter is enacted, all exponentials containing $\hat{A}$ occur before those containing $\hat{B}$ in the product of Eq. (39). If, on the other hand, the actions of the two meters overlap in time, the two kinds of exponentials are enmeshed. It follows from Eq. (38) that

$$\exp(a\hat{A})\exp(b\hat{B}) = \exp(b\hat{B})\exp(a\hat{A})\exp(abC), \tag{40}$$

and we can use this relation to rearrange the operator exponentials in such a way that all those containing $\hat{A}$ are to the right of those containing $\hat{B}$, as if the measurement of $\hat{A}$ preceded that of $\hat{B}$. The price of such a rearrangement is the reappearance of the commutator term, which, this time, cannot be neglected. Indeed, moving to the right the first exponential $\exp[-i\lambda_1\epsilon\beta_1(k\epsilon)\hat{A}]$ sandwiched between similar terms containing $\hat{B}$ produces the factor $\exp[-i\lambda_1\lambda_2|C|\beta_1(k\epsilon)\beta_2(k\epsilon)\epsilon^2]$, moving the second one produces two similar factors, and so on. The net result is

$$\begin{aligned}|\Psi(t|\underline{f})\rangle = (2\pi)^{-2}\int d\underline{\lambda}\,\exp(i\underline{\lambda}\underline{f})G(\underline{\lambda})\exp[-i\lambda_1\lambda_2|C|I(t)]\\ \times\exp(-i\lambda_1\hat{B})\exp(-i\lambda_1\hat{A})|\Psi_0\rangle,\end{aligned} \tag{41}$$

where for $\epsilon\to 0$ the factor $I(t)$ does not vanish but becomes the double integral

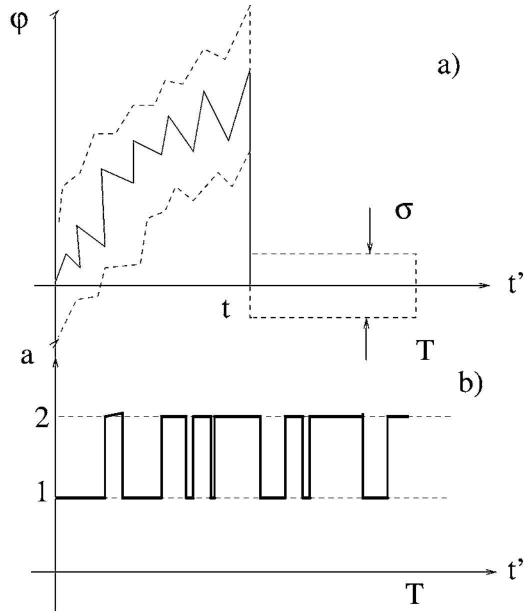

FIG. 1. (a) Rectangular switching functions for the two meters as given in Eq. (43) and (b) the region of integration in Eq. (42) for the switching functions shown in (a).

$$I(t) \approx \int_0^t dt' \int_0^{t'} dt'' \beta_1(t')\beta_2(t''), \tag{42}$$

and we have assumed

$$\int_0^t \beta_i(t')dt' = 1, \quad i = 1,2.$$

For example, the switching functions β_i can be chosen as rectangular "windows," shifted relative to each other,

$$\beta_1(t) = T^{-1}\theta_{T,2T}(t), \quad \beta_2(t) = T^{-1}\theta_{\tau,\tau+T}(t), \tag{43}$$

as shown in Fig. 1(a) and we choose $t \geqslant 3T$. Now for $\tau=0$ the measurement of $\hat{B}$ occurs before that of $\hat{A}$, for $\tau=2T$ $\hat{A}$ is measured before $\hat{B}$, and choosing $\tau=T$ yields a simultaneous joint measurement of the two operators, similar to that studied by Arthurs and Kelly [14]. In all other cases, the two measurements partially overlap in time. With this choice of the switching function, the integral (42) becomes [see Fig. 1(b)]

$$I_\tau(t) = 1 - \tau^2/2T \quad \text{for} \quad 0 \geqslant \tau \geqslant T \tag{44}$$

and

$$(2 - \tau/T)^2/2 \quad \text{for} \quad T \geqslant \tau \geqslant 2T.$$

Let us assume next that we have an experimental setup to perform a sequential measurements of the two variables with $\hat{A}$ always measured first, i.e., $\tau=2T$ and the two uncorrelated pointers prepared initially in a product state $G_1(f_1)G_2(f_2)$, so that in Eq. (41)

$$G(\underline{\lambda}) = G_1(\lambda_1)G_2(\lambda_2). \tag{45}$$

It is readily seen now that preparing the two meters in a correlated entangled state,

$$G(\underline{\lambda}) = G_1(\lambda_1)G_2(\lambda_2)\exp(-i\lambda_1\lambda_2|C|I_\tau) \tag{46}$$

or, equivalently,

$$G(\underline{f}) = \int G_1(f_1 - f_1')G_2(f_2 - f_2')\exp\left(\frac{if_1'f_2'}{|C|I_\tau}\right)dfdf', \tag{47}$$

allows us to obtain the results corresponding to the measurements performed with uncorrelated meters in any temporal order, even though, in reality, the system is always coupled first to the first and after that to the second pointer. This interesting possibility to "manipulate the order of the measurements" by entanglement arises from the fact that for the conjugate variables the commutator of exponential operators produces an additional number factor in the Fourier transform (39), which can then be absorbed in the initial meter state. Note that the meter state (46) is broad if the states $G_1(f_1)$ and $G_2(f_2)$ are sharp, thus giving large uncertainties in the initial positions of the two pointers.

We conclude this section by giving the expression for the distribution of the results of a similar measurement of the position, $\hat{A} \equiv \hat{x}$, and momentum, $\hat{B} \equiv \hat{p}$, $(t \geqslant 3T)$

$$\rho(f_1,f_2) \equiv \frac{\langle \Psi(t|f_1,f_2)|\Psi(t|f_1,f_2)\rangle}{\int d\underline{f}\langle \Psi(t|f_1,f_2)|\Psi(t|f_1,f_2)\rangle}, \tag{48}$$

conducted by two uncorrelated Gaussian meters,

$$G(\underline{f}) = \exp\left(\frac{-f_1^2}{\sigma_1^2}\right)\exp\left(\frac{-f_2^2}{\sigma_2^2}\right), \tag{49}$$

on a particle, initially prepared in a Gaussian state

$$\langle x|\Psi_0\rangle = \exp\left(\frac{-x^2}{\Delta x^2}\right). \tag{50}$$

It is given by the product of two Gaussian functions (details of the derivation are given in Appendix B)

$$\begin{aligned}\rho(f_1,f_2,\Psi_0) = 2\pi^{-1}&\left[\frac{(D-1)^2}{\sigma_1^2} + \frac{\sigma_2^2}{4} + \frac{1}{\Delta x^2}\right]^{1/2}\\ &\times\left(\frac{D^2}{\sigma_2^2} + \frac{\sigma 12^2}{4} + \frac{\Delta x^2}{4}\right)^{1/2}\\ &\times\exp\left[\frac{-f_1^2}{2}\left(\frac{D^2}{\sigma_2^2} + \frac{\sigma_1^2}{4} + \frac{\Delta x^2}{4}\right)\right]\\ &\times\exp\left\{\frac{-f_2^2}{2}\left[\frac{(D-1)^2}{\sigma_1^2} + \frac{\sigma_2^2/4 + 1}{\Delta x^2}\right]\right\}\end{aligned} \tag{51}$$

where $D(\tau) \equiv I_\tau$ given by Eq. (44) so that for $D=0$ the coordinate is measured first, while for $D=1$ it is the momentum. The choice $D=1/2$, and the special "balanced" choice of the accuracies of the two meters $\sigma_1 = 1/\sigma_2$, corresponds to the simultaneous measurement of the two quantities, as analyzed by Arthurs and Kelly [14]. Note that in this special case the probability $\rho(f_1,f_2)|$ can be written as square of the modulus of a single probability amplitude (see Eq. (8) of Ref. [14]). However, in general, Eq. (48) must be used.

VI. SUM OF TWO CONJUGATE VARIABLE: WEAK MEASUREMENTS

Next we consider a von Neumann-like measurement of an operator $\hat{S}$ which is a linear combination of the particle's position and momentum

$$\hat{S} = a\hat{x} + b\hat{p}, \tag{52}$$

where we will put the constants a and b, required to insure that $\hat{S}$ has correct units, to unity, $a=b=1$. Again, a classical analogy is helpful. A classical meter coupled to a particle with the Hamiltonian

$$H(p,x) + \lambda[\beta_1(t)p + \beta_2(t)x] \tag{53}$$

measures the values of $p+x$ provided both x and p do not change appreciably during the measurement and $\int\beta_i(t)dt=1$, $i=1,2$,

$$f = \int_0^t [\beta_1(t')p(t') + \beta_2(t')x(t')]dt' \approx p(0) + x(0). \tag{54}$$

Since a classical measurement does not perturb the particle, the switching functions $\beta_{1,2}(t)$ can be chosen, for example, so that the pointer measures the momentum first, then the coupling is turned off and the same pointer is coupled to the position x. Alternatively, this order can be reversed or, what may seem more natural, the pointer can be coupled it both to x and p at the same time, using $\beta_1(t)=\beta_2(t)$. It is easy to check that all this choices give the same result: the pointer's position will be correlated with the value $p+x$ after the measurement has taken place.

Quantally, it is not so. A meter which is coupled to measure the momentum first, would perturb the particle's position, and therefore, the result of the measurement. Thus, different couplings would, in principle, give different results. As in Sec. V, we will neglect the system's own Hamiltonian during the period it is coupled to a single meter designed to measure $\hat{p}+\hat{x}$ so that

$$i\partial_t|\Psi(t|f)\rangle = -i\partial_f[\beta_1(t)\hat{x} + \beta_2(t)\hat{p}]|\Psi(t|f)\rangle \tag{55}$$

with the switching functions β_i defined as in Sec. V. Note that the meter measures the value of $x+p$ in such a way that the individual values of x and p remain indeterminate. Proceeding as in Sec. V, we write the measurement state as

$$|\Psi(t|\underline{f})\rangle = (2\pi)^{-1}\int d\lambda\, \exp(i\lambda f)G(\lambda)\exp[-i\lambda^2|C|I(t)] \times\exp(-i\lambda\hat{p})\exp(-i\lambda\hat{x})|\Psi_0\rangle, \tag{56}$$

where, as before, $G(\lambda)$ is the Fourier transform of the initial meter state, and we have rearranged the operators so that those containing $\hat{x}$ occur first. The information about their original ordering is contained in the overlap of the switching functions $I(t)$ defined earlier in Eq. (42). Consider now the measurement amplitude for the particle prepared in a position state $|X\rangle$ and postselected after the measurement in a momentum state $|P\rangle$, $A(f,P,X)\equiv\langle P|\Psi(t|\underline{f})\rangle$ with the rectangular switching functions given in Eq. (43). From Eq. (56), we have

$$A(f,P,X) = (2\pi)^{-3/2}\exp(-iPX)\int d\lambda G(\lambda) \times\exp(-i\lambda^2 I_\tau)\exp\{i\lambda[f-(P+X)]\}, \tag{57}$$

where $I_\tau(t)$ is given by Eq. (44). If the pointer is first coupled to the particle's coordinate, $\tau=2T$, $I_\tau=0$ and the second Gaussian term in Eq. (56) disappears. The integral can be carried out to give the initial pointer state shifted by the value $P+X$,

$$A_{PX}(f) = (2\pi)^{-1/2}\exp(-iPX)G[f-(P+X)] \tag{58}$$

so that, for a system preselected with a known position and postselected with a known momentum, our measurement is sharp and yields the sum of the two. Note that this is achieved by choosing a particular order in which the couplings are enacted. The possibility to obtain such a result, or the lack of it, was discussed by Aharonov *et al.* in [22], who concluded that to have the values P, X and $P+X$ in the interim one may decrease the strength of the coupling by choosing the initial meter state to be very broad in the position space. We conclude this section by giving a simple illustration to their result. Suppose the pointer is coupled to measure both x and p at the same time, $I_\tau=1/2$, so that we have an additional term $\exp(-i\lambda^2/2)$ in the integral (56). As in Sec. V, the fact that the commutator of $\hat{x}$ and $\hat{p}$ is a number, allows us to cancel it by modifying the Fourier transform of the initial meter state

$$G'(\lambda) = G(\lambda)\exp\left(\frac{i\lambda^2}{2}\right). \tag{59}$$

With such an initial state, and the two couplings occurring simultaneously, the result of the measurement will be just Eq. (57). However, if the original initial state was, say, a narrow Gaussian

$$G(f) = (2\pi)^{-1}\int G(\lambda)\exp(i\lambda f)d\lambda = \exp\left(\frac{-f^2}{\sigma^2}\right),$$

the modified initial state will have the form

$$G'(f) = [\pi(\sigma^2+2i)]^{1/2}\exp\left[-\frac{f^2}{\sigma^2+4/\sigma^2} - i\frac{2f^2}{\sigma^4+4}\right] \tag{60}$$

and will, therefore, be broad, which can also be seen from the fact that multiplying Fourier transforms in Eq. (59) leads to a convolution in the f variable. In this way, we have obtained the result $P+X$ for a system starting in $|X\rangle$ and finishing in $|P\rangle$ at the cost of making the initial position of the pointer highly uncertain, i.e., by conducting, as suggested in [22], a particular type of a weak measurement.

VII. CONCLUSIONS AND DISCUSSION

To summarize, we have shown that a quantum von Neumann-like meter, like its classical counterpart, measures the value of a particular functional $F[\text{path}]$ defined on the paths, or a path, traced by the value of the measured quantity. Classically, only one such path exists, and the meter's pointer position coincides with the value of F. Quantally, there is a variety of virtual histories, and obtaining, in an accurate measurement, the pointer position f means that the system's state is as if it has evolved along only those paths that give the value f to $F[\text{path}]$. The paths, like the Feynman paths [1], are continuous but highly irregular and may only take values from the spectrum of the operator, which represents the measured quantity. This general picture applies also if several noncommuting variables are being measured. However, already for two noncommuting variables the situation is more complex. For a single variable A, the path decomposition is obtained by factorizing each term of the product $\Pi_{j=1}^{N}\exp(-i\hat{H}\epsilon-i\lambda\hat{A}\epsilon)$ as $N\to\infty$. It is easy to see that no matter how it is done, the final result is essentially the same. For two noncommuting variables, one needs to factorize $\Pi_{j=1}^{N}\exp(-i\hat{H}\epsilon-i\lambda_1\hat{A}\epsilon-i\lambda_2\hat{B}\epsilon)$, and this can be done in a variety of ways. For $\hat{A}=\hat{x}$ and $\hat{B}=\hat{p}$, one such way leads to the conventional phase space path integral, which, if restricted, can be used to construct measurement amplitudes for joint measurements of $\hat{A}$ and $\hat{B}$ as well as for a sum $\hat{C}=\mathcal{F}_1(\hat{x})+\mathcal{F}_2(\hat{p})$. It fails, however, for a product of the position and momentum operators. There is a simple reason for that. Although the above operator $\hat{C}$ is uniquely defined by its classical expression, a product allows for different operator realizations, depending on the order in which the noncommuting $\hat{x}$ and $\hat{p}$ are placed. Clearly, the conventional path integral cannot provide for all such choices. Path sums of different kinds can be constructed so that the measurement amplitude for a particular product can be obtained by a restriction. This is done at the price of changing the amplitudes assigned to each path and, in some cases, by redefining the paths themselves. As a result, these path sums usually lose the simplicity and elegance of the conventional path integral. Needless to say, any function of the position and momentum can be treated as an operator in its own right and a path representation can be constructed from its eigenvalues as in Sec. II, without a recourse to simultaneous histories for its constituent parts. However, restricted phase space integrals have been used to describe quantum measurements (see, for instance, [16]) and we thought it worthwhile to clarify this matter.

Furthermore, for a single variable, a conventional projection measurement [9] at $t=t_0$ is recovered when the functional is just the value of the path at t_0. Classically, the existence of a smooth classical path ensures that a "simultaneous" measurement of the two variables can, in fact, be conducted in any order, e.g., A can be measured just before B or vice versa. Quantally, because of the chaotic nature of the virtual paths, the amplitude to pass first through a and then through b is different from that for passing through a after b. In general, one has to consider a whole class of measurements that partially overlap in time, as has been done in Sec. III, where each such measurement can be seen as distinguishing between different classes of simultaneous paths. It is interesting to note that for two conjugate variables, such as $\hat{x}$ and $\hat{p}$ and $\hat{H}=0$, entangling the two meters with each other (but not with the system) allows one to "alter the order in which the measurements occur," without changing the way in which the actual coupling occurs between the pointers and the system.

Similarly, we note that a classical measurement of a single quantity represented by a sum of two variables can be conducted in different ways. For example, coupling the same meter to measure first A and then B, first B and then A, or A and B at the same time would give the same result, $A+B$. Not so quantally, where three different functionals corresponding to the three cases yield different amplitudes. For the coordinate and momentum, however, we can emulate an accurate measurement for any order by choosing an appropriate unsharp initial state of the meter, i.e., by making the measurement "weak," as was first suggested in [12].

Finally, for a particular functional to be measurable, the corresponding restricted sum should must evolve in time in a unitary manner, so that its equation of motion can be interpreted as a Schroedinger equation for the system coupled to a measuring device. It is easy to produce restricted path sums whose evolution is nonunitary [cf. Eq. (A4)]. Thus far, we have been unable to ascribe any physical significance to such sums or interpret them as probability amplitudes of any kind.

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APPENDIX A

Consider the equation of motion for a restricted path integral

$$|\Phi'(t|f)\rangle \equiv \int DpDx\,\delta\left(f-\int_0^t p^2x^2dt'\right)|\Phi(t|[p,x])\rangle, \tag{A1}$$

where the substates $|\Phi(t|[p,x])\rangle$ are defined in Eq. (32). In order to do so, we write the δ function as a Fourier integral, $(2\pi)^{-1}\int d\lambda\,\exp(i\lambda f-i\lambda\int_0^t p^2x^2dt')$, and discretize the time. With the integral in the exponent replaced by its Riemann sum, the Fourier transform of Eq. (A1) becomes

$$|\Phi'(t|\lambda)\rangle=\prod_{j=1}^{N}\exp(-i\hat{H}\epsilon)\int dp_jdx_j\exp(-i\lambda p_j^2x_j^2\epsilon)|p_j\rangle\langle p_j| \times|x_j\rangle\langle x_j|\Psi_0\rangle. \tag{A2}$$

Evaluating the difference $\Delta\Phi'\equiv\epsilon^{-1}[|\Phi'(t+\epsilon|\lambda)\rangle-|\Phi'(t|\lambda)\rangle]$,

$$\Delta\Phi' = \epsilon^{-1}\Big\{ \exp(-i\hat{H}\epsilon)\int dpdx\,\exp(-i\lambda p^2x^2\epsilon)|p\rangle \times\langle p|x\rangle\langle x| - 1 \Big\}|\Phi'(t|\lambda)\rangle = \Big\{ -i\hat{H} - i\lambda\int dp|p\Big\rangle p^2\Big\langle p|\int dx|x\Big\rangle x^2\langle x|\Big\}|\Phi'(t|\lambda)\rangle + O(\epsilon), \tag{A3}$$

and performing the Fourier transform from λ to f shows that $|\Phi'(t|f)\rangle$ satisfies the Schroedinger-like equation

$$i\partial_t|\Phi'(t|f) = [\hat{H} - i\partial_f\hat{p}^2\hat{x}^2]|\Phi'(t|f)\rangle. \tag{A4}$$

It is readily seen that the last term in the square bracket is not Hermitian since the momentum and the position do not commute. Thus, the Eq. (A4) does not describe an interaction of the particle with any physical meter. Note that the order in which the operators $\hat{p}^2$ and $\hat{x}^2$ occur in the right-hand side of Eq. (A4) is determined by the choice made while defining the measurement substates in Eq. (30) when we factorized the operator exponential as (28)

$$\exp[-i\epsilon(\hat{H}+\lambda\hat{p}+\lambda\hat{x})] \approx \exp(-i\hat{H}\epsilon)\exp(-i\lambda\hat{p}\epsilon)\exp(-i\lambda\hat{x}\epsilon). \tag{A5}$$

However, this choice is by no means unique. A physical interaction could be obtained, for example, if the product $\hat{p}^2\hat{x}^2$ is replaced by its Hermitian part, $\{\hat{p}^2\hat{x}^2\}_H \equiv [\hat{p}^2\hat{x}^2+\hat{x}^2\hat{p}^2]/2$. Repeating the steps leading to Eq. (A4) shows that the restricted sum corresponding to a measurement of $\{\hat{p}^2\hat{x}^2\}_H$ can be obtained by redefining the substates $|\Phi(t|[p,x])\rangle$ so that in Eq. (28) the operator exponential $\exp\{-i\epsilon[\hat{H}+\lambda\hat{x}+\lambda\hat{p}]\}$ is resolved as

$$\exp[-i\epsilon(\hat{H}+\lambda\hat{p}+\lambda\hat{x})] \approx \frac{\exp(-i\hat{H}\epsilon)[\exp(-i\lambda\hat{p}\epsilon)\exp(-i\lambda\hat{x}\epsilon)+\exp(-i\lambda\hat{x}\epsilon)\exp(-i\lambda\hat{p}\epsilon)]}{2}. \tag{A6}$$

Equation (A6), which is obtained by dividing the initial exponential into two equal halves and then choosing different orders of $\hat{p}^2$ and $\hat{x}^2$, is another form, valid for a sufficiently small ϵ. Inserting these new substates into Eq. (A1), we obtain a valid path decomposition for the measurement amplitude, but the simple expression (33) for the phase assigned to each path is lost due to the presence of two terms, rather than just one, added within each time slice. In the same vein, we can show that to obtain the measurement states of $\hat{x}\hat{p}^2\hat{x}$ we require the resolution

$$\exp[-i\epsilon(\hat{H}+\lambda\hat{p}+\lambda\hat{x})] \approx \exp(-i\hat{H}\epsilon)\exp\left(\frac{-i\lambda\hat{x}\epsilon}{2}\right)\exp(-i\lambda\hat{p}\epsilon)\exp\left(\frac{-i\lambda\hat{x}\epsilon}{2}\right) \tag{A7}$$

and an even more drastic overhaul of the path sum (32) because now within each time slice the path must be specified by two positions and momenta for each time slice. We conclude by giving the expression for the corresponding path sum

$$|\Phi_{xp^2x}(t|f)\rangle = \int Dp'DpDx'Dx\,\delta\left(f-\int_0^t x'p^2xdt'\right) \times|\Phi(t|[p',px',x])\rangle, \tag{A8}$$

where

$$|\Phi(t|[p',px',x])\rangle \equiv \lim_{N\to\infty}\int dx_{N+1}|x_{N+1}(2\pi)^{-2N} \times\exp\Big\{ i\sum_j \Big[p'_j(x_{j+1}-x'_j) + p_j(x'_j-x_j)\frac{-p'^2}{2m\epsilon} - V(x')\epsilon \Big]\Big\}\langle x_1|\Psi_0\rangle. \tag{A9}$$

In general, these examples demonstrate that, for two noncommuting variables, there is no unique simultaneous path decomposition and a different ones need to be chosen when constructing restricted path sums for various ordered products of these variables.

APPENDIX B

We need to estimate the integrals

$$\int dq|\langle q|\Psi(t|\underline{f})\rangle|^2, \tag{B1}$$

where, from Eq. (39), we have ($D \equiv |C|\int_0^t dt'\int_0^{t'} dt''\beta_1(t')\times\beta_2(t'')$)

$$\Psi(t|\underline{f}) = (2\pi)^{-2}\int d\underline{\lambda}\tilde{G}_1(\lambda_1)\tilde{G}_2(\lambda_2)\exp(i\underline{\lambda}\underline{f} - i\lambda_1\lambda_2 D) \times\langle q|\exp(-\lambda_2\hat{p})\exp(-\lambda_1\hat{x})|\Psi_0\rangle, \tag{B2}$$

and we have introduced the tildes to distinguish the Fourier transform of the meter's initial state $\widetilde{G}_i(\lambda_i)$ from the state $G_i(f_i)$ itself. Writing the two operator exponentials as

$$\int\int dxdp \exp(-i\lambda_1 x - i\lambda_2 p - ipx)|p\rangle\langle x|,$$

performing the integrals over $d\underline{\lambda}$ and dp, and introducing the new variable $z \equiv x - q$ yields

$$\Psi(t|\underline{f}) = \int dx G_1[f_1 - q - (D+1)z]\widetilde{G}_2(z) \times \exp(if_2 z)|\Psi_0(q+z)\rangle. \quad \text{(B3)}$$

For $G_1(f_1) = \exp(-af_1^2)$, $\widetilde{G}_2(\lambda_2) = \exp(-b\lambda_2^2)$, and $\Psi_0(x) = \exp(-cx^2)$ Eqs. (B1) and (B3) become Gaussian integrals, which, fortunately, need not be evaluated explicitly. Indeed, the variable f_2 enters the exponent of the integrand linearly, in combination with the first power of z and with a purely imaginary coefficient. Thus, its only real valued contribution to the exponent of the integral is of the form $-f_2/k$, where the coefficient k is proportional to that multiplying the second power of z and is easily evaluated to be

$$k = 2[a(D+1)^2 + b + c]. \quad \text{(B4)}$$

To obtain a similar coefficient k' for the term containing $-f_1^2$ we may interchange the order of the operator exponentials in (B2), i.e., write $\exp(-i\lambda_1\hat{x})\exp(-i\lambda_2\hat{p})$ and repeat the above analysis projecting $|\Psi(t|\underline{f})\rangle$ on a momentum state $|p\rangle$. A detailed inspection shows that it leads to replacing the coordinate width of the initial state by that in the momentum space, $c \to 1/4c$, and, similarly, $b \to 1/4b$, $a \to 1/4a$. Also, we must change $D \to D' \equiv D - 1$, due to the change in the sign of the commutator and interchange β_1 with β_2, which yields $k' = 2(D^2/4b + 1/4a + 1/4c)$. Thus, the normalized distribution takes the form

$$\rho(f_1, f_2) = (\pi^2 kk')^{-1/2} \exp\left(\frac{-f_1^2}{k'} - \frac{f_2^2}{k}\right), \quad \text{(B5)}$$

which, after inserting the values of a, b, and c, becomes Eq. (48).

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