

Role of initial state reconstruction in short- and long-time deviations from exponential decay

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We consider the role of the reconstruction of the initial state in the deviation from exponential decay at short and long times. The long-time decay can be attributed to a wave that was, in a classical-like, probabilistic sense, fully outside the initial state or the inner region at intermediate times, i.e., to a completely reconstructed state, whereas the decay during the exponential regime is due instead to a nonreconstructed wave. At short times, quantum interference between reconstructed and nonreconstructed paths is responsible for the deviation from the exponential decay. We may thus conclude that state reconstruction is a “consistent history” for long-time deviations but not for short ones.

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I. INTRODUCTION

The decay from unstable quantum states is a recurrent and important topic in several fields of physics. The exponential decay, which is by far the most common type, is surrounded by deviations at short and long times [1]. The short-time deviations have been much discussed, in particular in connection with the Zeno effect, and experimental observations have been performed quite recently [2]. An exactly solvable analytical model for a single resonance that describes these regimes follows for a decaying particle of mass m from some general considerations of the survival amplitude [3],

$$A(T) = \langle \psi_0 | e^{-iHT} | \psi_0 \rangle = \frac{i}{2\pi} \int_C e^{-ik^2 T} a(k) dk, \quad (1)$$

(the units employed are $\hbar=2m=1$), where the contour C goes from $c+i\infty$ to $ic+i\infty$, with $c>0$ a small constant, along the upper half complex k plane, and ψ_0 is the initial state. If $a(k)$ is given by a single resonance, $a(k)=2k[D_r/(k-k_r)+D_r^*/(k+k_r^*)]$, where $k_r^2=\varepsilon_r-i\Gamma_r/2$ and $D_r=k_r/\text{Re}(k_r)$. It follows from the conditions $A(T=0)=1$ and $a(k=0)=0$ that [3]

$$A(T) = \frac{1}{2} D_r e^{-y_r^2} - \frac{1}{2} [D_r w(-y_r) + D_r^* w(y_{-r})], \quad (2)$$

where $y_r = -\exp(-i\pi/4)k_r T^{1/2}$, $y_{-r} = \exp(-i\pi/4)k_r^* T^{1/2}$, and $w(z) = \exp(-z^2) \text{erfc}(-iz)$. The second term in Eq. (2) dominates at short and long times compared with the lifetime. For other analytical or semianalytical examples, see [4–6], which show that in general the exponential decay may be related to a pole residue and the deviation to a line integral resulting from a contour deformation along a steepest-descent path in the complex momentum plane. The long-time deviations may also be associated with the boundedness of the spectrum [1] and the short-time deviations with the existence of the

energy moments [5], but clearly a more physical and intuitive understanding is desirable [7,8]. In this respect, Fonda and Ghirardi [7] put forward an interesting connection following Ersak’s previous insight [9]: “Concluding, we can say that the physical mechanism producing deviations from the exponential decay law of an unstable system is therefore provided by the regeneration of the initial unstable state. In this regeneration process, a very decisive role is played by the decayed states, which, as a consequence of the evolution of the system, are able to reconstruct partially the initial unstable state through a process of rescattering” [7]. While that paper was centered about long-time deviations, in fact the arguments given are also valid for short times, or arbitrary deviations [10], as we shall see. Let us decompose the evolution up to the final time T into different “paths” with the aid of the following complementary projectors:

$$P = |\psi_0\rangle\langle\psi_0|, Q = 1 - P, \quad (3)$$

at an intermediate time t ,

$$A(T) = \langle \psi_0 | e^{-iH(T-t)} (P + Q) e^{-iHt} | \psi_0 \rangle \\ = A(T-t)A(t) + \langle \psi_0 | e^{-iH(T-t)} Q e^{-iHt} | \psi_0 \rangle. \quad (4)$$

Except for the last term, $A(T)$ would be exponential [7,9]. Since the last term involves a path that goes out of the initial state at the intermediate time, the deviation from exponential decay is attributed to state reconstruction.

The generality of this argument, however, diminishes somehow its explanatory power, and we may rightly wonder what is the difference, if any, between the physical origin of short- and long-time deviations, in particular because some of the mathematical arguments provided to explain the deviations are different for long (Paley and Wiener theorem) and short times (behavior at the origin, and existence of energy moments). It also seems reasonable to demand a detailed description of the reconstruction process. In addition, a key conceptual question should be addressed: since Eq. (4) is an equation for *amplitudes*, the language used for interpreting it is possibly not fully appropriate, and has to be made more precise. As is well known, Feynman paths interfere, so that, in general, regarding a sequence of events (a history) of the system previous to a final or measured event as something that has actually occurred is not justified, in the sense

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that probabilities cannot be assigned consistently, in general, to the set of alternative histories. This has been emphasized lately by the consistent histories (CH) interpretation, in which a time sequence of events can only be considered a “consistent” (classical-like) history under specific conditions, in a nutshell when the alternative paths do not interfere [11].

By applying the resolution of the identity $1 = P + Q$ at an intermediate time t , the survival probability is decomposed, using Eq. (4) and its complex conjugate, as

$$S(T) = |A(T)|^2 = [PP]_t + [PQ]_t + [QP]_t + [QQ]_t, \quad (5)$$

where we have introduced the notation

$$[XY]_t = \langle \psi_0 | e^{-iH(T-t)} X e^{-iHt} | \psi_0 \rangle \times \langle \psi_0 | e^{iHt} Y e^{iH(T-t)} | \psi_0 \rangle \quad (6)$$

(or $[XY]$ for short), in which X and Y can be any of the projectors P or Q . It is a simple exercise to see that all the terms that contribute to the survival probability in Eq. (5) can be reduced to products of survival amplitudes for different times,

$$\begin{aligned} [PP]_t &= |A(T-t)A(t)|^2, \\ [PQ]_t &= A(T-t)A(t)A^*(T) - |A(T-t)A(t)|^2, \\ [QP]_t &= A(T-t)^*A(t)^*A(T) - |A(T-t)A(t)|^2, \\ [QQ]_t &= |A(T)|^2 - A(T-t)A(t)A^*(T) \\ &\quad - A(T-t)^*A(t)^*A(T) + |A(T-t)A(t)|^2. \end{aligned} \quad (7)$$

The diagonal terms $[PP]_t$ and $[QQ]_t$ have simple “event” interpretations: for $[PP]_t$ the particle is initially in $|\psi_0\rangle$, it is again in $|\psi_0\rangle$ at the intermediate time t , and ends up in the same state at T ; for $[QQ]_t$ the particle begins and ends in $|\psi_0\rangle$ but it is in the complementary subspace spanned by Q at the intermediate time t . This is therefore the term associated with initial state reconstruction. The nondiagonal terms, however, are quantum interferences without a classical counterpart. Thus the “histories” represented by $[PP]_t$ and $[QQ]_t$ terms are only “consistent” if the contribution from the interference terms is negligible, i.e., if the survival probability may be attributed essentially to these diagonal terms only. In other words, when the histories are consistent we may assign to them probabilities, without the need to invoke virtual paths and complex amplitudes. There remains of course an arbitrariness in the definition of “negligible,” since the interference terms are often small compared to the others but rarely zero. One should accept, clearly, that the “consistency” or “classicality” of the histories is not absolute and sharply defined but a gradual quality that may, however, be precisely quantified. Some applications and even basic premises of the consistent histories approach have been much debated and criticized [12,13]. In particular, the association between classicality and consistency is unfortunately not simple (there may be nonclassical and consistent histories), and the interpretation of the probability of a history in an objective, classical way may lead to contradictions when considering different families of histories [12]. From our purposes here, we shall not need the CH approach and its implications in their full glory, but only some of the elementary mathematical

manipulations, basically the exact decomposition of the density operator performed by the complementary projectors P and Q at an intermediate time. The controversial aspects may be circumvented by adopting an operational viewpoint, in accordance with the standard interpretation of quantum mechanics. Note that the decomposition in Eq. (5) is exact. Our operational interpretation of “consistency” amounts to actually performing ideal projective measurements at the intermediate time t , to determine whether the particle is or is not in the subspaces P, Q . Consistency will then occur when the results at the final time T do not depend on the intermediate measurement, which will be true provided the interference terms vanish. As before, this operational consistency criterion admits different degrees of fulfillment which may be quantified. In practice, the accuracy of the final measurement will set an effective and finite consistency threshold.

II. THE MODELS

The exact single resonance model for decay as the one discussed before, see [3], is possibly the simplest one for decomposing the survival since $A(t)$ reduces to a known function. One pays a price though, as the single resonance model does not contain an explicit form of the Hamiltonian or the initial state. The numerical calculations presented in the figures correspond to a more explicit δ -function Hamiltonian model described below. Note, however, that all basic properties of the survival decomposition found for the δ model have also been reproduced for the simple single resonance model, although the results for the latter are not shown to avoid unnecessary repetitions. The results have also been reproduced for a double barrier potential on the full line. The agreement of at least three different models strongly suggests that the conclusions that we shall draw are of general validity.

In the δ -function model, the particle is confined to the half-axis $x > 0$ with boundary condition $\psi(x=0)=0$, and subjected to a delta-function potential,

$$V = \eta\delta(x-1). \quad (8)$$

In all calculations, the initial state is the ground state of the infinite well,

$$\psi(x, t=0) = \sqrt{2} \sin(\pi x). \quad (9)$$

The evolved wave function $\psi(x, t)$ is calculated by using the resolution of the identity based on the eigenfunctions of the stationary Schrödinger equation, δ -normalized in k space. Since their overlap with the initial function can be calculated in the form of an explicit function, $\psi(x, t)$ is reduced to a single integral that is performed numerically. The calculation may also be carried out by expanding $\psi(x, t)$ in terms of the resonant-state basis [4].

Consider now a model that represents the effect of a detector by adding to the potential in Eq. (8) an external absorbing potential $-iV_0$ for $x \geq 1$ [14]. The solutions of the stationary Schrödinger equation which are right eigenvectors of the Hamiltonian with eigenvalue $E = q^2 - iV_0$ can be written as

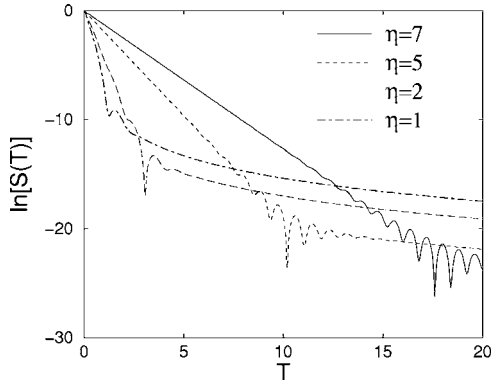


FIG. 1. \ln of the survival probability $S(t)$ for different δ -function strength factors.

$$\phi_q(x) = \frac{1}{(2\pi)^{1/2}} \begin{cases} C_1(e^{ikx} - e^{-ikx}), & 0 \leq x \leq 1 \\ e^{-iqx} - \mathcal{S}e^{iqx}, & x \geq 1, \end{cases} \quad (10)$$

where $k=(q^2-iV_0)^{1/2}$ is the wave number inside and q is the wave number outside. For scattering-like solutions, q is positive. Note the two branch points of k in the complex q plane. We shall take the branch cut joining these points. Similarly, the root in $q=(k^2+iV_0)^{1/2}$ is defined with a branch cut joining the two branch points in the k plane. The matching conditions determine the amplitudes C_1 and \mathcal{S} . The right eigenvectors of a complex Hamiltonian have biorthogonal partners $\hat{\phi}_q(x)$, which are left eigenvectors of H with the same eigenenergy E [15]. Alternatively, they may also be regarded as right eigenvectors of H^\dagger with eigenvalue E^* .

Aside from the scattering solutions, there may be localized solutions for complex q_j and positive imaginary part, normalized with respect to their biorthogonal partners as $\langle \hat{u}_j | u_i \rangle = \delta_{ij}$. The time evolution in this case is obtained from the biorthogonal resolution of the identity,

$$1 = \sum_j |u_j\rangle\langle \hat{u}_j| + \int_0^\infty dq |\phi_q\rangle\langle \hat{\phi}_q|. \quad (11)$$

III. RESULTS AND DISCUSSION

Typical survival curves for different η factors are shown in Fig. 1. ($V_0=0$ for the time being.) Note the clear transition between exponential and long-time deviations (the short-time deviation is not seen in the scale of the figure).

First we decompose the survival at a large time T in which the decay is not exponential for different intermediate times t according to Eq. (5). Figure 2 shows the clear dominance of the reconstructed term. The other terms are negligible near the middle intermediate time point, $5 < t < 15$, so that the reconstruction is a consistent history, in fact the only significant one, in this range of intermediate times, whereas the interference and nonreconstructed terms are not negligible for extreme intermediate times.

No fundamental difference is observed if instead of the survival probability $S(T)$ the nonescape probability $N = \int_0^\infty dx |\psi(x,t)|^2$ is decomposed into terms $[P'P']_t$, $[Q'Q']_t$,

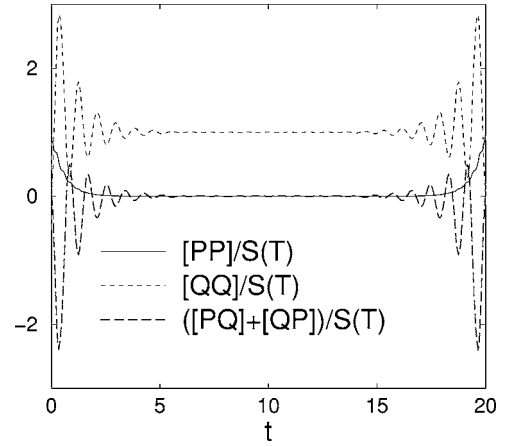


FIG. 2. Decomposition of the survival probability $S(T)$ as a function of the intermediate time t for a final time $T=20$ in the long-time, nonexponential decay regime. $\eta=5$.

and interference contributions as before, but using now the new projector

$$P' = \int_0^1 dx |x\rangle\langle x| = \sum_{j=1}^{\infty} |E_j\rangle\langle E_j|, \quad (12)$$

where $|E_j\rangle$ are the eigenstates of the infinite well, and $Q' = 1 - P'$. In practice the sum is only taken up to a finite number of terms until convergence is achieved. The calculation reduces then to evaluate matrix elements of the general form $\langle E_n | e^{-iHt} | E_j \rangle$. The resulting figure for $\eta=5$ and $T=20$, not shown, is almost identical to Fig. 2.

At variance with the long-time case, in the region of exponential decay the nonreconstructed term $[PP]_t$ dominates, see Fig. 3, the other ones being negligible at all intermediate times.

In the region of short-time deviations, the $[PP]_t$, nonreconstructed term dominates too. Nevertheless, there is a deviation from exponential decay that must be explained by the difference with the survival $S(T) - [PP]_t$. This difference essentially coincides, and it is numerically indistinguishable in our calculations, with the interference term $[PQ]_t + [QP]_t$ for

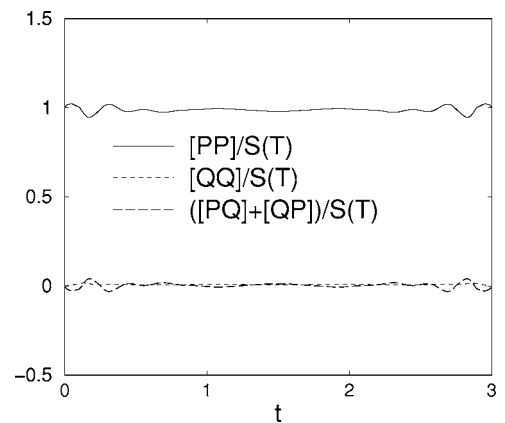


FIG. 3. Decomposition of the survival probability for a final time $T=3$ where exponential decay holds. $\eta=5$.

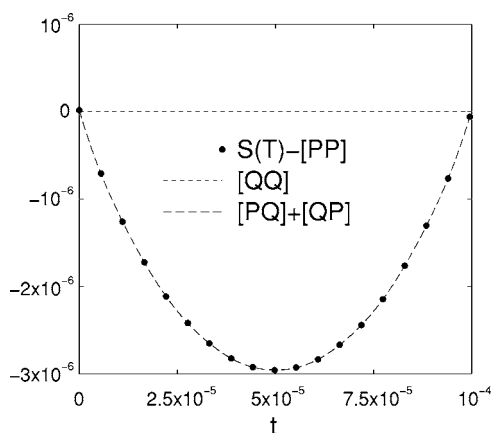


FIG. 4. Importance of the interference at short times: Decomposition of the survival probability for $T=0.0001$. $\eta=5$.

all intermediate times t , see Fig. 4. Therefore, initial state reconstruction in this regime cannot be regarded as a consistent history. It is an interference effect.

Fonda, Ghirardi, and Rimini [10] pointed out that the state reductions associated with measurement interactions could hinder initial state reconstruction and make the decay purely exponential [10]. They substantiated this idea with a measurement model producing state reductions at random times. Several models have been used afterwards by different authors to show the effects of a detector in the exponential decay and its deviations [16,17]. Here we propose a complex imaginary potential as a simple way to represent atomic detection by fluorescence induced by laser (this is justified in [14]). Figure 5 shows that the presence of a weak detector pushes the transition between the exponential and long-time deviation to larger times with increasing V_0 , showing the gradual suppression of initial state reconstruction. This can also be seen explicitly in Fig. 6, where the dominant term in the decomposition is $[PP]$ instead of $[QQ]$, compare with Fig. 2. For the same values of V_0 , however, the short-time deviations are not affected because there is not enough time for absorption to occur. Note that the slope of the long-time

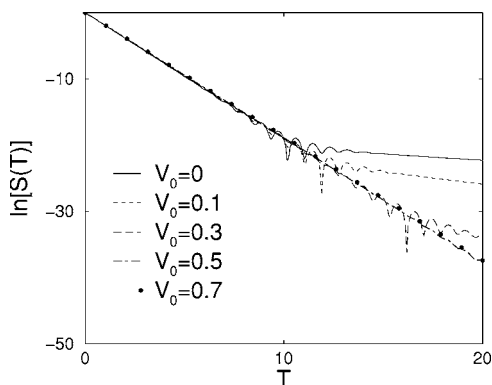


FIG. 5. Effect of a detector, represented by an imaginary potential $-iV_0$ outside the internal region limited by the δ . $\eta=5$.

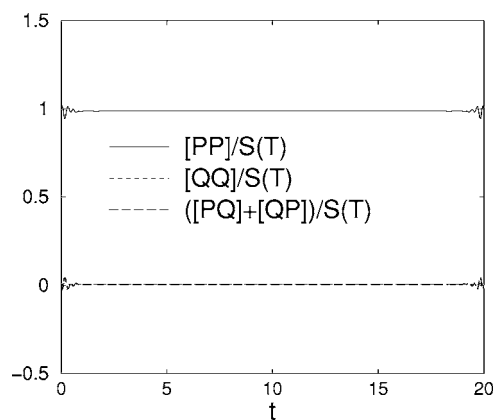


FIG. 6. Dominance of the nonreconstructed $[PP]$ term in the survival when the detector is on. $\eta=5$, $V_0=1$, and $T=20$. Compare with Fig. 2.

deviation increases with V_0 . In fact, around $V_0 \approx \Gamma$ the deviation disappears. (A more extensive analysis of this measurement model will be presented elsewhere, in particular for strong interactions, which produce important changes in the lifetime, but are not so relevant for our present discussion.)

IV. DISCUSSION

There is in summary a significant difference between short- and long-time deviations from exponential decay regarding the role of state reconstruction, which becomes a “consistent history” for long times but not for short times. This means that the short-time deviation is an interference effect between wave components coming from the initial state or region and their complementary subspaces, whereas the long-time deviation is of a more classical nature, with no interference contribution. An operational consistency criterion has been put forward in agreement with the standard interpretation of quantum mechanics, namely consistency of the histories may be checked and quantified by the agreement between the results at the final observation time with and without an intermediate projective measurement on the system. This is of course because the intermediate measurement destroys interference. The stated criterion falls short of the original aim of the CH approach (to formulate probabilities without the observer), but is enough for our analysis and may be useful in other contexts and applications.

Note added in Proof: The relevance of our results has been greatly enhanced by the recent experimental observation of deviations from exponential decay at long times (see Ref. [18]).

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- [1] L. A. Khal'fin, Zh. Eksp. Teor. Fiz. **33**, 1371 (1957) [Sov. Phys. JETP **6** 1053 (1958)].
- [2] S. R. Wilkinson *et al.*, Nature (London) **387**, 575 (1997); Q. Niu and M. G. Raizen, Phys. Rev. Lett. **80**, 3491 (1998).
- [3] G. García-Calderón, V. Riquer, and R. Romo, J. Phys. A **34**, 4155 (2001).
- [4] G. García-Calderón, J. L. Mateos, and M. Moshinsky, Phys. Rev. Lett. **74**, 337 (1995); **90**, 028902 (2003).
- [5] J. G. Muga, G. W. Wei, and R. F. Snider, Europhys. Lett. **35**, 247 (1996).
- [6] J. G. Muga, G. W. Wei, and R. F. Snider, Ann. Phys. (N.Y.) **252**, 336 (1996).
- [7] L. Fonda and G. C. Ghirardi, Nuovo Cimento Soc. Ital. Fis., A **7A**, 180 (1972).
- [8] T. G. Douvropoulos and C. A. Nicolaides, Phys. Rev. A **69**, 032105 (2004).
- [9] L. Ersak, Yad. Fiz. **9**, 458 (1969) [Sov. J. Nucl. Phys. **9**, 263 (1969)].
- [10] L. Fonda, G. C. Ghirardi, and A. Rimini, Rep. Prog. Phys. **41**, 587 (1978).
- [11] R. B. Griffiths, J. Stat. Phys. **36**, 219 (1984); R. Omnès, *The Interpretation of Quantum Mechanics* (Princeton University Press, Princeton, 1994); M. Gell-Mann and J. B. Hartle, Phys. Rev. D **47**, 3345 (1993); J. J. Halliwell, in *Time in Quantum Mechanics*, edited by J. G. Muga, R. Sala, and I. L. Egusquiza (Springer, Berlin, 2002).
- [12] A. Bassi and G. Ghirardi, J. Stat. Phys. **98**, 457 (2000).
- [13] M. Schlosshauer, Rev. Mod. Phys. **76**, 1267 (2004).
- [14] A. Ruschhaupt, J. A. Damborenea, B. Navarro, J. G. Muga, and G. C. Hegerfeldt, Europhys. Lett. **67**, 1 (2004).
- [15] J. G. Muga, J. P. Palao, B. Navarro, and I. L. Egusquiza, Phys. Rep. **395**, 357 (2004).
- [16] B. Elattari and S. A. Gurvitz, Phys. Rev. A **62**, 032102 (2000).
- [17] R. E. Parrot and J. Lawrence, Europhys. Lett. **57**, 632 (2002).
- [18] C. Rothe, S. I. Hintschich, and A. P. Monkman, Phys. Rev. Lett. **96**, 163601 (2006);