Connecting the generalized robustness and the geometric measure of entanglement

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The main goal of this paper is to provide a connection between the generalized robustness of entanglement (R_g) and the geometric measure of entanglement (E_{GME}). First, we show that the generalized robustness is always higher than or equal to the geometric measure. Then we find a tighter lower bound to $R_g(\rho)$ based only on the purity of ρ and its maximal overlap to a separable state. As we will see it is also possible to express this lower bound in terms of E_{GME}.

Since it was first noted [1,2] the issue of quantum correlations has been largely studied and debated. However, it was not until entanglement was recognized as a physical resource that this theme got a solid status. From this point of view, entanglement was shown to allow several tasks such as quantum cryptography [3], teleportation [4], and quantum algorithms [5]. On the other hand, entanglement has also given us new insights for understanding important physical phenomena including superconductivity [6], super-radiance [7], quantum phase transitions [8–12], and the appearance of classicality [13].

One of the greatest challenges concerning entanglement is how to properly quantify this resource. Although this problem is well understood for bipartite pure states, in a more complex scenario (multipartite systems or mixed states) a complete theory on the quantification of entanglement is still lacking.

Among the difficulties of dealing with multipartite entanglement is the fact that systems composed by various parts can exhibit many kinds of entanglement. This is because one may be interested in the entanglement according to a specific partition of the whole system. So a state can present some entanglement in relation to a given partition, while it can be separable according to another one.

In the past few years many candidates of entanglement quantifiers were proposed. Generically speaking, the ways of quantifying entanglement can be divided into two classes: quantifiers with a geometrical interpretation, and ones with an operational meaning. In the first class we can cite the relative entropy of entanglement [14,15], the geometric measure of entanglement [16,17], the negativity [18–20], and the robustness of entanglement [21–23]. The entanglement cost [24,25], the distillable entanglement [24,26], and the singlet fraction [27] are examples of operational measures.

The purpose of this paper is to point out a connection between two well-discussed entanglement quantifiers, the generalized robustness (R_g) [22,23], and the geometric measure of entanglement (E_{GME}) [16,17]. That these quantifiers are related is not obvious *a priori*, for these functions rely on distinct geometrical interpretations. While E_{GME} measures the minimum angle between an entangled state and a sepa-

rable one, R_g can be treated as a kind of "distance" between an entangled state and the set of unentangled states. Furthermore, both quantifiers are able to deal with the various types of entanglement that a multipartite system can present.

Let us first present the language we shall adopt to talk about multipartite entanglement. Suppose a state ρ can be written as a convex combination of states which are products of k tensor factors. The state ρ is then said to be a k-separable state. One should note that in a system of n parts, n-separability is separability itself and that every state is trivially 1-separable. The set of k-separable states will be denoted by S_k. It is clear that $S_n \subset S_{n-1} \subset \cdots \subset S_1 = D$, where D denotes the set of density operators.

We are now able to understand why R_g and E_{GME} can distinguish the types of entanglement a system contains. The geometric measure is a pure-state entanglement quantifier given by

$$E_{GME}^k(\psi) = 1 - \Lambda_k^2(\psi), \tag{1}$$

where

$$\Lambda_k^2 = \max_{\phi \in S_k} |\langle \phi | \psi \rangle|^2. \tag{2}$$

Thus $E_{GME}^k(\psi)$ measures the sine squared of the minimum angle between $|\psi\rangle$ and a k-separable state [36]. It is known that this quantity is an entanglement monotone [28], i.e., it is a nonincreasing function under local operations and classical communication.

The relation between E_{GME}^k and the notion of k-entanglement witnesses [29] (observables with positive mean value to all k-separable states, but negative to some $\rho \notin S_k$) (see Ref. [17]) has also been determined. This results from the fact that one can always construct a k-entanglement witnesses W^k for a pure state $|\psi\rangle$ of the type

$$W^k = \lambda^2 - |\psi\rangle\langle\psi|. \tag{3}$$

As this operator must have a positive mean value for every k-separable state, the relation

$$\lambda^2 \geq \max_{|\phi\rangle \in S_k} \|\langle\phi|\psi\rangle\|^2 = \Lambda_k^2 \tag{4}$$

must hold. Thus the optimal entanglement witness of the form (3) is reached when $\lambda = \Lambda_k^2$, and we can write
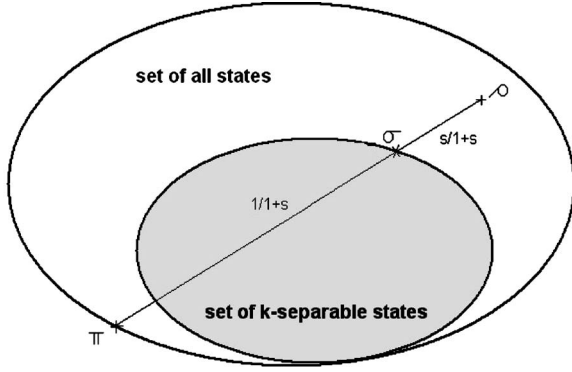
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FIG. 1. Geometrical interpretation of R_g^k. The straight line represents the convex combination $\rho + s\pi/1 + s$. We see that for a given state π and a value of s this combination becomes k-separable. $R_g^k(\rho)$ is defined as the minimum s, considering all possible states π.

$$W_{opt}^k = \Lambda_k^2 - |\psi\rangle\langle\psi|. \quad (5)$$

Here optimality is defined in the sense of getting the highest value to $|\langle\psi|W^k|\psi\rangle|$.

In a different fashion, the robustness of entanglement of a state ρ quantifies how robust the entanglement of ρ is under the presence of noise. Thus the robustness of ρ in relation to the state π, $R(\rho\|\pi)$, is the minimum s such that the state

$$\sigma = \frac{\rho + s\pi}{1 + s} \quad (6)$$

is k-separable. We will be interested in an extension of the relative robustness, namely the *generalized robustness*. This entanglement quantifier is obtained by the minimization of the relative robustness over all states π [22]. Recently, an interesting operational interpretation to R_g^k was given in terms of the percentual increase a state can provide to teleportation processes [30]. The generalized robustness can also be viewed as a "distance" of ρ to the set S_k in the space of states (see Fig. 1) [31], and thus allow both a geometrical and an operational interpretation. Moreover R_g^k was used to investigate the shape of entangled states sets [31] and was shown to exhibit a kind of *polygamy of entanglement* [32].

As well as the geometric measure, R_g^k is intimately connected to the notion of entanglement witnesses. In fact, R_g^k can be expressed as

$$R_g^k(\rho) = \max\{0, -\min_{W^k \in \mathcal{M}} \text{Tr}(W^k\rho)\}, \quad (7)$$

where $\mathcal{M}$ is the set of operators $M \leq I$ and W^k is a k-entanglement witness [33].

As the witness (5) obviously satisfies the condition $W^k \leq I$ we can attest the following:

$$R_g^k(\psi) \geq E_{GME}^k(\psi). \quad (8)$$

Some points concerning the inequality (8) should be stressed at this stage. First, it is a relation valid to all kinds of multipartite entanglement. Moreover, this relation will be strict whenever the witness (5) is a k-entanglement witness, which solves the minimization problem in (7). Finally, one could argue that relation (8) may be, in fact, a consequence of

standard results from matrix analysis relating different distance measures between operators (as commented, both R_g^k and E_{GME}^k are related to such distances). However, it must be clear that $R_g^k(\psi)$ is not simply the distance between ψ and its closest state $\sigma \in S_k$, but one should keep in mind that this distance is taken with relation to the state π as a reference [37] (recall Fig. 1). This makes the closest k-separable state usually different for R_g^k and E_{GME}^k.

In fact, it is possible to give a tighter relation between R_g^k and E_{GME}^k. Recall the Lemma 1 shown in Ref. [32]. We now give a clearer proof of it, and interpret it as a lower bound to R_g^k.

Lemma 1. *For every state $\rho \in D$,*

$$R_g^k(\rho) \geq \frac{\text{Tr}(\rho^2)}{\max_{\sigma \in S_k}\text{Tr}(\rho\sigma)} - 1. \quad (9)$$

Proof. First of all let us show that $\max_{\sigma \in S_k}\text{Tr}(\rho\sigma)$ is equal to the minimum value of λ (λ_{min}) such that $W = \lambda I - \rho$ is a k-entanglement witness. As $\text{Tr}(W\sigma) \geq 0 \,\forall\, \sigma \in S_k$,

$$\text{Tr}[(\lambda I - \rho)\sigma] = \lambda - \text{Tr}(\rho\sigma) \geq 0. \quad (10)$$

It is thus straightforward to see that $\lambda_{min} = \max_{\sigma \in S_k}\text{Tr}(\rho\sigma)$.

Note that

$$W' = \frac{W}{\lambda_{min}} = I - \frac{\rho}{\lambda_{min}} < I. \quad (11)$$

So it is possible to see that $R_g^k(\rho) \geq -\text{Tr}(W'\rho)$, from which follows the required result. $\square$

The lower bound to R_g^k expressed by (9) can be easily interpreted: $\text{Tr}\,\rho^2$ measures the purity of ρ, and $\text{Tr}(\rho\sigma)$ is the Hilbert-Schmidt scalar product between ρ and σ. It is expected that the more mixed ρ is, the lower the value of $\text{Tr}\,\rho^2$ gets, and the state becomes less entangled. Similarly, the larger $\max_{\sigma \in S_k}\text{Tr}(\rho\sigma)$ is, closer to the set S_k ρ gets, and the system will show less entanglement.

But now we note that in the special case of pure states the relations $\text{Tr}(\rho^2) = 1$ and $\max_{\sigma \in S_k}\text{Tr}(\rho\sigma) = \Lambda_k^2(\rho)$ hold and therefore we have the general relation

$$R_g^k(\psi) \geq \frac{1}{\Lambda_k^2(\psi)} - 1 \quad (12)$$

and we can see the relation we are looking for

$$R_g^k(\psi) \geq \frac{E_{GME}^k}{1 - E_{GME}^k}. \quad (13)$$

It is interesting that two entanglement monotones with different geometric interpretation are actually related, and furthermore this relation allows an analytic lower bound to the generalized robustness for all states whenever $\Lambda_k^2(\rho)$ can be analytically computed. This is the case, for example, of completely symmetric states, Werner states, and the isotropic states [17,34].
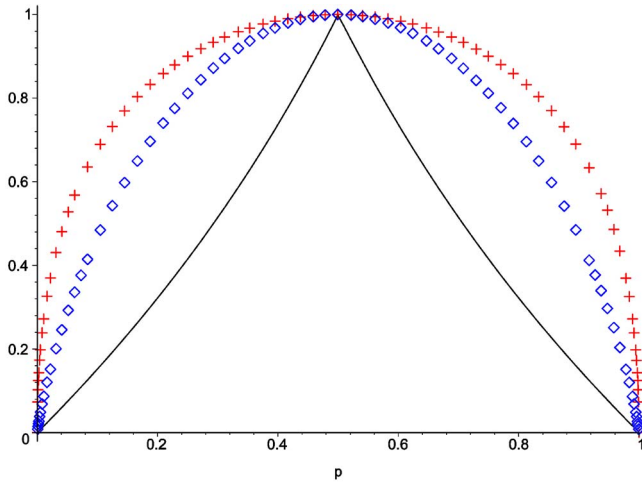
We can furthermore see from (12) that

FIG. 2. (Color online) Red crosses: logarithmic generalized robustness of entanglement. Blue diamonds: relative entropy of entanglement. Black line: lower bound given in Eq. (14).

$$\log_2(1 + R_g^k) \geq -2\log_2\Lambda_k. \tag{14}$$

The left side of this expression is the logarithmic robustness of entanglement (LR_g^k), another entanglement quantifier with interesting features [33]. Curiously, this is exactly the same lower bound expressed to the relative entropy of entanglement (E_R^k) in [34]. Numerical and analytical results (see, for example, Fig. 2 and Table I) suggest that $LR_g^k \geq E_R^k$, in general, but at the moment this is just a conjecture.

For bipartite pure states all the quantities considered so far can be analytically computed. While the relative entropy of entanglement equals the entropy of entanglement (given by the von Neumann entropy of the reduced state) [15], the generalized robustness is given by

$$R_g^k(\psi) = \left(\sum_i c_i\right)^2 - 1, \tag{15}$$

being $\{c_i\}$ the spectrum of Schmidt of $|\psi\rangle$ [22]. In this context it can be noted that Λ_k is given by the modulus of the highest Schmidt coefficient of $|\psi\rangle$ [17]. To visualize and compare these entanglement measures we calculate the relative entropy of entanglement, the logarithmic generalized robustness, and the lower bound expressed in (14) for the state

$$|\psi(p)\rangle = \sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle. \tag{16}$$

The plots are available in Fig. 2.

As the presented relations between R_g^k and E_{GME}^k are also valid to multipartite entanglement it would be useful to illustrate the results in this context as well. We choose to study some completely symmetric states for this aim. These states

TABLE I. A comparison among multipartite entanglement of some states (17), given by different entanglement quantifiers. E_{GME}^n (geometric measure of entanglement, given in Ref. [17]), R_g^n (numerical calculation), LR_g^n [logarithmic robustness of entanglement, given by the left side of Eq. (14)], and E_R^n [given by the left side of Eq. (14)].

	$\|S(2,1)\rangle$	$\|S(3,2)\rangle$	$\|S(4,3)\rangle$	$\|S(4,2)\rangle$
E_{GME}^n	0.5	0.55	0.58	0.25
R_g^n	1	1.25	1.7	1.83
LR_g^n	1	1.17	1.43	1.5
E_R^n	1	1.17	1.24	0.41

are referred to as Dicke states or, some times, as generalized *W* states and appear naturally as eigenstates of various models such as the η-pairing model [6] and the Dicke model [7]. Following Ref. [17] we will label these states according to the number of 0's, as follows:

$$|S(n,k)\rangle = \sqrt{\frac{k!\,(n-k)!}{n!}}\,S|\underbrace{000\cdots0}_{k}\underbrace{11\cdots1}_{n-k}\rangle, \tag{17}$$

where S is the total symmetrization operator. Wei and Goldbart showed an analytical expression to $E_{GME}^n(|S(n,k)\rangle)$ (i.e., the geometric measure of $|S(n,k)\rangle$ with relation to the completely separable states) [17]. Additionally, it was shown that the relative entropy of entanglement is exactly equal to the lower bound given in Eq. (14) [34]. On the other hand we do not have an analytical way of calculating R_g^n for the states (17). We thus computed it numerically using the robust semidefinite programming approach presented in Refs. [33,35]. The results are displayed in Table I.

In brief, we have shown some relations between the geometric measure of entanglement and the generalized robustness of entanglement. We reached a lower bound to R_g^k with nice interpretations and wrote it in terms of E_{GME}^k. These relations also allowed us to compare two other entanglement quantifiers, the logarithmic generalized robustness and the relative entropy of entanglement. Examples were given to illustrate the results.

Because many entanglement quantifiers exist it is important to understand their relation and this, we believe, should be a major goal in the theory of entanglement. We hope that this discussion can help in this sense.

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[36] The extension of $E_{GME}^k(\psi)$ to mixed states is made through the convex-roof construction, which makes this quantifier hard to be computed, in general. Furthermore this construction makes the geometrical interpretation of E_{GME}^k not so clear. Thus we will focus on pure states unless otherwise specified.

[37] Besides that there is a minimization among all possible states π.