Factorization in breakup and recombination processes for atoms with a large scattering length

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Breakup and recombination processes for loosely bound molecules composed of atoms with a large scattering length a necessarily involve interactions that are nonperturbative in the exact two-body interaction. If these processes involve atoms with relative momenta much larger than $\hbar/|a|$, the leading contributions to their rates can be separated into short-distance factors that are insensitive to a and long-distance factors that are insensitive to the range of the interaction. These factorization contributions can be obtained from the leading term in a perturbation expansion in the exact atom-atom scattering amplitude. The short-distance factors are atom-atom cross sections at a lower collision energy. In the special case of inclusive breakup cross sections for atom-molecule scattering, the long-distance factors simply count the number of atoms in the molecule.

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Nonrelativistic particles with short-range interactions that have been tuned, either by the experimenter or fortuitously by nature, so that their S-wave scattering length a is much larger than the range, have universal low-energy properties that depend on a but are otherwise insensitive to their interactions at short distances. (See Ref. [1] and references therein.) If the particles form loosely bound two-body or higher N-body clusters whose sizes are comparable to $|a|$, the clusters also have universal properties. A classic example in atomic physics is ^{4}He atoms whose scattering length $a \approx 100$ Å is much larger than their effective range $r_s \approx 7$ Å. The ^{4}He dimer and the excited state of the ^{4}He trimer both have sizes comparable to a. A classic example in nuclear physics is nucleons. The deuteron is a bound state of the neutron and proton associated with a large scattering length in the spin-triplet channel.

The large scattering length implies that interactions between atoms whose relative momenta are comparable to $\hbar/|a|$ are nonperturbative in the two-body scattering amplitude. A simple consequence of these strong interactions with $a>0$ is the existence of a loosely bound dimer whose binding energy is given by

$$E_D = \hbar^2/(ma^2). \tag{1}$$

In some cases (for example, identical bosons), the strong interactions produce the *Efimov effect*: as $a \to \pm\infty$, there are increasingly many loosely bound trimers with an accumulation point at the three-atom threshold [2]. In the limit $a = \pm\infty$, the trimers have an asymptotically exponential spectrum:

$$E_T^{(n)} \to (e^{-2\pi/s_0})^{n-n_*}\hbar^2\kappa_*^2/m \quad \text{as } n \to \infty, \tag{2}$$

where s_0 is a numerical constant whose value for identical bosons is 1.006 24, n_* is an arbitrary integer, and κ_* is a three-body parameter [1]. The strong interactions can also lead to intricate dependence on the scattering length a. An example in the case of identical bosons is the event rate constant in the low-energy limit for three-body recombination into the loosely bound dimer:

$$K_3 = \frac{768\pi^2(4\pi - 3\sqrt{3})\hbar a^4/m}{\sinh^2(\pi s_0) + \cosh^2(\pi s_0)\cot^2[s_0 \ln(a\kappa_*) + 1.16]}. \tag{3}$$

The log-periodic dependence on a in Eq. (3) was discovered in Ref. [3]. The completely analytic expression was derived more recently by Petrov [4]. The coefficient of $\hbar a^4/m$ on the right side of Eq. (3) can range from 0 to 402.7 depending on the value of the three-body parameter κ_*. The dimer breakup cross section in atom-dimer scattering at a collision energy E just above the threshold E_D has the same log-periodic factor:

$$\sigma_{AD}^{(\text{breakup})}(E) = \frac{\sqrt{3}}{96\pi}\left(\frac{mK_3}{\hbar a^4}\right)a^2\frac{(E-E_D)^2}{E^{1/2}E_D^{3/2}}. \tag{4}$$

To reproduce such intricate dependence on a requires accurate numerical methods that are capable of solving the problem "exactly," that is, to any desired precision. The Schrödinger equation for two or three atoms interacting through a short-range potential can be solved exactly. Systems of four atoms are at the frontiers of few-body physics: the binding energies of tetramers can be calculated accurately, but there are some scattering observables for which effective calculational methods have yet to be developed. In any process that involves loosely bound molecules, strong interactions involving the momentum scale $\hbar/|a|$ are unavoidable. This suggests that accurate calculations of the rates for such processes may be possible using present methods only if the total number of atoms involved is at most 3 or 4.

In this paper, we point out that this obstacle can be avoided for processes involving atoms whose momenta relative to the loosely bound molecules are set by a scale $Q \gg \hbar/|a|$. Examples of such processes are the breakup of the molecule in a collision with an energetic atom and the formation of the molecule in a collision involving energetic atoms. The separation of scales $Q \gg \hbar/|a|$ can be exploited by using *factorization* to separate the rate into *short-distance* factors that are insensitive to a and *long-distance* factors that do not involve the scale Q. The strong interactions associated with large a enter only in the long-distance factors. The rate can be calculated accurately using present methods provided

the long-distance factors involve at most thre or four atoms. In some cases, even this limitation is unnecessary because the long-distance factors can be determined analytically. Thus factorization significantly expands the list of processes whose rates can be calculated accurately using present methods.

Factorization has proved to be a powerful tool in quantum chromodynamics (QCD). The Nobel Prize in physics for 2004 was awarded to Gross, Politzer, and Wilczek for the discovery of the *asymptotic freedom* of QCD [5]. Asymptotic freedom refers to the decrease of the strength of the interaction between quarks as their separation decreases, or equivalently, as their relative momentum increases. For selected observables, asymptotic freedom can be exploited by using factorization to separate the observables into short-distance factors that involve only weak QCD interactions between quarks and gluons and long-distance factors that involve strong QCD interactions [6]. The short-distance factors involve a large momentum transfer Q and can be calculated using perturbation theory in the running coupling constant $\alpha_s(Q)$ of QCD. The long-distance factors involve only momentum transfers small compared to Q. In most cases, systematic methods for calculating the long-distance factors have not been developed. The bulk of the quantitative evidence that QCD describes the strong force is the experimental verification that the short-distance factors are correctly predicted by perturbative QCD.

Atoms with a large scattering length have a property analogous to asymptotic freedom: the strength of their interaction, as measured by the magnitude of the elastic scattering amplitude, decreases as their relative momentum $\hbar k$ increases. This behavior of the atom-atom interaction motivates our use of the factorization strategy that has been so successful in QCD. The S-wave elastic scattering amplitude has the form

$$f_k = \left[\left(-1/a + \frac{1}{2} r_s k^2 + \cdots \right) - ik \right]^{-1}, \tag{5}$$

where r_s is the effective range. For collision energies E at which higher partial waves can be neglected, the elastic cross section for identical bosons is

$$\sigma_{AA}(E) = \frac{8\pi}{\left(-1/a + \frac{1}{2} r_s mE/\hbar^2 + \cdots \right)^2 + mE/\hbar^2}. \tag{6}$$

The cross section has its maximum value $8\pi a^2$ at $E=0$. In the scaling region $\hbar^2/ma^2 \ll E \ll \hbar^2/mr_s^2$, the cross section scales like $1/E$:

$$\sigma_{AA}(E) \approx \frac{8\pi\hbar^2}{mE}. \tag{7}$$

Consider the breakup of a loosely bound dimer through a collision with an atom. We take all three atoms to be identical bosons. The two atoms inside the dimer have typical separation a. The atom-dimer collision energy is $E=\frac{3}{4}Q^2/m$, where Q is the momentum of the atom or the dimer in the center-of-mass frame. If $Q \gg \hbar/a$, the leading contribution to the breakup of the dimer comes from one of its constituent atoms being knocked out by the energetic incoming atom, while the other acts as a spectator. This process can be represented by the Feynman diagram in Fig. 1, which is the leading term in a perturbation expansion in the exact atom-atom scattering amplitude. The expression for the T-matrix element is

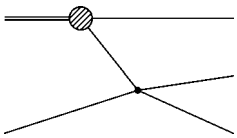

FIG. 1. Feynman diagram for the breakup of a dimer by the hard scattering of one of its constituent atoms and a colliding atom. The blob represents the wave function of the dimer and the dot represents the exact atom-atom scattering amplitude.

$$\mathcal{T} = \sum_{(123)} \frac{(32\pi\hbar^3/m)(\pi/a)^{1/2}\left(1 + \frac{3}{4} r_s/a + \cdots\right)}{(q_3^2 + mE_D)\left[\left(-1/a + \frac{1}{2} r_s k_{12}^2 + \cdots\right) - ik_{12}\right]}, \tag{8}$$

where $E_D=\hbar^2/(ma^2)(1+r_s/a+\cdots)$ is the dimer binding energy including range corrections, q_3 is the relative momentum of the two atoms in the dimer (one of which is the outgoing atom labeled 3), and $\hbar k_{12}=\frac{3}{4}Q$ is the relative momentum of the incoming atom and the scattered atom from the dimer (which is also the relative momentum of the outgoing atoms labeled 1 and 2). The sum is over cyclic permutations of the three outgoing atoms. The effects of subsequent interactions between the scattered atoms and the spectator atom in the dimer enter through higher-order diagrams in the perturbation expansion in the exact atom-atom scattering amplitude. These diagrams are all suppressed by a factor of $\hbar/(aQ)$ and can be neglected if the collision energy is sufficiently high. If we take the limit $Q \gg \hbar/a$, the T-matrix element in Eq. (8) can be expressed as the product of a short-distance factor that involves only the large momentum scales Q and $\hbar/r_s$ and a long-distance factor that involves only the small momentum scales q and $\hbar/a$:

$$\mathcal{T} \approx \sum_{(123)} \frac{4\hbar^2(\pi/a)^{1/2}}{q_3^2 + \hbar^2/a^2} \frac{8\pi\hbar/m}{\left(\frac{1}{2} r_s k_{12}^2 + \cdots\right) - ik_{12}}. \tag{9}$$

The long-distance factor is proportional to the momentum-space wave function of the dimer in the zero-range limit. The cross section is obtained by squaring the T-matrix element and integrating over the momenta of the three identical outgoing atoms. The cross section is dominated by the squares of the terms in Eq. (9), which we call diagonal terms. The terms in the cross section that correspond to the interference between terms in Eq. (9), which we call cross terms, are suppressed by a power of $\hbar/(aQ)$. We can take the limit $Q \gg \hbar/a$ in the momentum integrals. The result for the breakup cross section at collision energy $E \gg \hbar^2/ma^2$ can be expressed in the factored form

$$\sigma_{AD}^{(\text{breakup})}(E) \approx 2\sigma_{AA}\left(\frac{3}{4}E\right). \tag{10}$$

The coefficient 2, which is the long-distance factor, is obtained by using the normalization integral for the dimer wave function. The factorization form in Eq. (10) applies at arbitrarily large E provided σ_{AA} is the total atom-atom cross section. In the scaling region $E_D \ll E \ll \hbar^2/mr_s^2$, this breakup cross section scales as $1/E$:

$$\sigma_{AD}^{(\text{breakup})}(E) \approx \frac{64\pi\hbar^2}{3mE}. \tag{11}$$

The result can be generalized to the inclusive breakup cross sections for Efimov trimers and other loosely bound molecules whose constituents all have separations of order $|a|$. The leading contribution comes from one of the atoms in the molecule being knocked out by a collision with the energetic incoming atom. If the molecule contains N loosely bound atoms, the atom-molecule collision energy is $E = \frac{N+1}{2N}Q^2/m$, where Q is the momentum in the center-of-mass frame. The short-distance factor in the T-matrix element is the same as in Eq. (9), except that the relative momentum is $\hbar k = \frac{N+1}{2N}Q$. For any specific final state, the long-distance factor may be a very complicated function of a and the three-body parameter κ_* that appears in Eq. (3). It involves the wave function of the molecule and the wave function of the $(N-1)$-atom system that remains after the hard scattering. Some of those $N-1$ atoms may be bound into dimers or other clusters. However, if we sum over all these $(N-1)$-atom states, we can use completeness relations to show that the long-distance factor is simply the number N of atoms in the molecule. Our result for the inclusive breakup cross section as a function of the collision energy E is

$$\sigma_{AM}^{(\text{breakup})}(E) \approx N\sigma_{AA}\left(\frac{N+1}{2N}E\right). \tag{12}$$

This expression is valid when the collision energy satisfies $E \gg \hbar^2/ma^2$ and $E \gg E_M$, where E_M is the binding energy of the molecule with respect to the N-atom threshold.

The cross section for the breakup of ^{4}He dimers in collisions with Xe atoms has been studied in Ref. [7] for collision energies E ranging from 46 to 348 K. Since the Xe atom is so much heavier than a He atom, each of the atoms in the He dimer carries approximately $\frac{1}{2}$ the collision energy E. The analog of our factorization formula in Eq. (10) predicts that if E is much larger than the He dimer binding energy, which is about 1.6 mK, the He_2-Xe breakup cross section should be approximately twice the He-Xe cross section at collision energy $\frac{1}{2}E$. This prediction agrees reasonably well with the VCC-RIOS approximation [8] studied in Ref. [7]. By comparing the cross section $\sigma_{\text{tot}}^{\text{VCC-RIOS}}$ in Table 2 of Ref. [7] with the appropriate interpolated values of $\sigma_{\text{HeXe}}^{\text{el}}$, we can see that the difference is less than 9% in the energy range from 93 to 348 K. In the independent-atom model considered in Ref. [7], the total cross section is twice the He-Xe cross section at collision energy E instead of $\frac{1}{2}E$. This model underestimates the VCC-RIOS cross section by an amount that ranges from 11% at 93 K to 30% at 348 K.

Factorization can also be applied to the rate for three-body recombination into the loosely bound dimer when the three incoming atoms all have relative momenta much greater than $\hbar/a$. The leading contribution comes from a hard scattering of two of the atoms that scatters one of them into a state with small momentum relative to the third atom, followed by the coalescence of those two atoms into a dimer. The coalescence probability is substantial only if the relative momentum of the scattered atom and the third atom is of order $\hbar/a$. At leading order in the perturbation expansion in the exact atom-atom scattering amplitude, the T-matrix element for this process can be expressed as the sum of three Feynman diagrams obtained by time-reversing the diagram in Fig. 1 and summing over cyclic permutations of the three incoming atoms. We work in the center-of-mass frame, taking the momenta of the three incoming atoms to be $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$ and the momenta of the outgoing dimer and atom to be $+\vec{Q}$ and $-\vec{Q}$. The collision energy is $E=(p_1^2+p_2^2+p_3^2)/(2m)$. For large collision energy E, the leading contribution to the T-matrix element is given by Eq. (9). Each of the three terms in the sum is the product of a short-distance factor and a long-distance factor. The hard scattering of atoms 1 and 2 followed by the coalescence of one of them with atom 3 gives the term shown explicitly in Eq. (9), where $\hbar\vec{k}_{12}=\frac{1}{2}(\vec{p}_1-\vec{p}_2)$ is the large relative momentum of atoms 1 and 2 and $\vec{q}_3=\vec{p}_3-\frac{1}{2}\vec{Q}$ is the small relative momentum between atom 3 and the atom it coalesces with. The other two terms are obtained by cyclic permutations of 1, 2, and 3. The three-body recombination rate is obtained by squaring the T-matrix element and integrating over the final momenta of the outgoing atom and dimer. In the limit $E \gg \hbar^2/(ma^2)$, the rate R for forming dimers per volume and per time is

$$R \approx \frac{256\pi^2\hbar^3}{m^2}\sigma_{AA}\left(\frac{3}{4}E\right)\sum_{(123)}\delta(E-3p_3^2/m). \tag{13}$$

The three terms in the sum are diagonal terms that come from the squares of the terms in Eq. (9). The cross terms corresponding to the interference between terms in Eq. (9) have been neglected because they are suppressed by a factor of $(E_D/E)^{1/2}$.

The event rate R in Eq. (13) can be reduced to a function K_3 of the collision energy E by averaging over the momentum hyperangle α_3 defined by $p_k=(\frac{4}{3}mE)^{1/2}\cos\alpha_k$ and over the angle β_3 between the vectors $\vec{p}_3$ and $\vec{p}_1-\vec{p}_2$. The hyperangular average of R can be expressed as

$$\langle R\rangle = \frac{2}{\pi}\int_0^{\pi/2} d\alpha_3 \sin^2(2\alpha_3)\int_0^{\pi} d\beta_3 \sin(\beta_3)\, R. \tag{14}$$

The hyperangular average of the rate in Eq. (13) is

$$K_3 \equiv \langle R\rangle \approx \frac{384\sqrt{3}\pi\hbar^3}{m^2E}\sigma_{AA}\left(\frac{3}{4}E\right). \tag{15}$$

This result is valid when $E \gg E_D$. In the scaling region $E_D \ll E \ll \hbar^2/mr_s^2$, the rate in Eq. (15) scales as $1/E^2$:

$$K_3 \approx \frac{4096\sqrt{3}\pi^2\hbar^5}{m^3E^2}. \tag{16}$$

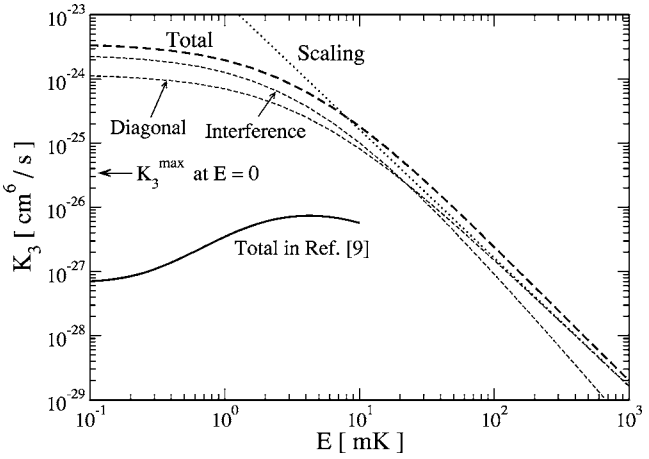

FIG. 2. Three-body recombination event rate K_3 (in units of cm^6/s) as a function of the collision energy E (in units of mK). The solid lines are the total result in Fig. 3 of Ref. [9]. The dashed lines are the hyperspherical averages of the R in Eq. (17), the diagonal terms, and the cross terms. The dotted line is the scaling approximation in Eq. (16). The horizontal arrow indicates the maximum possible value of K_3 at $E=0$ allowed by Eq. (3).

Suno *et al.* have calculated the three-body recombination rate of ^{4}He atoms into the dimer at collision energies up to 10 mK [9] using accurate solutions of the three-body Schrödinger equation for the HFD-B3-FCI1 potential model [10]. In this model, the scattering length is $a=91.0$ Å, the effective range is $r_s \approx 7$ Å, and the dimer binding energy is $E_D=1.6$ mK. The scattering length deduced from the universal formula for the dimer binding energy in Eq. (1) is $a_D=87.0$ Å. In Fig. 2, we compare the result of Ref. [9] for K_3 with the scaling approximation in Eq. (16). At $E=10$ mK, the highest collision energy considered in Ref. [9], our scaling approximation in Eq. (16) is larger than the total result in Ref. [9] by a factor of 28.8. Our factorization approximation in Eq. (15) is larger than the total result in Ref. [9] by a factor of 23.7.

We now consider why the scaling approximation in Eq. (16) overestimates the three-body recombination rate at 10 mK by more than an order of magnitude. We will show that the scaling approximation can be accurate only for collision energies that are at least an order of magnitude greater than 20 mK. We calculate the sum of the contributions to R at leading order in the perturbation expansion in the exact atom-atom scattering amplitude without making the factorization approximations. The T-matrix element is given by Eq. (8). We set the effective range r_s to zero to make the scaling behavior of the individual contributions at large energy E more evident. The resulting expression for the three-body recombination rate is

$$R = \frac{32\sqrt{3}\pi\hbar^3}{m^2E^2}[E_D(E+E_D)]^{1/2}\sum_{(123)}\Bigg(\sigma_{AA}(E\sin^2\alpha_3) \times\left[\left(\cos^2\alpha_3-\frac{1}{4}+\frac{1}{2}E_D/E\right)^2+\frac{3}{4}(E_D/E)(1+E_D/E)\right]^{-1} + 2\frac{\sigma_{AA}(E\sin^2\alpha_1)\sigma_{AA}(E\sin^2\alpha_2)}{\sigma_{AA}(E\sin\alpha_1\sin\alpha_2)} \times\int_0^1 dt\left\{\left[\cos^2\alpha_{12}(t)-\frac{1}{4}+\frac{1}{2}E_D/E\right]^2+\frac{3}{4}(E_D/E) \times(1+E_D/E)+3t(1-t)(1+E_D/E)\sin^2\alpha_3\right\}^{-1}\Bigg). \quad (17)$$

The angle $\alpha_{12}(t)$ is defined by

$$\cos^2\alpha_{12}(t) = t\cos^2\alpha_1 + (1-t)\cos^2\alpha_2. \quad (18)$$

The two terms inside the sum in Eq. (17) are a diagonal term, which corresponds to the square of a term in Eq. (8), and a cross term, which corresponds to the interference between two terms in Eq. (8). In the diagonal term, the last factor peaks at $\cos\alpha_3=\frac{1}{4}-\frac{1}{2}E_D/E$ with a width proportional to $[(E_D/E)(1+E_D/E)]^{1/2}$. In the limit $E\gg E_D$, this factor can be approximated by a δ function $\delta\left(\cos^2\alpha_3-\frac{1}{4}\right)$ multiplied by $(2\pi/\sqrt{3})(E/E_D)^{1/2}$. The resulting expression for R agrees with that in Eq. (13). The hyperangles α_1 and α_2 in Eq. (17) can be expressed as functions of α_3 and the angle β_3 between the vectors $\vec{p}_3$ and $\vec{p}_1-\vec{p}_2$:

$$\cos^2\alpha_{1,2} = \frac{1}{4}\cos^2\alpha_3+\frac{3}{4}\sin^2\alpha_3 \mp \frac{1}{2}\sqrt{3}\cos\alpha_3\sin\alpha_3\cos\beta_3. \quad (19)$$

The hyperangular average of the rate R in Eq. (17) can then be calculated using Eq. (14). In Fig. 2, we show $K_3=\langle R\rangle$ as a function of the collision energy E, as well as the contributions to K_3 from the diagonal terms and from the cross terms. The cross terms are larger than the diagonal terms at low energies and smaller at higher energies, with the crossover occurring near $E=20$ mK. At very high energy, the cross terms scale like $E^{-5/2}$. They eventually become negligible compared to the diagonal terms, which scale like E^{-2}. The factorization approximation in Eq. (15) is a high-energy approximation to the contribution from the diagonal terms. From the crossover point and the scaling behavior, we can infer that the factorization approximation can be a good approximation only if the collision energy exceeds the 20 mK by more than an order of magnitude.

We can get further insights into the factorization approximation by considering the angular-momentum decomposition of the three-body recombination rate. The rate calculated by Suno *et al.* in Ref. [9] was obtained by adding the contributions with total angular momentum quantum number $J=0,1,2$, and 3. These contributions are shown in Fig. 3. Note that at $E=10$ mK, the highest energy for which they were calculated, the $J=0$ and 1 contributions are increasing and they are smaller than the $J=2$ and 3 terms which are decreasing. The angular-momentum decomposition of the rate in Eq. (17) is

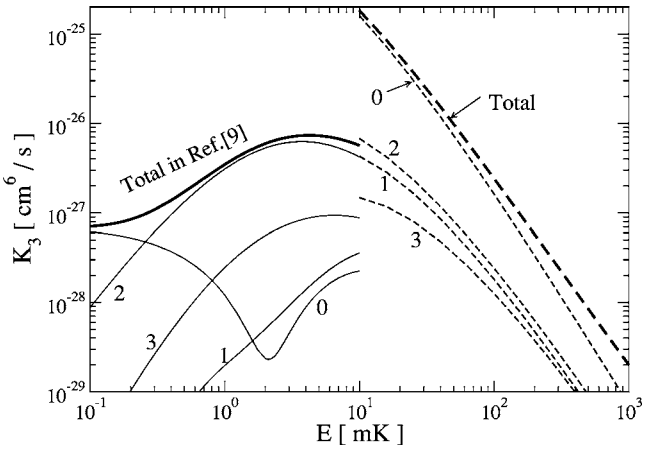

FIG. 3. Three-body recombination event rate K_3 (in units of cm^6/s) as a function of the collision energy E (in units of mK). The solid lines are the results in Fig. 3 of Ref. [9]. The dashed lines are the hyperspherical averages of the R in Eq. (17) and the $J=0$, 1, 2, and 3 terms in Eq. (20).

$$R = \frac{32\sqrt{3}\pi\hbar^3}{m^2 E}\left(\frac{E_D}{E+E_D}\right)^{1/2}\sum_{J=0}^{\infty}(2J+1) \times \sum_{(123)}\left(\sigma_{AA}(E\sin^2\alpha_3)\frac{Q_J(z_3)^2}{\cos^2\alpha_3} + 2\frac{\sigma_{AA}(E\sin^2\alpha_1)\sigma_{AA}(E\sin^2\alpha_2)}{\sigma_{AA}(E\sin\alpha_1\sin\alpha_2)}\frac{Q_J(z_1)Q_J(z_2)}{\cos\alpha_1\cos\alpha_2}P_J(z_{3,12})\right), \tag{20}$$

where the $Q_J(z)$ are Legendre functions of the second kind with branch cuts on the interval $-1<z<+1$. Their arguments are

$$z_i = \frac{\cos^2\alpha_i + \frac{1}{4} + E_D/E}{\sqrt{1+E_D/E}\cos\alpha_i}. \tag{21}$$

The argument of the Legendre polynomial $P_J(z)$ in Eq. (20) is

$$z_{3,12} = \frac{\cos^2\alpha_3 - \cos^2\alpha_1 - \cos^2\alpha_2}{2\cos\alpha_1\cos\alpha_2}. \tag{22}$$

The individual contributions from $J=0,1,2$, and 3 are shown in Fig. 3. In the high-energy limit, each individual contribution scales like $E^{-5/2}$. Since the total $\langle R\rangle$ scales like E^{-2}, the sum over J must provide the additional factor of $E^{1/2}$. At $E=10$ mK, the $J=0$, 1, 2, and 3 terms are larger than the results of Ref. [9] by factors of 711, 12.0, 1.6, and 1.7, respectively. The mismatch in the $J=2$ and 3 curves in Fig. 3 is small. The large mismatch in the $J=1$ curves is easy to understand. The cross terms in Eq. (20) are constructive for even J and destructive for odd J, and they are much smaller than the diagonal terms for $J\geqslant 3$. Thus the $J=1$ channel is the only one with substantial destructive interference. The sum of the diagonal and cross terms is 47% of the diagonal terms at 100 mK, 22% at 10 mK, and 4.5% at 1 mK. Because of the destructive interference at low energies, small corrections to the amplitude can give a large correction to the rate. Thus only small corrections to the amplitudes for $J=1$, 2, and 3 would be required for the terms in Eq. (20) to match smoothly on to the results of Ref. [9]. These corrections presumably come from higher orders in the perturbation expansion in the exact atom-atom scattering amplitude.

We now discuss the much larger mismatch between the $J=0$ curves in Fig. 3. In this channel, there is constructive interference between the diagonal terms and the cross terms in the leading-order result in Eq. (20). The corrections from higher orders in the exact atom-atom scattering ampitude could bring the curves into agreement if they interfere destructively with the leading-order term. The destructive interference would have to decrease the $J=0$ term at $E=10$ mK by a factor of about 700. We argue that such strong destructive interference is plausible by noting that there is strong destructive interference in the $J=0$ channel in the calculation of Ref. [9]. The maximum possible value for K_3 at $E=0$ can be obtained from Eq. (3) by setting $a_D=87.0$ Å. This value, $3.7\times10^{-26}\,cm^6/s$, is indicated by the horizontal arrow in Fig. 2. The value of K_3 at $E=0$ in Ref. [9] is smaller by a factor of 52, which indicates strong destructive interference. The $J=0$ result in Ref. [9] exhibits even stronger destructive interference at an energy near 2 mK, where it is smaller than at $E=0$ by a further factor of 31. Thus the suppression from destructive interference at this energy is comparable to that required to make the $J=0$ curves in Fig. 3 match smoothly at 10 mK. A nontrivial three-body calculation would be required to verify that higher orders in the exact atom-atom scattering amplitude provide the necessary destructive interference. The result would depend not only on the scattering length a or a_D, but also on the three-body parameter κ_* that appears in Eqs. (2) and (3).

The production of ^{4}He dimers, trimers, and tetramers has been studied in experiments involving the free expansion of a jet of cold pressurized ^{4}He atoms [11]. The experiments are analyzed using coupled rate equations for various formation and breakup processes. The rate constants for processes involving dimers were estimated by using interpolations and extrapolations of previous theoretical and experimental results. Our results for dimer breakup in Eq. (10) and for three-body recombination in Eq. (15) can be used to estimate the rates for those processes at large collision energies. The rate constants in Refs. [11] for processes involving trimers were based on geometrical estimates only. Our result in Eq. (12) for the inclusive breakup cross section can be applied to both the excited state and the ground-state of the trimer. Since the binding energy of the ground state trimer is about 50 times larger than that of the excited state, the condition $E\gg E_T$ requires much larger collision energy in the case of the ground-state trimer.

Factorization can also be used to calculate the rate for three-body recombination of two atoms and a loosely bound dimer into a loosely bound trimer when the three incoming particles all have relative momenta much greater than $\hbar/a$. The leading contribution comes from a hard scattering of the atoms that scatters one of them into a state with small momentum relative to the dimer, allowing it to subsequently coalesce with the dimer to form a trimer. The short-distance

factor has a form similar to that in Eq. (13). The long-distance factor involves an overlap integral of trimer and dimer wave functions. The evaluation of this factor requires a nontrivial three-body calculation, but it is much simpler than the direct calculation of the recombination rate by solving the four-atom Schrödinger equation. The factorization approximation could be used to calculate the production rate of both the excited state and the ground state of the ^{4}He trimer in the experiment of Ref. [11].

The production of deuterons, He nuclei, and the corresponding antinuclei have been observed in ultrarelativistic heavy-ion collisions [12]. These collisions are believed to produce a thermalized state with very high energy density which, as it expands and cools, makes transitions to a quark-gluon plasma and then to a hadron gas. Analogs of the factorization formulas in Eqs. (10) and (15) can be applied to the breakup and formation of deuterons and antideuterons in the hadron gas phase.

The strong interactions between atoms with large scattering lengths implies that high numerical accuracy is required to calculate rates for processes involving loosely bound molecules. Using traditional methods, the list of processes for which accurate calculations are possible is restricted to those involving at most three or four atoms. By using factorization, the list can be expanded to include all those for which the long-distance factors involve $N \leqslant 3$ or 4 atoms. In some cases, such as the breakup cross sections in Eq. (12), the long-distance factors can be determined analytically and accurate calculations are possible even for $N > 4$.

In the examples of ^{4}He molecules and the deuteron, the large scattering length a arises from a fortuitous fine tuning by nature. Another exciting application of factorization is to alkali atoms near a Feshbach resonance, which allows a to be tuned to arbitrarily large values by varying a magnetic field. The factorization formulas give the leading term in a systematic expansion in powers of $1/a$. As the scattering length is tuned to be increasingly large, the factorization approximation becomes increasingly accurate and it applies at increasingly lower collision energies.

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