Correspondence between quantum and classical information: Generalized quantum measurements

Boris A. Grishanin* and Victor N. Zadkov
International Laser Center and Faculty of Physics, M.V. Lomonosov Moscow State University, 119899 Moscow, Russia
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The concept of generalized quantum measurement is introduced as a transformation that sets a one-to-one correspondence between the initial states of the measured object system and final states of the object-meter system with the help of a classical informational index, unambiguously linked to a classically compatible set of quantum states. It is shown that the generalized quantum measurement concept covers all key types of quantum measurement—standard projective, entangling, fuzzy, and generalized measurements with a partial or complete destruction of initial information associated with the object. A special class of soft quantum measurements as a basic model for the fuzzy measurements widespread in physics is introduced and its information properties are studied in detail. Also, a special class of partially destructive measurements mapping all states of the Hilbert space of a finite-dimensional quantum system onto the basis states of an infinite-dimensional quantum system is considered.

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I. INTRODUCTION

One of the fundamental transformations in quantum physics is the projective measurement transformation, which sets a correspondence between a basis set of the quantum object states and classical results of its observation [1–3]. This transformation leaves the eigenstates of the measured quantum variables unperturbed and, therefore, corresponds to the *nondemolition* measurement. Despite the fact that its experimental realization causes significant experimental difficulties [4,5], the fundamental value of such a transformation in quantum physics as a standard procedure for obtaining physically meaning information is beyond controversy [6].

Nowadays, tremendous progress in experimental quantum physics has given us new, powerful tools for quantum-state engineering and arbitrary manipulation of quantum information [7]. This new experimental environment serves as the ground for further development of quantum theory and, specifically, leads to the necessity of revising the standard concept of quantum measurement in accordance with the needs of modern quantum information science [8–10].

In our view, the first step towards a generalization of the concept of the standard quantum measurement could be waiving the limitation on the measuring device (the meter) as a (quasi)classical system, which is necessary for a proper analysis of modern experiments in the field of quantum information engineering with the use of such objects as photons and trapped atoms or ions. This generalization has been recently given in the form of the *entangling* measurement concept [11,12] in which the measurement results in entanglement between the measured object and the meter. In this paper, we take the next step towards a generalization of the standard quantum measurement concept introducing the *soft* quantum measurements that adopt the concept of *fuzzy* quantum measurements [13,14], but give the results of the measurements in an approximate form, with some uncertainty, due only to their quantum indistinguishability—i.e., to the nonorthogonality of their wave functions.

As the next step, we consider a special class of so-called *partially destructive* quantum measurements, which give measurement results at the cost of information loss about initial states of the measured object. This information loss is only due to the representation of the output information about classically indistinguishable nonorthogonal measured states in a classically distinguishable form.

All these classes of generalized quantum measurements are based on an analogy with projective measurements and the idea of a one-to-one correspondence between the initial states of the object and final states of the meter.

In the classical case, one can easily fulfill the requirement of the absolute accuracy of the measurement result with the requirement of the absolute absence of perturbations in the object system during the measuring procedure. By contrast, in the quantum case, both these requirements cannot be fulfilled, simply due to the specific properties of the set of quantum states forming a linear space. Any interaction with a quantum system inevitably changes at least part of its quantum states [15].

For a quantum system, the requirement of the absolute absence of perturbations can be fulfilled only with respect to the chosen simultaneously measurable variables via the choice of a nondemolition type of measurement. Due to the uniqueness of the entire quantum information (thanks to the no-cloning principle [16]) the nondemolition measurement completely destroys the *coherent*—i.e., essentially quantum—information about the initial state of the object [11], which in the case of completely coherent measurements is distributed among the object and meter and does not exist in the separate systems of the object and meter.

Considered in this paper the class of soft quantum measurements allows a partial preservation of the coherent information in the object. On the contrary, the transmission of the major portion of the coherent information towards the meter can be realized only via destructive (demolition) quantum

*Electronic address: grishan@comsim1.phys.msu.ru

measurements, when the state of the object is inevitably perturbed. A limiting case of maximally destructive measurements, which entirely destroy the initial information, can be illustrated with the purely coherent transition of the excitation from one oscillator to another one in a set of coupled oscillators. Purely noncoherent measurements of the state of a two-level atom with the help of detection of the irradiated photon can serve as another example.

We consider here a more general class of partially destructive quantum measurements, which by contrast with the soft quantum measurements allows perturbation of the initial eigenstates of any physical variable. We analyze the most interesting, with respect to the qualitative content of the mapped information, case of nonselective mapping of all states $|\alpha)_A$ of the Hilbert space H_A of a quantum system onto the classically distinguishable eigenstates $|\alpha\rangle_B$ of a continuous variable of another more complex system with the Hilbert space $H_B=L_2(H_A)$.

The generalized quantum measurements, along with revealing potential resources of the measurement transformations for developing new methods of quantum information engineering, is also of interest from the viewpoint of a qualitative interpretation of quantum theory. It helps to expose the most general relationships between the physical changes caused by the transformations applied to a quantum system and the classical information contained in the information index that sets a one-to-one correspondence between the initial quantum states and those after the measurement.

The paper is organized as follows. In Sec. II, we define the generalized quantum measurements and discuss their informational meaning. In Sec. III, the class of soft quantum measurements is introduced and discussed in detail, including its information analysis and concurrency between object and meter. Section IV discusses the partially destructive measurements. Finally, key results are summarized in the Conclusions.

II. DEFINITION AND INFORMATIONAL MEANING OF THE GENERALIZED QUANTUM MEASUREMENTS

In this section we introduce the generalized quantum measurements starting from the definition of the entangling measurement.

In the abridged description of a measurement process, only two quantum systems—i.e., the object and meter represented by the corresponding Hilbert spaces H_A and H_B—must be introduced explicitly. Taking into account that the initial state of the meter can be specially prepared, the measurement can be represented as a properly specified type of a physically meaning transformation $H_A \to H_A \otimes H_B$, which, in the general case, is described by a corresponding superoperator in order to take into account the interaction with the microscopical subsystem of the meter and, possibly, with an auxiliary environment [17]. The standard projective measurement then is given by the nondemolition *entangling* measurement superoperator

$$\mathcal{M}_e = \sum_{kl} R_{kl} |k\rangle_A |k\rangle_B \langle l|_B \langle l|_A \langle k|_A \odot |l\rangle_A, \tag{1}$$

with the specific choice $R_{kl}=\delta_{kl}$ of matrix R_{kl}. Here the symbol $\odot$ denotes the place to substitute with the initial object density matrix $\hat{\rho}^A$; the ket and brastates denote the *orthogonal* eigenbasis states of the measured object and meter pointer, correspondingly. The *entangling matrix* $R=(R_{kl}) \geqq 0$ with $R_{kk}\equiv 1$ (in the general case, $R_{kl}\neq \delta_{kl}$) takes into account the quantum nature of the meter and the resulting after-measurement entanglement between the meter and object.

The case of $R_{kl}\equiv 1$ corresponds to a purely coherent—i.e., isometric—transformation, which is called sometimes a *premeasurement* [14], which results in the total preservation of the initial coherency and transmission of all essentially quantum (coherent) information, initially stored in the object, onto the set of duplicated states $|k\rangle_A|k\rangle_B$ [18]. Inequality $R_{kl} \neq 1$ represents dephasing due to the interaction of the object-meter system [19] with the traced-out degrees of freedom: R_{kl} can be represented as the Gram matrix $(l|k)_D$ resulting from averaging over the *nonorthogonal* dephasing subsystem states $|k)_D$ that are coupled with the meter ones simply via the set of the joint product states $|k\rangle_B|k)_D$.

The entangling measurement generalizes the standard projective measurement onto the case when the quantum nature of the meter is taken into account and, as a result, the obtained information is provided in the form of a complete or partial entanglement between the object and meter leaving the measured states $|k\rangle_A$ unperturbed [20]. It may also be worth mentioning that even in the case of a purely coherent transformation $\mathcal{M}_e$ the corresponding reduced $A\to A$ transformation is given by the maximally incoherent projective measurement superoperator $\mathcal{M}_A=\mathrm{Tr}_B\mathcal{M}_e =\sum_k |k\rangle_A\langle k|_A \odot |k\rangle_A\langle k|_A$.

Physical meaning of the transformation (1) is that it maps the information about the object onto the meter by setting a one-to-one correspondence between the classically distinguishable quantum states of the object and quantum states of the meter with the help of the classical *information index* k. It is worth stressing here that this information index has an important physical meaning, being associated with the corresponding physical variables, that is possible due to the orthogonality of the corresponding quantum states and hence enables unlimited copying of information associated with this index. By contrast, for the case of an arbitrary set of nonorthogonal states $|k)$, this copying is impossible and, therefore, the information index k has no physical informational meaning. Undoubtedly, this approach, based on the classical information index connecting the quantum states of the object and meter systems, is important for quantum information and has apparently a fundamental value for the foundations of quantum mechanics.

One can generalize the entangling measurement by releasing two requirements—first, that the measured states are not to be perturbed after the measurement and, second, about the classical distinguishability of the states in both the object and meter systems. This can be readily done by replacing in Eq. (1) the exactly cloned orthogonal states $|k\rangle$ of the object and meter with the nonorthogonal states $|k)$, which contain the internal indeterminacy and cannot be cloned in principle. After this replacement, the superoperator (1) remains positively defined. However, as one can easily see, in order to preserve its normalization it is necessary and sufficient to fulfill the condition $R_{kl}Q^A_{kl}Q^B_{kl}=0$ for all k,l, where $Q^{A,B}_{kl}=(k|l)^{A,B}$ are the respective Gram matrices for the corresponding sets of object and meter states. This means that the orthogonality for the given k,l must be hold true at least in one of the subsystems of the object-meter-reservoir system, which is due to the unitarity of the mapping, considered in terms of the complete system evolution.

Respectively, for the completely coherent measurement ($R_{kl}\equiv 1$), the possibility of using the nonorthogonal resulting states of the object and meter has an alternative character; i.e., for the output object-system states instead of $||k\rangle\rangle_{AB} = |k\rangle_A |k\rangle_B$ we have to use either

$$||k\rangle\rangle_{AB} = |k\rangle_A |k)_B \quad \text{or} \quad ||k\rangle\rangle_{AB} = |k)_A |k\rangle_B. \tag{2}$$

These alternative equations define a condition that k is a physically meaning information index. They result in two specific types of generalized quantum measurements that are discussed below in the following sections of this paper.

First, we consider in Sec. III the generalization of the entangled measurement onto the case of the so-called soft quantum measurements, which maps distinguishable measured states of the object onto not entirely distinguishable states of the meter. Second, we consider in Sec. IV the generalization onto the case of so-called partially destructive quantum measurements, which change the initial basis states of the object. Both these classes of measurements are based on the isometric transformation of the form $A \to A+B+D$, where D is the dephasing subsystem (a reservoir). Therefore, both these generalized quantum measurements completely preserve an initial distinguishable information [21] on the "measured" states $|k)_A$ (or $|k\rangle_A$)—i.e., the probability distribution $P_k = \nu_k (k|\hat{\rho}^A|k)_A$ [ν_k is given by Eq. (23) later on in the paper]—in the form of joint states (2), regardless to the presence or absence of an external dephasing or decoherency.

III. SOFT QUANTUM MEASUREMENTS

A. Definition and physical essence of the soft quantum measurements

The most general nondemolition measurement transformation that preserves a complete set of classically compatible object states $|k\rangle_A$ is described by the superoperator of the form

$$\mathcal{M}_{\mathrm{nd}} = \sum_{kl} (|k\rangle_A \langle l|_A \otimes \hat{\rho}^B_{kl}) \langle k|_A \odot |l\rangle_A, \tag{3}$$

where the set of operators $\hat{\rho}^B_{kl}$ defines a positive block-type operator with the normalized diagonal terms $\mathrm{Tr}\, \hat{\rho}^B_{kk} = 1$ in a Hilbert space H_B, which describes variables of the meter that are essential for the measurement. This superoperator associates the object projectors $|k\rangle_A \langle l|_A$ with the kl elements of the block-type operator $(\hat{\rho}^{AB}_{kl}) = |k\rangle_A \langle l|_A \otimes \hat{\rho}^B_{kl}$ in the object-meter system, which, in the general case, results in the states of the meter entangled with the measured states of the object. Here, normalization of the diagonal terms ensures preserving the probability for the set of compatible measured object states $|k\rangle_A \langle k|_A$, which are not perturbed during this measurement.

For the trivial case of $\hat{\rho}^B_{kl} = \hat{\rho}^B_0$ we get $\mathcal{M}_{\mathrm{nd}} \hat{\rho}^A = \hat{\rho}^A \otimes \hat{\rho}^B_0$, and all states of the meter are associated with a single density matrix of the meter states; i.e., no measurement is performed. In the case of the entangling measurement (1), we have a set of $\hat{\rho}^B_{kl} = R_{kl} |k\rangle_B \langle l|_B$, which establishes a one-to-one correspondence between the measured states of the object with similar, orthogonal, and completely distinguishable states of the meter. Such a measurement is a distinct one in the sense that for the respective events, represented by the compatible states $\hat{P}^A_k = |k\rangle_A \langle k|_A \otimes \hat{I}_B$, $\hat{P}^B_l = \hat{I}_A \otimes |l\rangle_B \langle l|_B$ of the object and meter, the joint probability distribution, corresponding to the resulting joint density matrix $\hat{\rho}^{AB} = \mathcal{M}_{\mathrm{nd}} \hat{\rho}^A$, is singular: $P(k,l) = \mathrm{Tr}\, \hat{P}^A_k \hat{P}^B_l \hat{\rho}^{AB} = \delta_{kl} \rho^A_{kk}$; i.e., in the supporting subspace of the density matrix $\hat{\rho}^{AB}$ we have $\hat{P}^A_k = \hat{P}^B_k$ for all k.

We will consider here the most fundamental class of the nondemolition measurements (3), which is characterized by using a "fuzzy" set of nonorthogonal states $|k)_B$ that contain the internal quantum uncertainty for the measurement results in the form $\hat{\rho}^B_{kl} = R_{kl} |k)_B (k|_B$. This kind of measurement in the limiting cases of orthogonal or trivial [consisting of the only state $|k)_B \equiv |0\rangle_B$] sets is reduced to the above-described entangling (specifically, projective) measurement and the case where no measurement is performed, respectively. In the most general case, measurements of this type are usually called *fuzzy* measurements [13,14]. For this generalized measurement, the meter does not contain any specified physical variable, which can store exact information about the number k of measured object states $|k\rangle_A$, and the attained information is connected with the entire physical structure of the meter, being represented in an essentially quantum form [22].

We will call this type of measurement (3), described by the superoperator

$$\mathcal{M} = \sum_{kl} R_{kl} |k\rangle_A |k)_B \langle k|_A \odot |l\rangle_A (l|_B \langle l|_A, \quad (k\,|l)_B = Q_{kl}, \tag{4}$$

the *soft* quantum measurement. The resulting information is represented here by the pure states $|k)_B \in H_B$, the uncertainty of which has a purely quantum nature and is related to their nonorthogonality, which leads to the impossibility of setting a one-to-one correspondence between the measured physical variable $\hat{A} = \Sigma \lambda_k |k\rangle_A \langle k|_A$ and any analogous variable of the meter. Instead, the one-to-one correspondence is set between the distinguishable object states $|k\rangle_A$ and undistinguishable states of the meter, $|k)_B$.

The physical essence of the superoperator (4) is the transformation of the initial orthogonal basis object states $|k\rangle_A$ into also orthogonal—i.e., completely distinguishable—states $|k\rangle_A |k)_B$ of the bipartite object-meter system, which is independent from the initial meter state. At the same time, the phase relationships between the initial states are generally perturbed and their joint correlations are described by the matrix elements R_{kl}, whereas the Gram matrix $Q = (Q_{kl})$ describes the degree of quantum distinguishability of the measuring states $|k)_B$ of the meter. The soft character of the measurement is characterized by the difference of the matrix Q from the identity matrix I that corresponds to the conventional (distinct) entangling measurement. For the transformation (4), as well as in the case of a distinct measurement, the classical content of the measured object states $|k\rangle_A$ is not perturbed, whereas the quantum information initially stored in the initial object state $\hat{\rho}^A$ is redistributed between two subsystems and perturbed due to decoherence.

In the case of $R_{kl}\equiv 1$, when the coherency is preserved, the soft quantum measurement, considered as the transformation in the bipartite object-meter system with the initially "prepared" pure state $|0\rangle_B$ of the meter, is equivalent to the unitary transformation. It maps the set of initial orthogonal states of the form $|k\rangle_A|0\rangle_B$ onto the orthogonal states $|k\rangle_A|k\rangle_B$ and, obviously, can be redefined up to the unitary operator U_{AB} in the total space $H_A\otimes H_B$. The respective redefinition of the superoperator (4) then can be represented by the superposition $\mathcal{US}$ of the superoperator $\mathcal{S}=|0\rangle_B\langle 0|_B\mathrm{Tr}_B\odot$, which resets the meter into the initial state $|0\rangle_B$, and the unitary superoperator transformation $\mathcal{U}=U_{AB}\odot U_{AB}^{-1}$.

One can also easily see that in this case the entropy of the initial object state is entirely transferred into the entropy of the bipartite object-meter system, $S[\hat{\rho}^A]=S[\hat{\rho}^{AB}]$. Respectively, the coherent information [23], modified with respect to its transfer from H_A into $H_A\otimes H_B$ [18], is equal to its initial value $S[\hat{\rho}^A]$. All losses are due to the decoherence and respective violation of the isometricity of the transformation at $R_{kl}\neq 1$ only. Nevertheless, the corresponding marginal transformation $A\to A$ of the object system is ever incoherent due to the interaction-produced loss of initial coherence, even in the case of completely coherent measurements.

For the purely coherent case, it is not difficult to calculate now the Hamiltonian of the transformation of the infinitesimal soft quantum measurement in the object-meter system with the fixed initial state $|0\rangle_B$ of the meter, which can be chosen as one of the resulting states of the meter—i.e.,$|0\rangle_B=|0)_B$. Calculating the infinitesimal addition for a short time in the state of the object-meter system as the result of the corresponding unitary transformation with the "perturbation-free" Hamiltonian $\hat{\varepsilon}=\Sigma_k|k\rangle_A\langle k|_A\otimes\hat{\varepsilon}_B(k)$ and equating its result to the deviation, which is caused by the transformation (4), we do have $-i\frac{\Delta t}{\hbar}\hat{\varepsilon}|k\rangle_A|0\rangle_B=|k\rangle_A|\delta k)_B$, where$|\delta k)_B=|k)_B-|0\rangle_B$. From here, for the k-dependent Hamiltonian of the meter we receive $-i\frac{\Delta t}{\hbar}\langle l|\hat{\varepsilon}_B(k)|0\rangle_B=\langle l|\delta k)_B$, from which, due to the Hermitian property, follows the equation for the uniquely determined matrix elements:

$$\hat{\varepsilon}_B(k)=\sum_l\lim_{t\to 0}i\frac{\hbar}{\Delta t}(\langle l|\delta k)_B|l\rangle_B\langle 0|_B-(\delta k|l\rangle_B|0\rangle_B\langle l|_B). \quad (5)$$

Other elements can be defined arbitrary or set to zero.

B. Repeated soft quantum measurements

In this section, we will consider the results of the repeated quantum measurements of the object, which are illustrated on example of a two-level system in Appendix A.

Repeated application of the soft measurement to the result of the previous measurement does not increase the attained information, because the resulted information contains an additional indeterminacy in comparison with a single measurement. The indeterminacy produced by the latter does not vanish or decrease at the next interaction of the meter with the object.

FIG. 1. Physical implementation of the repeated soft quantum measurements in an optical dipole trap: an atom, a carrier of quantum information, moves along the linear chain of atoms, which are located in the potential microtraps of the optical dipole trap, each of which performs a measurement.

For the repeated measurements of the *object* with preservation of the measurement results in independent degrees of freedom of the multicomponent meter due to the n-fold application of the measurement transformation we receive the following resulting transformation:

$$\mathcal{M}^{(n)}=\sum_{kl}R_{kl}^n|k)\cdots|k)|k\rangle_A\langle k|_A\odot|l\rangle_A\langle l|_A(l|\cdots(l|. \quad (6)$$

Here $|k)\cdots|k)$ denotes the corresponding product state in the combined Hilbert space $H_B^n=H_B\otimes\ldots\otimes H_B$ of the identical copies of the initial meter system H_B with dimension D.

This transformation results in increasing incoherency and yields multiply duplicated fuzzy information about the value k. At the same time, the quantum character of the measured information is maximally preserved only in the entire system, which includes all the meter's subsystems. After averaging over $m<n$ output subsystems of the meter the remaining quantum information is affected by the decoherence and can be characterized by the entanglement matrix $R_{kl}^{(n-m)}=Q_{kl}^mR_{kl}^n$. This matrix defines the decoherence properties of the measurement even without any decoherence during the creation of the entanglement in separate measurements—i.e., for $R_{kl}\equiv 1$. In the active subspace $H_D=\mathrm{sp}\{||k\rangle\rangle, k=1,\ldots,D\}\sqsubseteq H_B^n$ of the *collective* states $||k\rangle\rangle=|k)\cdots|k)$ the measurement transformation (6) has the form of a single measurement, but with entanglement matrices and scalar products corresponding to n measurements.

Such a measurement can be illustrated with various examples of its physical implementation. For instance, if we select an atom from an atomic gas as a quantum object, we can consider, in general, all the surrounding atoms as the multipartite meter system. Then, collisions between atoms can be considered as separate measurement acts, which augment the measured information in the multipartite system. This situation is surely beyond the scope of the standard quantum measurement concept, which assures that all obtained information is accessible and can be used for any purpose. An optical dipole trap [24,25] can serve as another physical example, which clarifies our model of the repeated measurements more distinctly. In this trap, an atom, a carrier of quantum information, moves along the linear chain of atoms, which are located in the potential microtraps of the optical dipole trap, each of which performs a measurement (Fig. 1); i.e., we have the case of repeated measurements. Also, successful experimental realizations of the nondemolition projective measurements with single photons [5,26] give us a hope that the repeated measurements considered here would be implemented experimentally in the nearest future, not only with atoms, but also with photons.

The dimension of the Hilbert space of the active states of the meter does not exceed the dimension of the single meter space, despite the fact that the combined space of the meter's states unrestrictedly expands. For the appearance of new active states different from the set $|k\rangle_A|k)\cdots|k)$ (or to the unitary equivalent to it) it is necessary that the dynamics along the different degrees of freedom be independent and random. However, deviations of the bases used in different measurements lead to deviations of the resulting states of the meter from the indicated active space and, if we have no *a priori* information about these deviations, will result in losses of information about the object. Therefore, coding of information in the series of repeated soft measurements provides a resource for the latent storing of information with the unique quantum key.

When all the measurement results are preserved, the joint density matrix of the object-meter system can be written as

$$\hat{\rho}^{B^nA} = \sum_{kl} R^n_{kl}\rho^A_{kl}|k)\cdots|k)|k\rangle_A\langle l|_A(l|\cdots(l|. \tag{7}$$

In its turn, the density matrix of the meter, averaged over the states of the object, is equal to

$$\hat{\rho}^{B^n} = \sum \rho^A_{kk}|k)\cdots|k)(k|\cdots(k|, \tag{8}$$

i.e., represented with the weighted sum of D noncommuting projectors. In an orthogonal basis $||e_k\rangle\rangle$ of the active subspace of the collective states H_D, they can be rewritten in the form

$$\hat{\rho}^{B^nA} = \sum \rho^{B^nA}_{ki,lj}||e_k\rangle\rangle|i\rangle_A\langle j|_A\langle\langle e_l||, \tag{9}$$

$$\hat{\rho}^{B^n} = \sum \rho^{B^n}_{kl}||e_k\rangle\rangle\langle\langle e_l||, \tag{10}$$

with matrix elements corresponding to the respective equations (7) and (8) and the choice of basis $||e_k\rangle\rangle$.

The matrix of scalar products for the vector set $|k)\cdots|k)$ has the form $Q^{(n)}=(Q^n_{kl})$. In the case of the linearly independent set $|k)$ for $n\to\infty$ this matrix has the form of the identity matrix $Q^{(\infty)}=I$. The orthonormalized basis in H_D can be expressed via the duplicated states of the meter with the help of the following relationship:

$$||e_k\rangle\rangle = \sum_l (Q^{(n)-1/2})^*_{kl}|l)\cdots|l). \tag{11}$$

The corresponding formulas

$$\rho^{B^nA}_{ki,lj} = \langle\langle e_k||\langle i|\hat{\rho}^{B^nA}|j\rangle||e_l\rangle\rangle = R^n_{ij}\rho^A_{ij}\psi^{B^n}_i(k)\psi^{B^n*}_j(l), \tag{12}$$

$$\rho^{B^n}_{kl} = \langle\langle e_k||\hat{\rho}^{B^n}||e_l\rangle\rangle = \sum_j \rho^A_{jj}\,\psi^{B^n}_j(k)\psi^{B^n*}_j(l) \tag{13}$$

represent here the states of the meter with the matrix

$$\psi^{B^n}_i(k) = (Q^{(n)1/2})_{ki} \tag{14}$$

of fixed dimension $D\times D$. This dimension does not depend on the number of measurements, n, and respective total dimension D^n of the multipartite Hilbert space H^n_B of the meter. The entanglement matrix R_{ij} is essential only for constructing the bipartite density matrix of the object-meter system and does not affect the partial density matrix of the meter because after the tracing over the object states their coherence is not important. A set of nonorthogonal, in the general case, functions $\psi^{B^n}_i(k)$ makes sense of the ensemble of pure collective states of the meter, post-selected after n measurements and corresponding to the ith measured object states. All of them satisfy the normalization condition $(\psi^{B^n}_i,\psi^{B^n}_i)\equiv 1$, in which, due to definition (14) of collective states, the scalar products are reduced to the normalized on unit diagonal elements of the matrix $Q^{(n)}$.

C. Information analysis of the soft quantum measurements

In this subsection, we present a quantum information analysis of the soft quantum measurements with the help of both the coherent and semiclassical information.

1. Coherent information in the object-meter and object-object channels

The key features of the coherent information [23] exchange at the entangling measurement (1) are described in detail in Ref. [11]. The coherent information is purely quantum [27,28] and, therefore, cannot be copied or duplicated. Thus, when the states $|k\rangle_A$ are duplicated, the coherent information is transferred onto the superposition of bipartite states $|k\rangle_A|k\rangle_B$ and is equal to zero in the channels "initial-resulting state of the object" and "initial state of the object-resulting state of the meter."

In accordance with its definition, the coherent information preserved in the channel, which implements the superoperator transformation $\mathcal{N}$, can be written as

$$I_c = S[\mathcal{N}\hat{\rho}] - S[(\mathcal{N}\otimes\mathcal{I})\Psi\Psi^\dagger], \tag{15}$$

where the first term describes the entropy at the output of the channel and the second term the so-called exchange entropy that characterizes the entropy surge due to the irreversibility of the transformation; the pure state Ψ represents the so-called "purified state" at the input of the channel as the state of the bipartite system input+reference, which describes the mixed input state as the result of its tracing over the auxiliary reference system, and $\mathcal{I}$ is the identical transformation on the reference system state, which is not perturbed.

Let us consider how the coherent information is transformed at the soft measurement. In the two-time channel "object→object+meter," a fully coherent variance of the soft quantum measurement does not affect the coherent information, on account of the orthogonality of the states $|k\rangle_A|k\rangle_B$ of the object-meter system. All losses of coherent information in this channel are due to the decoherence only. By contrast, in the two-time "object-object" channel the influence of the soft measurement on the coherent information is a nontrivial one—in this channel the amount of preserved coherent information depends on both the decoherence and soft character of the measurement. The latter determines how the amount of information obtained by the meter affects the information preserved in the object.

Substituting in Eq. (15) the transformation $\mathcal{N}=\mathrm{Tr}_B\mathcal{M}$ and taking into account Eq. (4), we get $I_c=S[(R_{kl}Q_{kl}\rho^A_{kl})]$

$-S[(R_{kl}Q_{kl}\Psi_{ki}\Psi^*_{lj})]$, where the arguments in the parentheses are the matrix elements $\tilde{\rho}_{kl}$ and $\tilde{\rho}_{ki,lj}$ corresponding to the transformed density matrices of the object and object-reference systems; $\Sigma_i\Psi_{ki}\Psi^*_{li}=\rho^A_{kl}$. Therefore, for the considered channel contributions due to the decoherence and distinctness of the measurements, which are presented with the respected matrices R_{kl} and Q_{kl}, are totally equivalent. Simplifying the expression in the argument of the second term (for details, see Appendix B),one can rewrite the expression in the final form

$$I_c = S[(R_{kl}Q_{kl}\rho^A_{kl})] - S[(\sqrt{\rho^A_{kk}}R_{kl}Q_{kl}\sqrt{\rho^A_{ll}})]. \tag{16}$$

In the absence of decoherence—i.e., $R_{kl}\equiv 1$—and at the maximal softness degree—i.e., $|k)\equiv|0\rangle$—the second term in Eq. (16) vanishes, as far as its argument goes to the density matrix of a pure state, whereas the first term coincides with the entropy of the output; i.e., the coherent information is transmitted without disturbance to the object system only. In the opposite case, for the measurement with complete distinguishability of the states of the meter or for their maximal dephasing $R_{kl}Q_{kl}=\delta_{kl}$, both terms describe the entropy of the set of the measured states of the object $|k\rangle_A$, which is determined with the maximum entropy probability distribution $p_k=\rho^A_{kk}$, and, respectively, the coherent information vanishes due to the complete decoherence of the input information.

Let us illustrate calculation of the coherent information I_c on an example of a two-level system, for which we have

$$I_c = \frac{1}{2}[(1-x_1)\log_2(1-x_1) + (1+x_1)\log_2(1+x_1) - (1-x_2)\log_2(1-x_2) - (1+x_2)\log_2(1+x_2)], \tag{17}$$

where $x_{1,2}$ can be written with the only parameter $q=|R_{12}Q_{12}|$ of the matrix $R_{kl}Q_{kl}$, diagonal matrix element $p=\rho^A_{11}$, and the coefficient module of correlations $\mu=|\rho^A_{12}|/\sqrt{p(1-p)}$:

$$x_1 = \sqrt{1-4p(1-p)(1-q^2)}, \quad x_2 = \sqrt{1-4p(1-p)(1-q^2\mu^2)}.$$

The dependence corresponding to Eq. (17) for $p=1/2$ (for the maximally possible amount of information of the source equal to 1 bit) is shown in Fig. 2(a).

2. Semiclassical information in the object-meter channel

When one uses the object as a source of purely classical information in the most general form of the mixed ensemble $\{p_\lambda,\hat{\rho}^A(\lambda)\}$, the semiclassical information retrieved by the meter is described by the respective ensemble $\{p_\lambda,\hat{\rho}_\lambda\}$ resulting after the averaging over the object variables to

$$\hat{\rho}_\lambda = \sum_k \rho^A_{kk}(\lambda)|k)(k|. \tag{18}$$

Nonclassicality of this channel is related to the commutator $\hat{C}_{\lambda\mu}=[\hat{\rho}_\lambda,\hat{\rho}_\mu]=\Sigma_k[\rho^A_{kk}(\lambda)\rho^A_{ll}(\mu)-\rho^A_{kk}(\mu)\rho^A_{ll}(\lambda)]Q_{kl}|k)(l|$, which is nonzero only for the soft quantum measurements with $Q_{kl}\neq\delta_{kl}$. Nonorthogonality of the meter states $|k)$ in ensemble (18) leads to the respective reduction of the retrieved information.

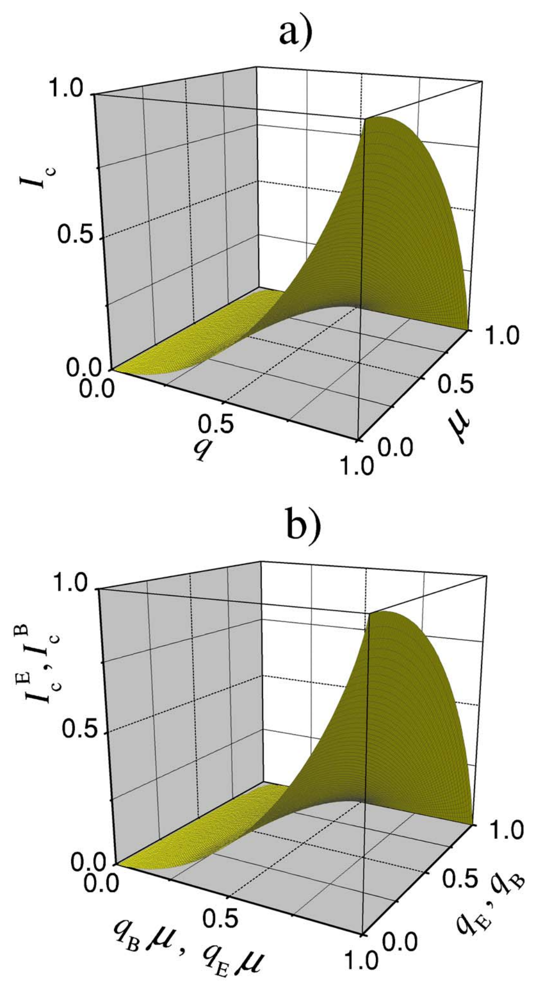

FIG. 2. (Color online) (a) The coherent information I_c, preserved in the object after the soft quantum measurement with the combined parameter q, which characterizes the level of the softness and the degree of coherency of the measurement, versus the function of the degree of coherency μ of the initial state of the object. (b) The coherent information I_c^E retrieved by Eve after the soft quantum measurement performed by Bob and the coherent information I_c^B retrieved by Bob after the soft quantum measurement performed by Eve with the same fuzziness parameter q_B. The graphs (a) and (b) numerically coincide, but they are different by their physical content.

Let us illustrate how the amount of information depends on the parameters of the soft quantum measurement for the case of the repeated measurements in the same basis for the input ensemble of pure states $\hat{\rho}^A_k=|k\rangle\langle k|$. As one can easily see, this ensemble corresponds to another one in the form of pure states $\hat{\rho}^{B^n}_k=||k\rangle\rangle\langle\langle k||$ in the active subspace of the meter, which is described in Sec. III B. As an adequate quantitative characteristic for this kind of the channels we can use the semiclassical Holevo information $I_s=S[\Sigma p_k\hat{\rho}^{B^n}_k]-\Sigma p_kS[\hat{\rho}^{B^n}_k]$ [29], which in this case is simply equal to the entropy $S[\hat{\rho}^{B^n}]$ of the resulting density matrix $\hat{\rho}^{B^n}=\Sigma p_k||k\rangle\rangle\langle\langle k||$, which has the matrix elements (13) with $\rho^A_{kk}\to p_k$, at the output of the measurement channel. For a two-level case, on account of

Eq. (A5) and for the *a priori* distribution $p_k=\{1/2,1/2\}$ we have the following amount of information:

$$I_s=-\frac{1}{2}\left(\log_2\frac{1-e^{-4\varkappa t}}{4}+e^{-2\varkappa t}\log_2\frac{1+e^{-2\varkappa t}}{1-e^{-2\varkappa t}}\right),$$

which monotonously changes from zero up to $I_{\max}=1$ with the change of the measurement time $0\leqslant\varkappa t\leqslant\infty$. The effects of the decoherence are not important in this case because related phases of the measured states of the meter are not essential.

D. Competition at the retrieval of information from the object

In this subsection we discuss the competition at the retrieval of the quantum information for the case when there is a single source of information and two receivers. This setting resembles the standard setup for quantum cryptography when Alice transmits a piece of information to Bob via a secure quantum or semiclassical channel and Eve tries to eavesdrop on the transmitted information [7].

1. Competition at the retrieval the coherent information

Let us consider now the restrictions on the soft quantum measurement information for the case when there is a single source of information (we will call it "Alice") and two receivers ("Bob" and "Eve"). The latter retrieve this information in series with the help of a repeated nondemolition quantum measurement with a different choice of the measured variables, in the general case (by contrast with Sec. III B). Mapping of the quantum states in the Hilbert spaces has the form $H_A\rightarrow H_A\otimes H_E\otimes H_B$ and the respective complete superoperator transformation can be written as

$$\mathcal{M}_{EB}=(\mathcal{I}_E\otimes\mathcal{M}_B)(\mathcal{M}_E\otimes\mathcal{I}_B), \tag{19}$$

where

$$\mathcal{M}_E=\sum R^E_{k_El_E}|k_E\rangle|k_E)(l_E|\langle l_E|\langle k_E|\odot|l_E\rangle,$$

$$\mathcal{M}_B=\sum R^B_{k_Bl_B}|k_B\rangle|k_B)(l_B|\langle l_B|\langle k_B|\odot|l_B\rangle$$

describe the measurements performed by Eve and Bob under the same object, but using different meters, and $\mathcal{I}_{E,B}$ is the respective identical transformation over the variable of the meter inaccessible in this measurement.

Expanding Eq. (19), we have

$$\mathcal{M}_{EB}=\sum R^E_{k_El_E}R^B_{k_Bl_B}\langle k_B|k_E\rangle\langle l_E|l_B\rangle|k_B\rangle|k_E)|k_B)(l_B|(l_E|\langle l_B|\times\langle k_E|\odot|l_E\rangle.$$

In a specific case of coinciding meter bases, $|k_B\rangle=|k_E\rangle$, we have

$$\mathcal{M}^0_{EB}=\sum R^E_{kl}R^B_{kl}|k\rangle|k)|k)(l|(l|\langle l|\langle k|\odot|l\rangle.$$

The difference between the above-considered case and that described in Sec. III B is in the independent use of information contained in the bipartite states $|k)|k)$ of the Eve-Bob system, which are partially coherently connected to the states $|k\rangle$ of the object.

The result of Eve's measurement does not depend on the subsequent Bob's measurement only for the marginal states $\hat{\rho}^B$, but not for the joint states $\hat{\rho}^{AE}$, which after Bob's measurement (for instance, in the same basis) are represented instead of the initial superoperator $\mathcal{M}_E=\sum R^E_{kl}|k\rangle|k)(l|\langle l|\langle k|\odot|l\rangle$ with the superoperator

$$\mathcal{M}'_E=\mathrm{Tr}_B\mathcal{M}^0_{EB}=\sum R^E_{kl}R^B_{kl}Q^B_{kl}|k\rangle|k)(l|\langle l|\langle k|\odot|l\rangle,$$

which contains an additional decoherency factor $R^B_{kl}Q^B_{kl}$. An absence of the back action—i.e., the equality $\mathcal{M}'_E=\mathcal{M}_E$—in the case of $R^E_{kl}\neq 1$, when Eve retrieves the information in essentially quantum form, occurs only for the completely coherent measurement by Bob, which contains no resulting information ($R^B_{kl}\equiv 1$, $Q^B_{kl}\equiv 1$). In the case of completely incoherent measurements by Eve there is never any reaction after Bob's measurement, which ensures the stability of classical information against its copying. A similar action does the measurement by Eve on Bob's measurement, which has after Eve's measurement the form $\mathcal{M}'_B=\mathrm{Tr}_E\mathcal{M}^0_{EB}=\sum R^B_{kl}R^E_{kl}Q^E_{kl}|k\rangle|k)(l|\langle l|\langle k|\odot|l\rangle$, i.e., contains an additional factor $R^E_{kl}Q^E_{kl}$compare to the case without Eve's measurement.

Such a reaction of the quantum operation can be adequately described with a respected reduction of coherent information due to its reception by a new receiver. In this case, due to the quantum measurement, the meter receives coherent information about the object states only after the measurement and the received information can be considered as the corresponding degree of quantum entanglement in the object-meter system, which is measured, for instance, in the system $A+B$ with the help of the difference $I_c=S[\mathrm{Tr}_B\mathcal{M}\hat{\rho}_A]-S[\mathcal{M}\hat{\rho}_A]$, which is always positive in the case of the soft quantum measurement.

By contrast with a similar definition used in Ref. [11] for the special case of distinct entangling measurements, the first term here determines the entropy of the object, but not the meter, because the entropies of the object and meter do not coincide for the case of the soft quantum measurement.

Calculating the respective information for Eve and Bob, we have

$$I^E_c=S[(\rho^A_{kl}R^E_{kl}R^B_{kl}Q^E_{kl}Q^B_{kl})]-S[(\rho^A_{kl}R^E_{kl}R^B_{kl}Q^B_{kl})],$$

$$I^B_c=S[(\rho^A_{kl}R^E_{kl}R^B_{kl}Q^E_{kl}Q^B_{kl})]-S[(\rho^A_{kl}R^B_{kl}R^E_{kl}Q^E_{kl})].$$

The coherent information retrieved by Eve after Bob's measurement and the amount of information received by Bob are shown for the two-dimensional case in Fig. 2(b) as functions of the softness parameters for the Bob's ($q_B=|Q^B_{12}|$) and the Eve's ($q_E=|Q^E_{12}|$) measurements at $R^B_{12}=R^E_{12}=1$ and for the density matrix of the object $\rho^A_{11}=\rho^A_{22}=1/2$, $\rho^A_{12}=\rho^A_{21}=\mu/2$.

The respective analytical expressions for I^E_c,I^B_c can be obtained using Eq. (17) for $I_c(q,\mu)$, which determines the coherent information about the reference system, which is preserved in the object with the initial density matrix $\hat{\rho}^A$ after the soft measurement with the softness parameter q. This case corresponds to the change of variables $\{q\rightarrow q_B\mu,\ \mu\rightarrow q_E\}$ when calculating the information retrieved by Eve and, respectively, $\{q\rightarrow q_E\mu,\ \mu\rightarrow q_B\}$ for the calculation of the information retrieved by Bob. In these calculations, q_B

$=1$ corresponds to the case when Bob practically does not perform the measurement and retrieves the unperturbed amount of Eve's information I_c^E (and vice versa), which is decreased with decreasing q_B due to the competence.

Note that the impact of the coherency parameter of the initial state μ on the coherent information (17), shown in Fig. 2(a), and the competitive information $I_c^{E,B}$, shown in Fig. 2(b), is the opposite. Whereas the value of Eq. (17) with increasing μ falls due to the decrease of the initial entropy of the density matrix, which determines the entanglement between the object and associated reference system, the information $I_c^{E,B}$ increases with increasing μ due to the respective increase of the object-meter entanglement after the measurement. This entanglement does not exist for the incoherent mixture of pure states $\{p_k, |k\rangle\}$, and because of their imperturbability, the states $|k\rangle$ even at the completely coherent measurement are described with the incoherent mixture of independent states $|k\rangle|k)$ of the object-meter system.

The competition at the selection of the coherent information reveals an opposite action of the parameters q_B and q_E on the information I_c^E retrieved by Eve, for instance: with decreasing q_E—i.e., with increasing accuracy of Eve's measurement—her information increases, whereas with decreasing q_B it decreases up to zero at $q_B=0$ due to the partial transfer of the information to Bob.

2. Competition at the selection of the classical information

The competition at the retrieval of the quantum information is revealed also in the case of the semiclassical channels $A\to E$ and $A\to B$ with a given ensemble of input states. An ensemble corresponding via Eq. (18) to the first channel, as can be easily checked by respective averaging of the superoperator (19), is not modified after the secondary measurement by Bob because he does not affect the input state. However, Bob's measurement result depends on the basis choice, which is used for Eve's measurement. The respective interdependence of the resulting quantum transformations is the foundation of quantum cryptography [7].

Whereas the transformation $\mathcal{M}_{A\to E}=\Sigma_k|k)(k|\langle k|\odot|k\rangle$ includes only parameters of the measurement performed by Eve—namely, the measured states $|k\rangle$ and the states of the meter $|k)$—the transformation of the channel $A\to B$ depends also on the parameters of Eve's measurement, in accordance with Eq. (19), as

$$\mathcal{M}_{A\to B}=\sum_{kl} R_{kl}^E Q_{kl}^E (\mathcal{P}_B|k\rangle_E\langle l|_E)\langle k|_E\odot|l\rangle_E. \tag{20}$$

Here $\mathcal{P}_B=\Sigma_k|k)_B(k|_B\langle k|_B\odot|k\rangle_B$ is the superoperator of the *soft projection* from H_A onto H_B, which describes the result of the secondary measurement performed by Bob after Eve's measurement; $|k,l\rangle_{E,B}$ are the vectors of the measured basis states in the space H_A of the object states for the measurements by Eve and Bob.

The incoherence introduced by Bob due to the averaging over the states of the object does not affect the information retrieved by Bob, and the soft character of the measurement is described by the superoperator $\mathcal{P}_B$. At the same time, the incoherence of Eve's measurement and its soft character are

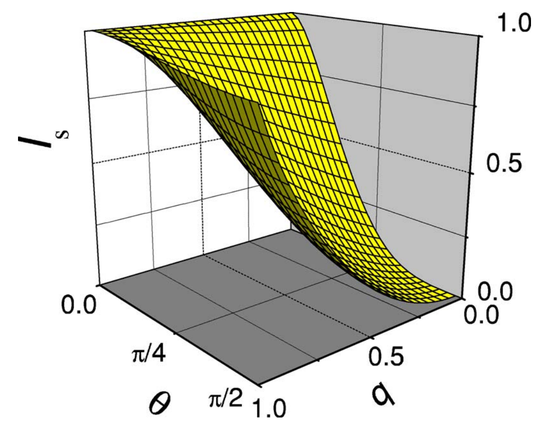

FIG. 3. (Color online) The semiclassical information I_s versus the softness parameter $q=|q_{12}|$ of the measurement performed by Eve and the orientation angle ϑ of Eve's basis on the Bloch sphere in respect to the input ensemble of two equiprobable pure states $|k\rangle_B$ at the rigid measurement by Bob with $|k)_B=|k\rangle_B$.

reflected with the common decoherence matrix (external in respect to Bob) $q_{kl}=R_{kl}^E Q_{kl}^E$, which describes the resulting degree of the "softness" of Eve's measurement. When the bases coincide—i.e., $|k\rangle_E=|k\rangle_B=|k\rangle$—Eve's measurement does not affect the information retrieved by Bob because $\mathcal{P}_B|k\rangle_E\langle l|_E=\delta_{kl}|k)_B(k|_B$ and the dependence on the parameters R_{kl}^E, Q_{kl}^E vanishes. In this case, both Eve and Bob use only classically compatible information about the object, which surely lacks a specific quantum nature of the competition in its selection, and this information can be copied independently. The dependence of the resulting channel $A\to B$ on Eve's transformations is related exclusively to the lack of coincidence of their measurements bases, which makes an essential quantum disturbance of the object state introduced by Eve at $q_{kl}\neq 1$.

The corresponding generalization of the semiclassical channel (18) on account of its modification (20) of the respective transformation of its quantum input has the form

$$\hat{\rho}_\lambda=\sum_{kl}\rho_{kl}^{A|E}(\lambda)q_{kl}\mathcal{P}_B|k\rangle_E\langle l|_E, \tag{21}$$

where $\rho_{kl}^{A|E}$ is the density matrix of the object in the basis Eve performs the measurement. The respective dependence of the semiclassical information I_s on the measurement parameters and on the input ensemble for the two-level example is shown in Fig. 3. At its maximum degree, the competition of the measurements is revealed at the orientation angle $\vartheta=\pi/2$, which, in the case of a physical realization of the Hilbert space of the object as the polarization degree of freedom of a photon, corresponds to the rotation of linear polarization of the photon at 45°.

IV. PARTIALLY DESTRUCTIVE QUANTUM MEASUREMENTS

Partially destructive quantum measurements, which we will discuss in this section, form a special class of generalized quantum measurements that map an overfull nonorthogonal set of states of the object quantum system onto an orthogonal basis set of the meter system. Of a special interest is a particular case when all states of a finite-dimensional Hilbert space are mapped onto a continuous orthogonal basis set of an infinite-dimensional Hilbert space.

We will start with an isometric transformation of the form

$$\mathcal{V}=\sum_{\alpha}\sqrt{\nu_\alpha}|\alpha\rangle_{AB}(\alpha|_A \qquad (22)$$

from the Hilbert space H_A of the object A onto the space $H_A\otimes H_B$ of the bipartite object-meter system $A+B$. Here the vectors $|\alpha\rangle_{AB}$ define the orthogonal basis in the Hilbert space of the object-meter system, indexed by the values α of an indicator variable. This basis allows a new representation of the initial quantum information that can be measured in the general case with the help of nonorthogonal "probe" states $(\alpha|_A$; the set of positive numbers ν_α characterizes the repetition factor of the elementary maps $|\alpha)_A\rightarrow|\alpha\rangle_{AB}$, of which the resulting transformation $\mathcal{V}$ is constructed as the coherent (i.e., depending on the phases of the wave functions) superposition of the respective generalized projectors.

The relation

$$\mathcal{V}^{+}\mathcal{V}\equiv\sum_{\alpha}\nu_\alpha|\alpha)_A(\alpha|_A=\hat{I}_A \qquad (23)$$

assures the isometric property of the transformation. It admittedly can be fulfilled if the set of mapped states $|\alpha)_A$ is a collection of orthogonal bases, randomly rotated with respect to each other. Specifically, for N equally represented bases we have $\nu_\alpha=1/N$.

The transformation (22) is a generalized modification of the canonical representation of the isometric mapping $\mathcal{V}=\Sigma|k\rangle_C\langle k|_A$ as the transformation of the entire orthogonal set in H_A into an orthogonal set in an arbitrary space H_C. This transformation, first of all, concretizes the structure of the mapping space as the space of the states of the bipartite object-meter system, $A+B$. Second, it uses in the general case an overfull set of states $|\alpha)$ for the representation of the set of the initial states. The isometric property is the condition for the physical realizability of the transformation in the form of dynamically reversible evolution in the bipartite object-meter system.

The index α in the transformation (22) accumulates in the classically measurable form information associated with a set of initial quantum states $|\alpha)_A$ of the object. The values of the index α are mutually uniquely mapped with the set of classically distinguishable states $|\alpha\rangle_{AB}$ of the bipartite system. This correspondence allows us to define the measurement transformation (22) as a sort of purely coherent measurement, which delivers the output information about the object in the form of entanglement of a linear combination of the states $|\alpha\rangle_{AB}$. On account of the decoherence effects, which are pronounced in the partial loss of coherence of the measurement results without loss of the classical information, such a measurement can be represented by the following superoperator [19]:

$$\mathcal{M}=\mathcal{D}(\mathcal{V}\odot\mathcal{V}^{+})=\sum_{\alpha\beta}R_{\alpha\beta}\sqrt{\nu_\alpha\nu_\beta}|\alpha\rangle_{AB}(\alpha|_A\odot|\beta)_A\langle\beta|_{AB}, \qquad (24)$$

where $\mathcal{D}=\Sigma_{\alpha\beta}R_{\alpha\beta}|\alpha\rangle_{AB}\langle\alpha|_{AB}\odot|\beta\rangle_{AB}\langle\beta|_{AB}$ is the decoherence superoperator and $R_{\alpha\beta}$ is an entangling matrix representing decoherence. At $R_{\alpha\beta}\equiv 1$—i.e., without decoherence—this superoperator simply describes the transformation $\mathcal{V}$ in terms of density matrix transformation.

In accordance with the two possibilities (2), representation (24) of the generalized measurement, as well as its purely coherent modification (22), include the following

(i) The standard projective and entangling measurements (1) at the choice of the mapped information in the form of a the complete set of classically compatible states, the orthogonal basis $|k\rangle_A$, and as $|\alpha\rangle_{AB}$—the duplicated basis $|k\rangle_A|k\rangle_B$.

(ii) The soft quantum measurement at the choice of $|\alpha\rangle_{AB}\rightarrow|k\rangle_A|k)_B$ with a nonorthogonal set $|k)_B$.

(iii) The measurement with partial destruction of the initial information at the choice $|\alpha\rangle_{AB}=|e_\alpha)_A|\alpha\rangle_B$, where the set of states $|e_\alpha)_A$ is arbitrary and $|\alpha\rangle_B$ is formed of orthogonal states and unambiguously maps the values of the information index α, whereas the set $(\alpha|_A$ can count in nonorthogonal states, as well.

The transformation (22) corresponding to the generalized quantum measurement takes the form

$$\mathcal{V}=\sum\sqrt{\nu_\alpha}|e_\alpha)_A|\alpha\rangle_B(\alpha|_A, \qquad (25)$$

where in the case of nonorthogonal set $(\alpha|_A$ the information index α is not unambiguously linked with the classically distinguishable states of A and its statistics includes the internal quantum uncertainty of the mapped states $|\alpha)_A$. It can be formally interpreted as the number of elementary coherent subchannels $|\alpha)_A\rightarrow|e_\alpha)_A|\alpha\rangle_B$, which link classically noncompatible input states of the object $|\alpha)_A$ with the states of the bipartite object-meter system $|e_\alpha)_A|\alpha\rangle_B$.

In the case of the soft quantum measurement, the orthogonality of the set $|e_\alpha)_A=|k\rangle_A$ leads to a unique correspondence to the informational index $\alpha=k$ and, respectively, to the nondemolition character of the measurement along the variables of the form $\hat{\lambda}=\Sigma\lambda_k|k\rangle_A\langle k|_A$, and to complete vanishing of the coherent information of the meter with respect to the initial state of the object.

Nonorthogonality of the set $|e_\alpha)_A$ leads, in its turn, to a reduction of the information remaining in the object—i.e., to the destructive measurement. During this measurement some coherent information is transferred into the meter states, the indicator α of which contains quantum uncertainty with respect to the object states $|\alpha)_A$ only if the latter have an internal quantum uncertainty despite being uniquely represented by the input states $|\alpha\rangle_B$ of the meter. In the limiting case $|e_\alpha)_A\equiv|0)_A$, the transformation (25) corresponds to the complete transmission of the initial information from A into B.

If one uses the orthogonal bases $|k\rangle_A$ for the sets $|e_\alpha)_A$, $|\alpha)_A$, the transformation (25) corresponds to the entirely coherent entangling measurement [11], which leads to the equitable probability distribution of the initial information between A and B and the complete absence of coherent information about the initial states in the subcomponents of the bipartite system $A+B$.

In the general case, the distribution of the initial information about the object among the object and meter is determined by the metric matrix $Q_{\alpha\beta}=(e_\alpha|e_\beta)_A$ of the vector set $|e_\alpha)_A$.

In the case of the overfull set $|e_\alpha)_A$, the representation of the operator (25) as a sum over α can be reduced into the superposition of D^2 projectors by shifting to the minimal orthogonal basis $|k\rangle_A$. The respective representation has the form

$$\mathcal{V}=\sum_{kl}|k\rangle_A|kl)_B\langle l|_A, \tag{26}$$

where $|kl)_B=\Sigma_\alpha\sqrt{\nu_\alpha}\langle\alpha|l\rangle_A\langle k|e_\alpha)|\alpha\rangle_B$ with the scalar product

$$(k'l'|kl)=\sum_\alpha\nu_\alpha(\alpha|l\rangle_A\langle k|e_\alpha)_A(e_\alpha|k'\rangle_A\langle l'|\alpha)_A,$$

which is determined only by the states in the Hilbert space of the object H_A. Therefore, representation (25) clarifies the transformation of the form (26) as setting the correspondence between the input and output via the classical information index, which surely contains the internal quantum uncertainty.

A. Relationship between the partially destructive quantum measurement transformation and its representation in the form of a positive operator valued measure

Let us now consider the superoperator (24), which applies the partially destructive isometric transformation (25) and additionally takes decoherence into account:

$$\mathcal{M}=\sum_{\alpha\beta}R_{\alpha\beta}\sqrt{\nu_\alpha\nu_\beta}|e_\alpha)_A|\alpha\rangle_B(\alpha|_A\odot|\beta)_A\langle\beta|_B(e_\beta|_A. \tag{27}$$

It corresponds to the probability distribution $P(\alpha)=\langle\alpha|_B\hat{\rho}_B|\alpha\rangle_B$, where $\hat{\rho}_B=\mathrm{Tr}_A\mathcal{M}\hat{\rho}_A$, for the results α of the measurement, which are physically realized in the form of meter quantum states. This distribution has the form

$$P(\alpha)=\mathrm{Tr}_A\hat{E}_\alpha\hat{\rho}_A, \tag{28}$$

with the positive operator valued measure (POVM) $\hat{E}_\alpha=\nu_\alpha|\alpha)_A(\alpha|_A$.

This expression does not depend either on the coherency of the transformation or on the form of its representation in the output state of the object and corresponding entanglement in the bipartite object-meter system after the measurement because it describes only classically compatible information of the object about its initial state. Equation (28) does not describe the quantum result of the measurement, but the resulting nonselected information preserved in the object in quasiclassical form.

The complete resulting information, though displayed in the classically distinguishable form $|\alpha\rangle_B$, is described by the contracted superoperator for the bipartite object-meter system:

$$\mathcal{M}_B=\sum_{\alpha\beta}R_{\alpha\beta}\sqrt{\nu_\alpha\nu_\beta}(e_\beta|e_\alpha)_A|\alpha\rangle_B(\beta|_A\odot|\beta)_A\langle\alpha|_B, \tag{29}$$

which takes into account quantum correlations with the initial state. Even for the completely coherent measurement, it contains the decoherence factor $R^A_{\alpha\beta}=(e_\beta|e_\alpha)_A$, which is due to ignoring the coherent information bundled in the form of entanglement in the bipartite object-meter system. In the case of complete decay of the output information at $R_{\alpha\beta}=\delta_{\alpha\beta}$ the transformation (29) is reduced to

$$\mathcal{M}_B=\sum_\alpha\nu_\alpha|\alpha\rangle_B(\alpha|_A\odot|\alpha)_A\langle\alpha|_B.$$

The respected probability distribution for this transformation is given by Eq. (28) on the algebra of classical events described by the set of compatible states $|\alpha\rangle_B$ and its subsets.

It is worth noting here that the generalized measurements in terms of the POVM have been widely discussed in the literature, particularly in connection with the problem of optimal measurement of continual quantum variables—e.g., coordinates and momenta [30–32]. However, the consideration presented here is qualitatively different, because in our terms a discussion of the measurements with a continuous output makes evident sense even in the case of a finite-dimensional object systems.

B. Selective and nonselective partially destructive quantum measurements

1. Selective partially destructive quantum measurements

A special case of the generalized quantum measurement (27) is the *selective* partially destructive quantum measurement, which has a different generalized set of output object states from the case of entangling measurement (1) but the same measured states:

$$|k\rangle_A\to|e_\alpha)_A=|k)_A,\quad (\alpha|_A=\langle k|_A,\quad |\alpha\rangle_B=|k\rangle_B,$$
$$k=1,\ldots,D,\quad \nu_\alpha=1.$$

The output object states $|k)_A$ differ by their nonorthogonality from the orthogonal basis states $|k\rangle_A$ of the measured variable, which prevents the object from preserving initial states of the form $|k\rangle_A$.

In the case of a purely coherent measurement $R_{kl}\equiv1$, the meter attains a nonzero coherent piece of information about the initial state of the object, which in the trivial limit $|k)_A\equiv|0\rangle_A$ is the complete information; i.e., the information equals numerically the initial entropy of the object. In this case, the information relationships for the object-meter mapping reproduce obviously the same relationships for the object-object mapping for the case of the soft quantum measurement, the transformation for which is described by the transformation $H_A\leftrightarrows H_B$ of the resulting states of the object and meter. Therefore, the respective dependences given already in the Sec. III D 1 for coherent information in the object-object channel for the two-level system retain their validity for the present case, as well. Quasiclassical information attained by the meter, due to the absolute accuracy of the

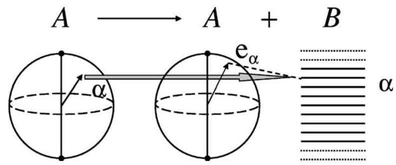

FIG. 4. Mapping of elementary states in the process of the nonselective partially destructive quantum measurement. New states $|e_\alpha)_A$ of the object are, generally, different from $|\alpha)_A$.

measurement, is always complete; i.e., the amount of information coincides with the entropy of the measured variable.

2. Nonselective partially destructive quantum measurements

The *nonselective* partially destructive quantum measurement is another special case of the generalized quantum measurements for which the set of mapped states $|\alpha)_A$ includes *all* quantum states of the object. In this case, the information index α unambiguously maps all physically different elements of the Hilbert space H_A and the appropriate representation of the set of its states is the unit $(2D-2)$-mensional sphere of the real Euclidean space.

Then, the respected generalized measurement is the map $H_A \to H_A \otimes H_B$ with the states of the meter $\psi_B(\alpha) \in H_B = L_2(H_A)$ the wave functions of the continual argument α. Multiplicity of the states, $d\nu = \Sigma_{dV}\nu_\alpha$, which corresponds to the elements in the set $\alpha \in dV$, has in this case the form $d\nu = DdV/V$, where V is the entire volume of the hypersphere of the physical states. Figure 4 illustrates the concept of the nonselective partially destructive quantum measurement in an example of a two-level system.

C. Distribution of information between the object and meter at the partially destructive quantum measurements

The amount of information preserved in the object is determined by the information capacity of the overfull basis $|e_\alpha)_A$, which duplicates information represented by the states of the meter $|\alpha\rangle_B$. In the general case, this information corresponds to a partial or complete loss of initial information $|\alpha)_A$ about the object. In the case of the completely coherent measurement—i.e., at $R_{\alpha\beta} \equiv 1$—the information capacity of the basis $|e_\alpha)_A$ for the pure input state $\hat{\rho}_A = |\psi\rangle_A\langle\psi|_A$ is determined by the entanglement $E[|\psi\rangle_{AB}]$ of the resulting state $|\psi\rangle_{AB} = \mathcal{V}|\psi\rangle_A$of the bipartite object-meter system. The meter in this case contains all *accessible* information [21] about the entire Hilbert space of the object states, which is represented in a quantum form including the entanglement with the object. This information is reduced to a classical form either after an additional projective measurement or after entirely decohering transformation $\mathcal{D}$, Eq. (24), at $R_{\alpha\beta} = \delta_{\alpha\beta}$, which are equivalent from an information point of view.

Let us illustrate the distribution of information among the object and meter in an example of a two-level system with $D=2$ using as $|e_\alpha)_A$ all states of a part of the Bloch sphere, which is formed by the mapping $\vartheta \to q\vartheta$, where $0 \leqq q \leqq 1$ is the compression coefficient of the initial Bloch sphere that is mapped onto its part corresponding to $0 \leqq \vartheta \leqq \pi q$. With this

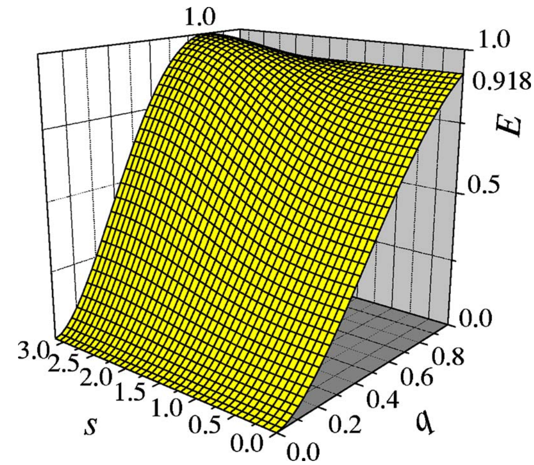

FIG. 5. (Color online) The degree of entanglement E (bits) versus the compression coefficient q of the Bloch sphere and angle $\vartheta = s$ of the initial state. The maximum value $E=1$ is achieved at $s=\pi$ and $q=0.7978$. At $q=1$ the degree of entanglement does not depend on s and is equal to the entropy $E_0=0.918$ for the pure state of the qubit after its complete depolarization.

choice of mapping, at $q<1$ there is some asymmetry with respect to the value of the polar angle s of the initial state $\alpha_0 = (s, \varphi_0)$. This asymmetry reaches its maximum at $q=0$ and vanishes at $q=1$.

The entanglement in the object-meter system that arises after the measurement can be written as the entropy $S[\hat{\rho}'_A] = -\mathrm{Tr}\hat{\rho}'_A \log_2 \hat{\rho}'_A$ of the partial density matrix $\hat{\rho}'_A = \mathrm{Tr}_B |\psi\rangle_{AB}\langle\psi|_{AB}$ of the transformed object state [33]. The corresponding dependence $E(s,q)$ is shown in Fig. 5.

The results of the analysis of the information distribution in the case of a two-level system for the completely nonselected representation of the final state of the object at $q=1$ and, respectively, $|e_\alpha)_A = |\alpha)_A$ are the obvious ones, even without calculations, because in this case $\hat{\rho}'_A$ corresponds to the entirely depolarized initial state [see, for example, Eq. (3.115) at $p=1$ in Ref. [34]]

$$\hat{\rho}'_A = (2/3)|\alpha_0\rangle\langle\alpha_0| + (1/3)|\overline{\alpha}_0\rangle\langle\overline{\alpha}_0|$$

($|\overline{\alpha}_0\rangle$ is orthogonal to $|\alpha_0\rangle$) and, independently from α_0, $E = E_0 = (2/3)\log_2(3/2) + (1/3)\log_2(3/1) = 0.918$ bit.

However, the result $E=1$ bit—i.e., the complete entanglement between the object and meter, which is achieved at the orientation of the initial state $s=\pi$, opposite to the Bloch sphere compression point $\vartheta=0$, and at the intermediate value of the compression coefficient—is not trivial and requires a qualitative elucidation.

We can do that easily because at the chosen orientation the problem is symmetrical with respect to the axis of the Bloch sphere; thus, the density matrix in the respective basis is a diagonal one and has the form $\hat{\rho}'_A = p_1|1\rangle_A\langle 1|_A + p_2|2\rangle_A\langle 2|_A$. Also, the direction $\vartheta=0$ is opposite to the direction of the initial state $s=\pi$ and, therefore, the probability p_1 in accordance with the given above equation for $q=1$ is simply $p_1=1/3$. If one changes the compression coefficient up to the value of $q=0$, which corresponds to the collapse of the

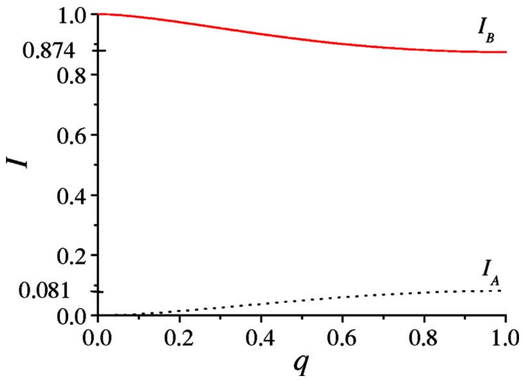

FIG. 6. (Color online) The measured amount of Holevo information (I_B, bit) and information preserved in the object (I_A) about the equally distributed ensemble of initial states $|\alpha)_A$ of the object (qubit) versus the degree q of the preserved information in the object. The maximum amount of information about the object I_A $=0.081$ bit corresponds to the minimum of measured information I_B=0.874 bit.

Bloch sphere onto the point $\vartheta=0$, the probability of the opposite state p_2 reduces up to zero and, respectively, the probability p_1 grows up to unity in accordance with the following analytical formula:

$$p_1 = 1 - p_2 = \frac{3 - 2q^2 + \cos \pi q}{4(1-q^2)} - \frac{1 - \cos \pi q}{4(4-q^2)}.$$

This probability, due to its continuity, passes the value of $p_1=1/2$, which corresponds to the maximum possible entanglement between two systems, one of which is the two-level system (qubit).

Note also that the degree of entanglement $E_0=0.918$ bit, achieved at exact reproduction by the object after the measurement of all states of the Hilbert space, is very close to the maximal entanglement $E=1$, which is achieved at the totally coherent nondemolition entangling measurement. However, the latter can be achieved only with the optimal choice of the initial wave function of the object. In the case of the completely nonselective measurement, the degree of entanglement is invariant with regard to the initial state $|\alpha_0)_A$ because all the states are due entirely equal.

D. Competition between the object and meter in the selection of nonselective partially destructive quantum information

In case of the nondemolition quantum measurement, there is no competition between the object and meter because classically compatible information retrieved at such measurement can be duplicated without bound. However, with the choice of nonselected information, which is connected with the nonorthogonal overfull set $|\alpha)_A$, as is typical, for instance, for the quantum key distribution protocols [35], competition arises. It is due to the impossibility of nondemolition duplication of the information about the nonorthogonal quantum states. Mathematically, such competition can be sufficiently treated with the Holevo information [29], which implicitly takes into account the quantum nature of the output information.

The corresponding Holevo information is defined for the semiclassical channel, which is characterized by the density matrix $\hat{\rho}(\alpha)$ depending on the continuous classical messages α at the input, as

$$I_h = S[\hat{\rho}] - \int P(d\alpha) S[\hat{\rho}(\alpha)], \quad \hat{\rho} = \int \hat{\rho}(\alpha) P(d\alpha), \quad (30)$$

where $P(d\alpha)$ defines the probability distribution or the frequencies of messages α. One can easily see that in the above-considered quantum measurement transformation the classical parameter α corresponds to the informational index of the initial states of the object $|\alpha)_A$ and two considered channels, object-object and object-meter, are described by the averaging of the pure state $\mathcal{V}|\alpha)_A(\alpha|_A\mathcal{V}^+$ of the combined object+meter system [or, in general, of the density matrix, which results after the incoherent transformation (24)] over the competing system. For the uniform distribution $P(d\alpha)$, the density matrices for the corresponding channels have the following form:

$$\hat{\rho}_A(\alpha) = \frac{D}{V} \int dV_\beta |(\beta|\alpha)_A|^2 |e_\beta)_A (e_\beta|_A,$$
$$\hat{\rho}_A = \frac{1}{V} \int dV_\beta |e_\beta)(e_\beta|, \quad (31)$$

$$\hat{\rho}_B(\alpha) = \sum_{\beta\beta'} \sqrt{\nu_\beta \nu_{\beta'}} (e_\beta|e_{\beta'})_A (\beta'|\alpha)_A (\alpha|\beta)_A |\beta'\rangle_B \langle\beta|_B, \quad (32)$$

$$\hat{\rho}_B = \frac{1}{D} \sum_{\beta\beta'} \sqrt{\nu_\beta \nu_{\beta'}} (e_\beta|e_{\beta'})_A (\beta'|\beta)_A |\beta'\rangle_B \langle\beta|_B \quad \to \quad (33)$$

$$\hat{\hat{\rho}}_B = \frac{1}{V} \int dV_\beta |\beta)_A |e_\beta^*)(e_\beta^*|(\beta|_A. \quad (34)$$

The latter equation is the isometric display of the continual density matrix of the meter into the discrete space $H_A \otimes H_A$, which realizes the active subspace of the states and which is used for the numerical calculations. The entropies of the density matrices $\hat{\rho}_A(\alpha)$ and $\hat{\rho}_B(\alpha)$ coincide with each other, so that there is no need to use the continual representation. The respective dependences for the information (30) of the meter and the object that are calculated with the help of Eqs. (31) and (34) are shown in Fig. 6. They demonstrate the relatively weak competition character, by contrast with the competition of the coherent information at the selective partially destructive quantum measurement, when preserving the entire information in the object corresponds to its complete absence in the meter [11].

V. CONCLUSIONS

In conclusion, we introduced the concept of generalized quantum measurement transformation of quantum states of the object onto quantum states of the bipartite object+meter system. It is based on mapping an orthogonal set of the object+meter states onto a classical information index that is additionally associated with an orthogonal—i.e., classically compatible and, hence, physically meaning—basis set either of the input object or the output meter states. This generalization accumulates in the natural way all of the most important classes of quantum measurements previously discussed in the literature: standard projective, entangling, soft, destructive, coherent, and partially incoherent measurements.

Two special classes of the generalized quantum measurements—the soft quantum measurements and the partially destructive quantum measurements—were defined and their information properties were analyzed in detail. For the soft quantum measurements, it was shown that they reveal the fundamental meaning of the quantum indistinguishability of the quantum states of the meter, which is used for storing the measurement results. It was also clarified how the quantum information about the object is obtained in the form of collective states of the multicomponent meter. For the partially destructive quantum measurements, our analysis shows that they reveal all specific features of quantum information transfer between two quantum systems (the object and meter) without the nondemolition condition and, additionally, the competitive nature of attaining the quantum information.

Finally, the concept of generalized quantum measurements developed in this paper helps to expose the most general relationships between the physical changes, caused by the transformations applied to a quantum system, and the classical information, contained in the information index that sets a one-to-one correspondence between the initial quantum states and those after the measurement. We also hope that this concept will be useful for further development of the foundations of quantum information theory, especially the part related to the conceptual setting and interpretation of experiments on engineering of quantum information.

ACKNOWLEDGMENTS

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APPENDIX A: REPEATED MEASUREMENTS OF AN EXAMPLE OF A TWO-LEVEL SYSTEM

Let us choose a representation in which $|0\rangle=(1,0)$, $|1\rangle=(0,1)$, and the fuzzy set of measured states has the form

$$|0) = (1,0), \quad |1) = e^{i\chi}\left(\cos\frac{\vartheta}{2}, e^{i\varphi}\sin\frac{\vartheta}{2}\right). \tag{A1}$$

Then, matrices of the scalar products

$$Q = \begin{pmatrix} 1 & e^{i\chi}\cos\frac{\vartheta}{2} \\ e^{-i\chi}\cos\frac{\vartheta}{2} & 1 \end{pmatrix},$$

$$Q^{(n)} = \begin{pmatrix} 1 & e^{in\chi}\left(\cos\frac{\vartheta}{2}\right)^n \\ e^{-in\chi}\left(\cos\frac{\vartheta}{2}\right)^n & 1 \end{pmatrix}$$

depend only on the angle ϑ between vectors $|1)$, $|2)$ and on their phase difference χ.

Applying relations (5) to the two-level case we have

$$\hat{\varepsilon}_B(0) = 0,$$

$$\hat{\varepsilon}_B(1) = \lim_{\Delta t\to 0}\frac{\hbar}{\Delta t}\begin{pmatrix} -2\sin\chi\cos\frac{\vartheta}{2} & ie^{i(\chi+\varphi)}\sin\frac{\vartheta}{2} \\ -ie^{-i(\chi+\varphi)}\sin\frac{\vartheta}{2} & 0 \end{pmatrix}$$

$$= \hbar\begin{pmatrix} -2\dot{\chi} & i\dot{\vartheta} \\ -i\dot{\vartheta} & 0 \end{pmatrix}, \tag{A2}$$

where $\dot{\chi}$ and $\dot{\vartheta}$ describe the rates of changes of the respective angles in the process of a single measurement. We do not take into account here the dependence on the second phase φ, which describes the freedom in the choice of the common phase for the states $|0)$, $|1)$.

For the matrix of wave functions of the ensemble under measurement we have, after n measurements

$$Q^{(n)1/2} = \frac{1}{2}\begin{pmatrix} \sqrt{1-\left(\cos\frac{\vartheta}{2}\right)^n} + \sqrt{1+\left(\cos\frac{\vartheta}{2}\right)^n} & e^{in\chi}\left[\sqrt{1+\left(\cos\frac{\vartheta}{2}\right)^n} - \sqrt{1-\left(\cos\frac{\vartheta}{2}\right)^n}\right] \\ e^{-in\chi}\left[\sqrt{1+\left(\cos\frac{\vartheta}{2}\right)^n} - \sqrt{1-\left(\cos\frac{\vartheta}{2}\right)^n}\right] & \sqrt{1-\left(\cos\frac{\vartheta}{2}\right)^n} + \sqrt{1+\left(\cos\frac{\vartheta}{2}\right)^n} \end{pmatrix}. \tag{A3}$$

The columns of this matrix play, in accordance with Eq. (14), the role of the wave functions describing n-fold excitations of the meter in the minimal basis of the two-dimensional $(D=2)$ space of the collective states.

Let us consider a sequence of n identical measurements, each performed with time period T with a small variation ϑ. We will also assume that relationships for the parameters, necessary for asymptotically continuous changes of the result of the n-fold measurement as a function of the continuous time t at $n \propto t$ are fulfilled. The respective continuous dynamics has a quantum diffusion character, which can be seen at short times. In addition to the usual classical diffusion quadratic diffusional change of state, a specific linear diffusional change in the nondiagonal matrix elements takes place, as well. The fluctuation character of the dynamics, which is typical for a description of the classical diffusion process with the use of stochastic equations, becomes apparent while considering the sequences of the respective classically compatible variables in the space H_B, which are in charge for the separate measurements in the whole measurement sequence. Their statistical description cannot be reduced to a reversible dynamics of the collective variables of the meter.

Matrix (A3) at $\vartheta^2 = 4\varkappa T \to 0$, $n = t/T \to \infty$, $n\vartheta^2 = 4\varkappa t = \text{const}$, and $\chi = \dot{\chi} T$ has a finite limit corresponding to the diffusion dynamics:

$$Q^{1/2}(t) = \begin{pmatrix} s_+(t) & e^{i\dot{\chi}t} s_-(t) \\ e^{-i\dot{\chi}t} s_-(t) & s_+(t) \end{pmatrix}, \tag{A4}$$

where $s_+(t) = \frac{1}{2}(\sqrt{1+e^{-\varkappa t}} + \sqrt{1-e^{-\varkappa t}})$ and $s_-(t) = \frac{1}{2}(\sqrt{1+e^{-\varkappa t}} - \sqrt{1-e^{-\varkappa t}})$.

A similar limit has the entangling matrix

$$R^{(n)} = (R_{ij}^n) = \begin{pmatrix} 1 & r^n \\ r^{*n} & 1 \end{pmatrix} \to \begin{pmatrix} 1 & e^{-\dot{r}t} \\ e^{-\dot{r}^* t} & 1 \end{pmatrix}, \quad \dot{r} = \lim_{T \to 0} \frac{1-r}{T}.$$

This matrix describes the decoherence of the measurement result due to the interaction of the meter with an external reservoir. Simultaneously, at long times, asymptotic diagonalization of the collective states of the meter occurs in accordance with the following asymptotic expression:

$$Q^{1/2}(t) \overset{t\to\infty}{\Rightarrow} \begin{pmatrix} 1 - \frac{1}{8} e^{-2\varkappa t} & \frac{1}{2} e^{-\varkappa t + i\dot{\chi}t} \\ \frac{1}{2} e^{-\varkappa t - i\dot{\chi}t} & 1 - \frac{1}{8} e^{-2\varkappa t} \end{pmatrix}.$$

This matrix describes how the soft quantum measurement transforms into the distinct completely coherent measurement ("premeasurement") with the orthogonal set of collective states of the meter $\|k\rangle\rangle \to |k\rangle$.

1. Partial density matrix of the meter

From Eqs. (8) and (A4) we obtain the partial density matrix of the meter in the process of continuous measurement in the form

$$\hat{\rho}^{B^n} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}\sqrt{1 - e^{-\varkappa t}}(\rho_{11}^A - \rho_{22}^A) & \frac{1}{2} e^{-\varkappa t + i\dot{\chi}t} \\ \frac{1}{2} e^{-\varkappa t - i\dot{\chi}t} & \frac{1}{2} - \frac{1}{2}\sqrt{1 - e^{-\varkappa t}}(\rho_{11}^A - \rho_{22}^A) \end{pmatrix}. \tag{A5}$$

At $t=0$, when no measurement is performed, for any initial state the resulting state has only one and the same value $|2) = |1) = |0\rangle$; the density matrix of the meter does not depend on $\hat{\rho}^A$ and is equal to the projector $|0\rangle\langle 0|$ onto $|0\rangle$, which in the given basis (11) for $n=1$ has the form $|0\rangle = (1/\sqrt{2}, 1/\sqrt{2})$. The choice of the basis states of the meter, which determines in the structure of the measurement superoperator (4) how the measured information is represented, is arbitrary.

At $t=\infty$—i.e., in the limit of infinitive series of the limiting continuous soft quantum measurements—matrix (A5) becomes a diagonal one with the matrix elements ρ_{11}^A and ρ_{22}^A, coinciding with the respective matrix elements of the measured object. This coincidence holds not only for the limiting continuous measurements, but also for the series of measurements with finite accuracy about the object states $|k\rangle_A$, which is determined with the tensor product of n states $|k) \cdots |k)$ in the form of multiparticle excitations of the n-fold copy of the meter system, the exact information in the limit $n \to \infty$ is retrievable.

This leads to diagonalization of the marginal density matrix of the meter, which is constructed as a bilinear combination of the duplicated states of the object-meter system $|k\rangle_A |k) \cdots |k)$ and to its coincidence with the density matrix of the object. The described coincidence of the representations $\hat{\rho}^{B^n} \to \hat{\rho}^A$ is due to the choice (11) of the basis linked to the object variable k. The entropy of the quantum state of the meter in the process of continuous measurement changes from zero up to the entropy of the object. At the same time, the zero internal entropy of the initial state of the meter is due to its preparation in a known pure state.

2. Joint density matrix of the object-meter system

The orthogonal set $|k\rangle_A|k)\cdots|k)$ of the object-meter states in the basis $|l\rangle_A||e_m\rangle\rangle$, in accordance with Eqs. (7), (12), and (A4), where ψ^{B^nA} are supplemented with the basis object states, has the form

$$\psi^{B^nA} = \begin{cases} s_+(t)||e_1\rangle\rangle|1\rangle + e^{i\dot{\chi}t}s_-(t)||e_2\rangle\rangle|1\rangle, \\ e^{-i\dot{\chi}t}s_-(t)||e_1\rangle\rangle|2\rangle + s_+(t)||e_2\rangle\rangle|2\rangle. \end{cases} \quad \text{(A6)}$$

Using this set and in accordance with Eqs. (7) and (12) the density matrix of the object-meter system can be written in the form of a 4×4 matrix

$$\hat{\rho}^{B^nA}(t) = \begin{pmatrix} \rho_{11}^A \begin{pmatrix} s_+^2 & s_+s_-e^{i\dot{\chi}t} \\ s_+s_-e^{-i\dot{\chi}t} & s_-^2 \end{pmatrix} & \rho_{12}^A e^{-\dot{r}t} \begin{pmatrix} s_+s_-e^{-i\dot{\chi}t} & s_+^2 \\ s_-^2e^{-2i\dot{\chi}t} & s_+s_-e^{-i\dot{\chi}t} \end{pmatrix} \\ \rho_{21}^A e^{-\dot{r}^*t} \begin{pmatrix} s_+s_-e^{i\dot{\chi}t} & s_-^2e^{2i\dot{\chi}t} \\ s_+^2 & s_+s_-e^{i\dot{\chi}t} \end{pmatrix} & \rho_{22}^A \begin{pmatrix} s_-^2 & s_+s_-e^{i\dot{\chi}t} \\ s_+s_-e^{-i\dot{\chi}t} & s_+^2 \end{pmatrix} \end{pmatrix}. \quad \text{(A7)}$$

APPENDIX B: CALCULATION OF THE ENTROPY OF THE OBJECT SYSTEM

First, we represent the joint density matrix $\hat{\hat{\rho}}^{AR} = (R_{kl}Q_{kl}\Psi_{ki}\Psi_{lj}^*)$ of the object-reference system in the form

$$\hat{\hat{\rho}}^{AR} = (\tilde{\rho}_{kl}\psi_i^{(k)}\psi_j^{(l)*}),$$

where $\tilde{\rho}_{kl} = \sqrt{\rho_{kk}^A}R_{kl}Q_{kl}\sqrt{\rho_{ll}^A}$ is the matrix argument of the entropy of the object system in Eq. (16) and $\psi_i^{(k)}$ are the normalized to unit k-dependent pure states of the reference system corresponding to the joint states Ψ_{ki}. The algebra of scalar functions of the matrix $\hat{\hat{\rho}}^{AR}$ is governed by the rule $(\hat{\hat{\rho}}^{AR})_{ki,lj}^n = (\hat{\tilde{\rho}}^n)_{kl}\psi_i^{(k)}\psi_j^{(l)*}$, which can be verified by induction, paying attention to the rule of combining the cofactors of the matrix product (i.e., the coincidence of the corresponding matrix indices of the first and second matrices).

Then, for the trace calculation of the n th power we get in abridged form $\mathrm{Tr}_{AB}(\hat{\hat{\rho}}^{AR})^n = \mathrm{Tr}_A\hat{\tilde{\rho}}^n$. Hence, the same rule holds for any scalar function $h(\hat{\hat{\rho}}^{AR})$, a particular example of which is the function $h(x) = -x\log(x)$ of the entropy functional $S[\hat{\rho}] = \mathrm{Tr}\,h(\hat{\rho})$ in Eq. (16). An isometric equivalence of the orthogonal set of the eigenstates $e_k^{(n)}$ of the object density matrix $\hat{\tilde{\rho}}$ and the object-reference system states $e_k^{(n)}\psi_i^{(k)}$forms the ground for the proved relation.

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