Bell-type inequalities embedded in the subgraph of graph states

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We investigate the Bell-type inequalities of graph states. In this paper, Bell-type inequalities can be derived based on two kinds of the associated subgraphs of the graph states. First, the star subgraphs lead to the maximal violation of the modified Seevinck-Svetlichny inequalities. Second, cycle subgraphs lead to maximal violation of Bell-type inequalities. As a result, once the associated graph of a graph state is given, the corresponding Bell operators can be immediatedly determined using stabilizing generators. In the above Bell-type inequalities, two measurement settings for each party are required.

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I. INTRODUCTION

In their celebrated paper, Einstein, Podolsky, and Rosen argue that quantum theory is incomplete [1]. Their premises have quite plausible propositions on locality, reality, and completeness [2]. Realism and locality constrain the correlation in any local hidden-variable model. According to the local realism hypothesis, measurement outcomes are predeterministic, and the measurement on one party of a multipartite system does not affect the other parties. However, quantum theory predicts that there are stronger correlations than the correlations of local hidden variables because of the fact that quantum theory is a nonlocal theory. The so-called Bell-type inequalities are exploited to distinguish the differences between classical correlation and quantum correlation. In general, the Bell-type inequalities put the upper bound on correlations based on local hidden-variable models. Certain quantum states can violate these inequalities. On the other hand, there are many recent alternative proofs of Bell's theorem without inequalities [3–5].

In this paper, we investigate the nonlocal behaviors of graph states. Graph states are defined as follows. Let G be a graph with a set of n vertices and some edges connecting them. For each vertex i the neighborhood $N(i)$ denotes the vertices that are connected to vertex i. In addition, each vertex i is associated with a stabilizing operator g_i,

$$g_i = X_i \prod_{j\in N(i)} Z_j. \tag{1}$$

Here X_i, Y_i, and Z_i denote the Pauli matrices σ_x, σ_y, and σ_z, respectively, acting on the ith qubit. The corresponding n-qubit state $|G\rangle$ of the graph G satisfies

$$g_i|G\rangle = |G\rangle \quad \forall \; i \in \{1,\ldots,n\}. \tag{2}$$

It is easy to verify that $g_i^2=\mathbf{I}$, where $\mathbf{I}$ is the identity operator. Moreover, $[g_i,g_j]=0$, where $[\cdots,\cdots]$ is a commutator. In addition, $|G\rangle\langle G|$ can be written as

$$\prod_{i=1}^{n} \frac{(\mathbf{I}+g_i)}{2}. \tag{3}$$

In this paper, we consider the connected graphs, i.e., graphs that do not decay into two unconnected subgraphs. Graph states are important in quantum-information science. For example, the fully connected n-qubit graph states correspond to n-qubit Greenberger-Horne-Zeilinger (GHZ) states. Special instances of graph states are codewords for various quantum error-correcting codes [6]. The linear cluster states are associated with a graph where all neighboring vertices are connected by their edges. It is demonstrated that cluster states can constitute a universal resource for quantum computation only when assisted by local measurements [7]. Scarani *et al.* have investigated the nonlocality of cluster states [8]. In addition, some researchers discuss the nonlocality of graph states. Plesch and Bužek considered the bipartite entanglement in graph states [9]. Gühne *et al.* derived a family of Bell inequalities [10]. Hein *et al.* characterized and quantified the multipartite entanglement in terms of the Schmidt measure of graph states [11,12].

Before proceeding further, we consider the simplest graph state $|G_2\rangle$ associated with only two interconnected vertices. The two stabilizing operators g_1 and g_2 for $|G_2\rangle$ are X_1Z_2 and Z_1X_2, respectively. Then g_1+g_2 is equal to $X_1Z_2+Z_1X_2$ and can be written as $\frac{1}{\sqrt{2}}\mathcal{B}_2$, where

$$\mathcal{B}_2 = AB + AB' + A'B - A'B', \tag{4}$$

and $A=X_1$, $A'=X_1$, $B=\frac{Z_2+X_2}{\sqrt{2}}$, $B'=\frac{Z_2-X_2}{\sqrt{2}}$. Here $\mathcal{B}_2$ is the Bell operator of the Clauser-Horne-Shimony-Holt (CHSH) inequality. For local hidden variable models, $|\langle\mathcal{B}_2\rangle| \leq 2$. However, $\langle G_2|\mathcal{B}_2|G_2\rangle = \langle G_2|\sqrt{2}(g_1+g_2)|G_2\rangle = 2\sqrt{2}$. That is, $|G_2\rangle$ can maximally violate the CHSH inequality. Another example is the graph state $|G_3^s\rangle$ with the associated three-vertex graph. Therein, vertices 2 and 3 are connected with vertex 1 but disconnected from each other. The corresponding stabilizing operators are $g_1=X_1Z_2Z_3$, $g_2=Z_1X_2$, and $g_3=Z_1X_3$. Consequently, $g_1g_2=Y_1Y_2Z_3$, $g_1g_3=Y_1Z_2Y_3$, and $g_1g_2g_3=-X_1Y_2Y_3$. On the other hand, the Bell-Klyshko operator for three qubits, denoted by $\mathcal{B}_3$, is

$$\mathcal{B}_3 = AB'C' + A'BC' + A'B'C - ABC. \tag{5}$$

The Bell-Klyshko inequality for the three parties reads [13–16]

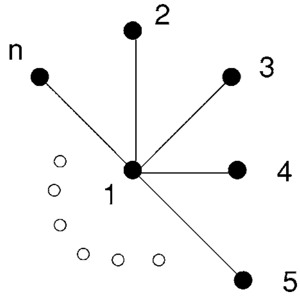

FIG. 1. Star graph.

$$|\langle \mathcal{B}_3 \rangle| \leqslant 2. \tag{6}$$

Equation (6) is based on the GHZ or "all versus nothing" argument [3]. Obviously,

$$g_1 + g_1 g_2 + g_1 g_3 + g_1 g_2 g_3 \tag{7}$$

can be recognized as $\mathcal{B}_3$, where $A = X_1$, $A' = Y_1$, $B = Y_2$, $B' = Z_2$, $C = Y_3$, and $C' = Z_3$. $\mathcal{B}_3$ can then reach 4 when evaluated on $|G_3^s\rangle$. Note that here we ignore the operators acting on the jth qubits ($j > 3$) in g_1, g_2, and g_3. This is because the operator acting on the jth ($j > 3$) qubit in g_i ($i = 1,2,3$) is either Z_j or $\mathbf{I}_j$, which does not affect the Bell-Klyshko inequality [8]. As an example, let vertex 4 be connected to vertex 3, but disconnected from vertices 1 and 2. In this case, qubits 3 and 4 can be grouped. As a result, the Bell-Klyshko operator $\mathcal{B}_3$ becomes $g_3 + g_3 g_1 + g_3 g_4 + g_1 g_3 g_4$, where $A = X_1$, $A' = Y_1$, $B = Y_2$, $B' = Z_2$, $C = Y_3 Z_4$, and $C' = Z_3 \mathbf{I}_4$ [8], and can reach 4. More details have been discussed in [8]. On the other hand, since vertex 2 is not connected to qubits 3 and 4, we cannot construct a Bell-Klyshko operator $\mathcal{B}_3$ based on g_2, g_3, and g_4. In the above examples, the Bell-type inequalities can be constructed as the linear combination of stabilizer operators.

In this paper, we address the following problem. If the associated graph and the corresponding stabilizing operators are given, can we immediately derived the linear combination of the stabilizer operators as Bell operators? Or can we derive nonlocal properties of the graph states by observing the associated graphs? The main results are twofold. First, in Sec. II, we study the star subgraph in Fig. 1, where the vertex 1 is connected with vertices $2, \ldots, n$. In the following discussion, the graph states associated with such n-qubit star subgraph are denoted by $|G_n^s\rangle$. We show that a given graph state $|G\rangle$ associated with a star subgraph can maximally violate some particular Bell-type inequalities. Second, in Sec. III, we consider the graph states associated with odd-vertex cycle subgraphs, where such states are denoted by $|G_c^n\rangle$, as shown in Fig. 2. It is shown that the nonlocality embedded in odd-vertex cycle subgraphs can be obtained using the all versus nothing argument [3,8,12]. In addition, some Bell-type inequalities are derived, which are also linear combination of stabilizer operators. Finally, some conclusions are made in Sec. IV.

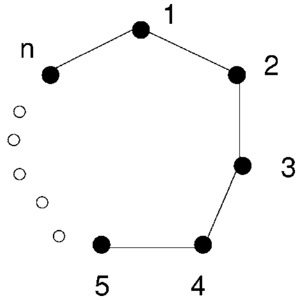

FIG. 2. Cycle graph.

II. VIOLATION OF BELL-TYPE INEQUALITIES EMBEDDED IN THE STAR SUBGRAPHS

In this section, we consider the violation of Bell-type inequalities embedded in the star subgraphs of the graph states. To proceed further, we introduce two alternative dichotomous observables $A_1^{(i)}$ and $A_2^{(i)}$ for each of the particles. In addition, we denote $B_3^+ = A_1^{(1)} A_2^{(2)} A_2^{(3)} + A_2^{(1)} A_1^{(2)} A_2^{(3)} + A_2^{(1)} A_2^{(2)} A_1^{(3)} - A_1^{(1)} A_1^{(2)} A_1^{(3)}$ and $B_3^- = A_2^{(1)} A_2^{(2)} A_2^{(3)} - A_1^{(1)} A_1^{(2)} A_2^{(3)} - A_1^{(1)} A_2^{(2)} A_1^{(3)} - A_2^{(1)} A_1^{(2)} A_1^{(3)}$. Here B_3^- is derived by replacing $A_1^{(1)}$ and $A_2^{(1)}$ in B_3^+ with $A_2^{(1)}$and $-A_1^{(1)}$, respectively. As a result, B_3^- is an alternative Bell-Klyshko operator $\mathcal{B}_3$ in Eq. (5), where $A = A_2^{(1)}$, $A' = A_1^{(1)}$, $B = A_1^{(2)}$, $B' = A_2^{(2)}$, $C = A_1^{(3)}$, and $C' = A_2^{(3)}$. Therefore, for any local hidden-variable model,

$$|\langle B_3^{\pm} \rangle| \leqslant 2. \tag{8}$$

Now we introduce the recursive relations of the Bell operators $B_n^{\pm}$

$$B_n^{\pm} = B_{n-1}^{\pm} A_2^{(n)} \pm B_{n-1}^{\mp} A_1^{(n)}, \quad n > 3. \tag{9}$$

Equation (9) is equivalent to the inequalities proposed by Seevinck and Svetlichny (SS inequalities) [17]. Here we take the $n = 4$ case as our example. Since

$$|\langle A_j^{(i)} \rangle| \leqslant 1 \tag{10}$$

for any i and $j \in \{1,2\}$, for any local hidden-variable model,

$$|\langle B_4^+ \rangle| \leqslant |\langle B_3^+ A_2^{(4)} \rangle| + |\langle B_3^- A_1^{(4)} \rangle| \leqslant |\langle B_3^+ \rangle| + |\langle B_3^- \rangle| \leqslant 4. \tag{11}$$

Therefore, for any local hidden variable model,

$$|\langle B_n^{\pm} \rangle| \leqslant 2^{n-2}. \tag{12}$$

The proof of Eq. (12) is straightforward. According to Eqs. (9) and (10), it is easy to verify that

$$|\langle B_n^{\pm} \rangle| \leqslant |\langle B_{n-1}^+ \rangle| + |\langle B_{n-1}^- \rangle|. \tag{13}$$

Recursively,

$$|\langle B_n^{\pm} \rangle| \leqslant 2(|\langle B_{n-2}^+ \rangle| + |\langle B_{n-2}^- \rangle|) \cdots \leqslant 2^{n-4}(|\langle B_3^+ \rangle| + |\langle B_3^- \rangle|) \leqslant 2^{n-2}. \tag{14}$$

It should be noted that the upper bounds in Eq. (14) are different from the original SS inequalities that were obtained in [17]. This is because, in the original SS inequalities, the recursive initial inequality is the CHSH inequality for the bipartite system, rather than the Bell-Klyshko inequality for the tripartite system.

In fact, it is the GHZ states that can maximally violate the original SS inequalities. It should be noted that graph states associated with the star graphs are equivalent to the GHZ states under local unitaries [11]. In fact, Seevinck and Svetlichny have verified that fully connected graph states can maximally violate the original SS inequalities [17]. However, the length (the number of operator terms) of the proposed n-qubit Bell operators in Eq. (9) is only half of that of the corresponding Bell operators in the original SS inequalities. Consequently, it is more efficient, in physical realization, to test the proposed SS inequalities rather than the original SS inequalities.

Now we consider the graph states associated with the n-vertex ($n>3$) star subgraph. In the following, $A_1^{(1)}=X_1$, $A_2^{(1)}=Y_1$, and the two alternative dichotomous observables $A_1^{(i)}$ and $A_2^{(i)}$ for qubit i ($i>2$) in Eq. (9) are set as Y_i and Z_i, respectively. For instance, in the $n=4$ case, $g_1+g_1g_2+g_1g_3+g_1g_2g_3=B_3^+A_2^{(4)}$ and $(g_1+g_1g_2+g_1g_3+g_1g_2g_3)g_4=B_3^-A_1^{(4)}$. Therefore, $\langle G_4^s|B_4^+|G_4^s\rangle=8$. We ignore the qubits that do not belong to $N(1)$. Using straight calculation, for any $|G_n^s\rangle$ associated with the star subgraph as shown in Fig. 1,

$$B_{n-1}^+Z_n = g_1 \prod_{i\in N(1), i\notin n} (\mathbf{I}+g_i)$$

and

$$B_{n-1}^-Y_n = g_1g_n \prod_{i\in N(1), i\notin n} (\mathbf{I}+g_i).$$

As a result,

$$\langle G_n^s|B_n^+|G_n^s\rangle = 2^{n-1}. \tag{15}$$

In this case, B_n^+ reaches the maximal value when evaluated on $|G_n^s\rangle$. In addition, the Bell operator B_n^+ can be expressed in terms of the stabilizing operators.

III. VIOLATION OF BELL-TYPE INEQUALITIES EMBEDDED IN THE CYCLE SUBGRAPHS

In this section, we consider the graph states associated with odd-vertex cycle subgraphs. As for $|G_c^3\rangle$, the corresponding stabilizing operators g_1, g_2, and g_3 are $X_1Z_2Z_3$, $Z_1X_2Z_3$, and $Z_1Z_2X_3$, respectively. Then $g_1g_2g_3=-X_1X_2X_3$. Again, we ignore the operators acting on the jth ($j>3$) qubit in g_1, g_2, and g_3. It is easy to verify that

$$g_1+g_2+g_3+g_1g_2g_3 \tag{16}$$

is another Bell-Klyshko operator in Eq. (5), where $A=X_1$, $A'=Z_1$, $B=X_2$, $B'=Z_2$, $C=X_3$, and $C'=Z_3$. It is worth noting that Bell-Klyshko operators in Eqs. (7) and (16) are built in different stabilizing operators. Now we consider the graph states with n-vertex ($n>3$) cycle subgraphs. Note that any n-vertex cycle subgraph can comprise n three-vertex star subgraphs. That is, $|G_c^n\rangle$ contains n Bell-Klyshko inequalities in Eq. (5) where the ith Bell-Klyshko operator corresponds to

$$g_i+g_ig_j+g_ig_k+g_ig_jg_k, \tag{17}$$

where $i\in\{1,\ldots,n\}$, j, $k\in N(i)$, and $j\neq k$. In addition, the GHZ-type or all versus nothing argument has been verified in considering the nonlocality of the cluster state [8]. Furthermore, the all versus nothing argument can be exploited for the odd-vertex cycle subgraph. Now consider the graph state $|G_c^n\rangle$ with n being odd. In this case, $g_k=Z_jX_kZ_{j'}$, where $j\neq j'$ and $j,j'\in N(k)$. In addition,

$$g_k|G_c^n\rangle=|G_c^n\rangle, \quad k=1,\ldots,n. \tag{18}$$

With straight algebra, we have

$$\prod_{k=1}^n g_k = -\prod_{k=1}^n X_k. \tag{19}$$

That is,

$$\prod_{k=1}^n X_k|G_c^n\rangle = -|G_c^n\rangle. \tag{20}$$

According to the local hidden-variable theory, since Eqs. (19) and (20) contain only local operations, classical communication is allowed, and the values of the elements of reality can be assigned as follows:

$$v(X_1)v(Z_2)v(Z_n)=1, \tag{21}$$

$$v(Z_{i-1})v(X_i)v(Z_{i+1})=1, \quad i\in\{2,\ldots,n-1\}, \tag{22}$$

$$v(Z_1)v(Z_{n-1})v(X_n)=1, \tag{23}$$

$$\prod_{k=1}^n v(X_k)=-1. \tag{24}$$

However, it is impossible to assign values, either 1 or −1, that satisfy Eqs. (21)–(24). This is because when we take the product of Eqs. (21)–(24), each of $v(Z_i)$ and $v(X_i)$ for any i appears twice in the left-hand side, while the right hand side is −1. Therefore, any local hidden-variable model cannot reproduce the prediction of the quantum theory for the graph states $|G_c^n\rangle$.

However, there are other Bell-type inequalities if n is odd with Bell operator $\mathcal{B}_n$,

$$\mathcal{B}_n = A_1A_2'A_n' + A_{n-1}'A_nA_1' + \sum_{i=1}^n A_{i-1}'A_iA_{i+1}' - \prod_{i=1}^n A_i. \tag{25}$$

Now A_i and A_i' are set as X_i and Z_i, respectively. In this case, $\mathcal{B}_n$ can be written as

$$\sum_{i=1}^n g_i + \prod_{i=1}^n g_i, \tag{26}$$

where n stabilizing operators are involved in Eq. (26). In Eq. (17), three stabilizing operators are involved. As a result,

$$\langle G_c^n|\mathcal{B}_n|G_c^n\rangle = \langle G_c^n|\sum_{i=1}^{n} g_i + \prod_{i=1}^{n} g_i|G_c^n\rangle = n+1. \qquad (27)$$

$\mathcal{B}_n$ in Eq. (26) can reach $n+1$ when evaluated on $|G_c^n\rangle$. On the other hand, for any possible assignment of each $v(Z_i)$ and $v(X_i)$ of the elements of reality in the local hidden-variable theory,

$$|\langle\mathcal{B}_n\rangle| \leq n-1. \qquad (28)$$

Therefore, the local hidden-variable theory contradicts the prediction of the quantum theory given by Eq. (27).

IV. CONCLUSION

In conclusion, this paper studies the nonlocality of graph states that is embedded in subgraphs. Therefore, once the associated graph G of the graph state $|G\rangle$ is given, the corresponding stabilizing operators are immediately known. We can analyze their nonlocality by decomposing G into star and cycle subgraphs. As for graph states associated with an n-vertex star subgraph, the modified SS inequalities, where the corresponding n-qubit Bell operator is B_n^+, are proposed to describe their nonlocality. As for graph states associated with an n-vertex cycle subgraph, the all versus nothing argument is exploited. In addition, the corresponding n-qubit Bell operator is $\mathcal{B}_n$. B_n^+ or $\mathcal{B}_n$ can be maximally violated when evaluated on $|G_s^n\rangle$ or $|G_c^n\rangle$, respectively, in which the corresponding operators can be expressed in terms of stabilizing operators and their products. It is worth noting that each of the Bell operators B_n^+ or $\mathcal{B}_n$ requires only two measurement settings for each party. Recently, Gühne *et al.* derived a family of Bell inequalities for graph states. However, three measurement settings are required for each party [10].

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