Sign and other aspects of semiclassical Casimir energies

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The Casimir energy of a massless scalar field is semiclassically given by contributions due to classical periodic rays. The required subtractions in the spectral density are determined explicitly. The semiclassical Casimir energies so defined coincide with those of zeta function regularization in the cases studied. Poles in the analytic continuation of zeta function regularization are related to nonuniversal subtractions in the spectral density. The sign of the Casimir energy of a scalar field on a smooth manifold is estimated by the sign of the contribution due to the shortest periodic rays only. Demanding continuity of the Casimir energy under small deformations of the manifold, the method is extended to integrable systems. The Casimir energy of a massless scalar field on a manifold with boundaries includes contributions due to periodic rays that lie entirely within the boundaries. These contributions in general depend on the boundary conditions. Although the Casimir energy due to a massless scalar field may be sensitive to the physical dimensions of manifolds with boundary. In favorable cases its sign can, contrary to conventional wisdom, be inferred without calculation of the Casimir energy.

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I. INTRODUCTION

Classically, the energy of a field is always positive. That need not be so for a Casimir energy $\mathcal{E}$, generally thought to have its origin in the vacuum fluctuations of the field [1]. The possibility of $\mathcal{E}$ being negative can be understood from the fact that it is the *difference* in the infinite zero-point energies of the field for two systems. For parallel plates, for example, one considers the energies in the presence and absence of the walls. Most calculations in the past were performed by directly evaluating this difference of infinite zero-point energies. There are many articles and texts that consider Casimir effects in this manner. See, for instance Refs. [2–4] for an overview.

The ultraviolet divergence of the zero-point energy in general reflects local properties of the system. From a path-integral point of view the divergence is due to contributions from arbitrarily short paths that begin and end at the same point. These zero-length paths probe the local radii of curvature or, for paths touching boundaries, local properties of the boundary [5–10]. In a few favorable situations, ultrashort paths do not contribute to the *difference* of zero-point energies. This in particular is the case for rigid disjoint boundaries that are moved relative to each other [11]. One sometimes also considers idealized boundaries whose local deformation does not cost energy. An example of the latter are smooth, perfectly conducting metallic surfaces of vanishing thickness [12] in three dimensions. The (local) surface tension of such an ideal surface vanishes [13,14] and no energy is required to deform it locally.

To be measurable, a Casimir energy should not be extremely sensitive to the (sometimes implicit) ultraviolet cutoff and should depend on global characteristics of the system only. It otherwise is difficult to disentangle the energy required to change the system as a whole from purely local effects, for instance, due to changes in the local curvature of the space or in the reflectance of a boundary.

There thus either is no contribution from ultrashort paths to a measurable Casimir effect or it must be possible to un-ambiguously isolate these local contributions to the vacuum energy. Most calculations of the Casimir energy based on spectral properties and Green function methods do not separate length scales explicitly. The regularization and subsequent subtraction of divergent contributions often are difficult to motivate physically and it is not always apparent how Casimir energies of different systems can be compared.

The semiclassical evaluation of Casimir energies advocated in Ref. [15] relies on an *ab initio* separation of scales. The approach separates the semiclassical contribution to the Casimir energy due to quadratic fluctuations about *classical* periodic rays (paths) from all others. Classical periodic rays are of (finite) extremal length and give a semiclassical approximation to the Casimir energy that depends on global characteristics of the system only. This part of the vacuum energy is naturally finite and does not include ultraviolet contributions from length scales that are much smaller than the shortest classical periodic ray. This is one of the principal conceptual differences to the "optical" approximation to Casimir energies [16]. The latter (in principle) takes all *closed* classical paths (not just periodic ones) into account.[1] Closed paths can be ultrashort in the vicinity of surfaces and lead to divergent Casimir self-energies. The optical approximation therefore has mainly been used to obtain a numerical estimate of the interaction energy between rigid bodies.

In Ref. [15] it was argued that a semiclassical evaluation of the Casimir energy is often particularly simple and gives

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[1]Although this appears to be an improvement over the semiclassical treatment, the optical approximation to Casimir energies also only includes quadratic fluctuations about classical rays. The optical approach in principle could provide a more uniform approximation in some cases (but not in all). It, in general, is no more accurate than the semiclassical approach and furthermore difficult to implement for chaotic systems. An objective comparison of the two methods [17,18] is complicated by the numerical limitations and approximations of this approach.

the leading asymptotic behavior when the Casimir energy is large. This is the experimentally most accessible region of parameter space [19].

However, the desired separation of length scales may not always be possible: changing the radius of a spherical shell invariably changes the local curvature of the surface as well. The energy required to achieve a change in radius in this case will include a possibly divergent contribution from the local change in curvature.

This suggests dividing systems into classes: the *difference* in vacuum energy of any two systems within the same class being finite. It would require an infinite amount of energy to compare systems belonging to different classes. Within a particular class, the finite Casimir energy has the universal interpretation of a vacuum energy: differences in Casimir energies *are* finite differences in vacuum energies.

The spectral density $\rho(E; \dots)$ is assumed to be a well-defined quantity for any system (at least for "free," noninteracting fields). The ellipsis here stands for the space $\mathcal{M}$, the types of field, the boundary conditions that are satisfied and any other qualifiers of the system. For systems $\mathcal{A}$ and $\mathcal{B}$ of the same class, the difference of spectral densities

$$\rho(E; \mathcal{A} - \mathcal{B}) := \rho(E; \mathcal{A}) - \rho(E; \mathcal{B}), \quad (1)$$

by definition has a finite first moment,

$$-\infty < \mathcal{E}_{\mathcal{A}-\mathcal{B}} := \frac{1}{2} \int_0^\infty \rho(E; \mathcal{A} - \mathcal{B}) E dE$$
$$= E_{\text{vac}}(\mathcal{A}) - E_{\text{vac}}(\mathcal{B}) < \infty. \quad (2)$$

$\mathcal{E}_{\mathcal{A}-\mathcal{B}}$ could be called the Casimir energy of system $\mathcal{A}$ with respect to system $\mathcal{B}$. There evidently are many equivalent definitions of the Casimir energy of a system within a particular class—they are distinguished by the spectral density used as reference. The Casimir energies determined by two such subtraction schemes, differ only by a finite amount that is the same for any system of a class. Such subtraction schemes are equivalent in all physical respects.

The semiclassical Casimir energy (SCE) is defined by a particular subtraction $\rho_0(E)$ in each class. I will take advantage of the fact that the semiclassical spectral density $\rho(E)$ is the sum of a part $\tilde{\rho}(E)$ determined by contributions from periodic rays and a (often classical) remainder [20] $\rho_0(E)$,

$$\tilde{\rho}(E; \dots) = \rho(E; \dots) - \rho_0(E; \text{class})$$
$$= -\frac{1}{\pi} \lim_{\varepsilon \to 0^+} \text{Im } \tilde{g}(E + i\varepsilon; \dots). \quad (3)$$

Here $\tilde{g}(E)$ is the part of the response function due to classical periodic rays. For a scalar field the remainder $\rho_0(E)$ at least includes the Weyl contribution to the spectral density proportional to the volume of $\mathcal{M}$. In addition $\rho_0(E)$ may depend on the type of field, the curvature, boundary conditions as well as other characteristics [5,6,14].

The SCE $\mathcal{E}_c$ then is defined as

$$\mathcal{E}_c(\mathcal{M}) = \int_0^\infty \frac{E}{2} \tilde{\rho}(E) dE = -\frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_0^\infty \frac{E}{2} \text{Im } \tilde{g}(E + i\varepsilon) dE. \quad (4)$$

Since the length of a periodic ray is finite, this contribution to the vacuum energy is free of ultraviolet divergences and in general is *finite* [15]. The SCE of Eq. (4) may be taken to (at least approximately) represent the vacuum energy within a class of systems for which the subtracted spectral density $\rho_0(E)$ is the *same*. It may happen (see the example of the Laplace-Beltrami operator on a half-sphere of Appendix A) that a particular class has just one member. The subtraction $\rho_0(E)$ in this case is not universal to several systems. The finite Casimir energy one extracts in this case is peculiar to the particular system and it is physically quite irrelevant: any (small) change in the system requires infinite energy. In the examples studied in Appendix A such nonuniversal subtractions are seen to be associated with poles in zeta function regularization [8,21,22]. The SCE of Eq. (4), on the other hand, coincides with the Casimir energy of zeta function regularization in those systems considered for which the subtraction has a more universal meaning.

Although Eq. (4) does not directly refer to $\rho_0(E)$, this implicit subtraction in the spectral density determines the class of systems and thus, in effect, the usefulness of the SCE. Other approaches, such as zeta function regularization often give finite answers without explicitly specifying what has been subtracted. Still other approaches, such as heat kernel expansion, subtract terms whose physical implications are not entirely clear [6,10] and the question whether one gains or looses vacuum energy by transforming an elongated ellipsoid into a sphere becomes difficult to answer. To escape this conundrum in the interpretation of a Casimir energy, Power [23] long ago considered a large rectangular box with a moveable wall to unambiguously define the original Casimir energy [1] for two parallel conducting plates. He in effect was considering a class of systems that all have the same total volume, total surface area, edge length and number of corners. We will see in Sec. IV B that the implicitly subtracted spectral density $\rho_0(E)$ for a three-dimensional parallelepiped in fact only depends on these characteristics. $\tilde{\rho}(E)$ of a parallelepiped can again be expressed in terms of periodic orbits [24–27].

Restricting the validity of a Casimir energy to a certain class of spaces for which the same subtraction in the spectral density gives a finite Casimir energy generalizes Power's procedure to slightly less obvious situations. As the example in Appendix A of a massless scalar field on S_4 demonstrates, (universal) subtractions can go beyond Weyl terms and for instance include contributions proportional to the integral of the (local) curvature over the whole space. These curvature dependent subtractions in the spectral density are proportional to coefficients that describe the short (proper-)time behavior of the heat kernel [10,28].

As emphasized in a perturbative setting by Barton [29], the physical interpretation of a Casimir energy depends almost entirely on the (implicit) subtraction. This is readily illustrated by a spherical cavity of radius R in three dimen-

sions. For dimensional reasons the Casimir energy is proportional to $\hbar c/R$ and apparently only the proportionality constant is of physical interest. However, the significance [30] of the electromagnetic Casimir energy (which was found to decrease with the radius of the cavity [12]), relies on the fact that this Casimir energy actually determines the physical pressure on the spherical surface of the cavity. This conclusion is possible only if the (implicit) subtractions in $\rho_0(E)$ do not depend on the surface area of the boundary. The finite Casimir energy otherwise could only be used to obtain the vacuum energy difference between cavities of the same surface area and *would not* determine the pressure on the cavity surface. That a subtraction proportional to the surface area is not required in the electromagnetic case is due to the ideal metallic boundary conditions [13,14]. The situation is less favorable for a *scalar* field [8,9] satisfying Dirichlet boundary conditions on such a spherical surface. The nonuniversality of the required subtraction was emphasized in Ref. [7].

A similar interpretational challenge arises when one considers De Sitter space as a conformal transformation of the Einstein universe. The Casimir stresses of these two spaces are not conformally related because the required subtractions are not [26]. De Sitter space and the Einstein universe belong to different classes and the classical periodic orbits of the two spaces are not mapped onto each other by the conformal transformation [31].

Defining the Casimir energy in terms of contributions due to periodic orbits rather than by any other subtraction of the spectral density has the advantage that this finite part of the vacuum energy may often be evaluated *approximately*. This is of practical use in situations where the exact spectrum is not, or is only numerically, known. A rather crude approximation will provide us with an estimate of the sign of the SCE in Eq. (4) without detailed knowledge of the periodic rays themselves.

The sign of Casimir energies is one of its many puzzles. Without explicit calculation, determining the sign of the *difference* of two divergent vacuum energies in general is quite hopeless. Obtaining the sign of the SCE, on the other hand, is much more promising due to the geometrical nature of this definition. The overall phase of the contribution to the response function from a particular periodic ray is given by a topological winding number [32]. The sign of the SCE can often already be inferred from the shortest periodic rays that contribute.

I first illustrate the approach for single valued (bosonic) fields on smooth d-dimensional manifolds without boundary such as S_d and T_d and verify the sign of the Casimir energy for these manifolds obtained by zeta function regularization [24–27,33,34]. I then generalize to manifolds with boundaries on which the bosonic field satisfies Dirichlet or Neumann conditions. Several examples show that classical periodic rays within the boundary must also be considered. In general the contribution of these rays to the SCE depends on the boundary condition. When the boundary is not smooth, as for a parallelepiped, contributions due to periodic rays in even lower-dimensional spaces have to be included as well.

II. GENERAL SPACES WITHOUT BOUNDARY

The conceptually simplest Casimir energy probably is that due to a massless single valued bosonic field on a smooth

$(d+1)$-dimensional Riemannian space-time without boundary. I will assume that the metric is static in a particular frame, that is it makes sense to speak of a d-dimensional spatial manifold $\mathcal{M}$ and of the energy of a particle. Periodic rays follow geodesics on $\mathcal{M}$ that close on themselves. The classical action for a periodic ray γ then is

$$S_\gamma = \oint \mathbf{p} \cdot d\mathbf{x} = p(E)L_\gamma, \tag{5}$$

where $\tau_\gamma = \partial S_\gamma/\partial E$ is the time for the ray to return to its starting point on $\mathcal{M}$. For a massless particle moving at the speed of light, $\tau_\gamma = L_\gamma/c$ and thus $p(E) = E/c$. Note that for periodic rays τ_γ is an integer multiple of the primitive period t_γ.

The contribution $\tilde{g}(E)$ of *isolated* periodic rays to the response function is of the form [20]

$$\tilde{g}(E) = \frac{1}{i\hbar} \sum_\gamma A_\gamma t_\gamma e^{iE\tau_\gamma/\hbar - i\sigma_\gamma \pi/2}. \tag{6}$$

In Eq. (6) the amplitude A_γ is determined by the monodromy matrix associated with the ray γ. It is a geometric quantity that (for massless particles in vacuum) does not depend on their energy E. A_γ furthermore is positive and real by definition. The integer $\sigma_\gamma \geq 0$ is the Maslov-like index of the stable and unstable manifolds of the periodic ray [32]. Important for us is that this index is a topological winding number. As such it is an additive integer that scales directly with the number of times an orbit is iterated. For practical calculations it will be useful that σ_γ may be written as the sum

$$\sigma_\gamma = \mu_\gamma + \nu_\gamma, \tag{7}$$

of the number of conjugate points μ_γ between the initial point $\mathbf{x}$ and the final point $\mathbf{x}' = \mathbf{x}$ of the periodic ray and of an integer ν_γ associated with the stability of the periodic orbit.

μ_γ gives the total phase retardation $\mu_\gamma \pi/2$ due to conjugate points (for manifolds without boundaries) encountered by the periodic ray. ν_γ in Eq. (7) is the number of *negative* eigenvalues of the matrix $\mathbf{W}$ of second variations of L_γ with respect to a change of the initial (=final) point of the periodic ray [20]

$$\delta L_\gamma(\mathbf{x} + \delta\mathbf{y}, \mathbf{x}' + \delta\mathbf{y})|_{\mathbf{x}=\mathbf{x}'} = \delta\mathbf{y}^T \cdot \mathbf{W}(\mathbf{x}) \cdot \delta\mathbf{y}. \tag{8}$$

Since one of the eigenvalues of the $d \times d$ matrix $\mathbf{W}$ always vanishes, $0 \leq \nu_\gamma \leq d-1$.

If all periodic rays are isolated, one can insert Eq. (6) in the definition of Eq. (4). Upon performing the energy integral, the SCE due to only isolated periodic rays is of the form

$$\mathcal{E}_c(\mathcal{M}) = -\hbar \sum_\gamma \cos(\sigma_\gamma \pi/2) \frac{A_\gamma t_\gamma}{2\pi \tau_\gamma^2}. \tag{9}$$

The sign of the contribution of a particular periodic ray to the SCE is determined by the integer σ_γ. Remarkably, periodic rays with odd σ_γ do not contribute to the Casimir energy. Although an expression like Eq. (9) is valid only for *isolated* periodic rays, it can also be used to infer the *sign* of the Casimir energy of integrable systems. The expression of

Berry and Tabor [35] for the spectral density of an integrable system in terms of periodic rays is more appropriate in this case (see below), but integrable systems are singular in the sense that small deformations of the manifold $\mathcal{M}$ destroy the symmetries and generically result in isolated periodic rays. The expression of Eq. (9) is robust in that the SCE changes continuously and in particular generally does not change sign if the deformation is small enough. Slightly deforming $\mathcal{M}$ to isolate the orbits therefore should allow us to obtain the sign of $\mathcal{E}_c$ from Eq. (9) even for integrable systems. In support of this conjecture note that the integer σ_γ of an individual periodic ray in Eq. (6) is a winding number that changes only when a new conjugate point appears or the stability of the periodic ray changes. If the contribution of a periodic ray to the Casimir energy does not vanish, its sign should not change for sufficiently small deformations of the manifold.

A unique determination of the sign of $\mathcal{E}_c$ is possible when $\cos(\sigma_\gamma \pi/2)$ does not depend on the periodic ray γ. In less favorable situations I resort to finding the sign of the contribution due to the *shortest* periodic rays to Eq. (9). The contribution in Eq. (9) of a periodic ray that winds n times about the geodesic generally decreases in magnitude as $1/n^2$ and in some cases decreases even faster. If the contribution from primitive periodic rays dominates the SCE in Eq. (9), one can attempt to obtain its sign from the *shortest* primitive periodic rays. For some spaces (see, for instance, S_{even} below), short primitive rays do not contribute to the Casimir energy at all or may give contributions of either sign. The overall sign of the Casimir energy in this case is ambiguous and this estimate fails. One nevertheless might expect $\mathcal{E}_c$ to be rather small in magnitude in these situations and I will write $\mathcal{E}_c \sim 0$ when the sign cannot be determined from the shortest periodic rays.

A. *d*-dimensional tori and spheres

Obtaining the sign of the SCE is straightforward for a massless scalar on a d-dimensional torus $T_d = S_1 \times S_1 \times \cdots \times S_1$. The curvature of T_d vanishes and it is a space without boundary. The subtracted spectral density therefore is the Weyl-term proportional to the volume of T_d only. However, due to the translational symmetries this is an integrable system and classical periodic rays are not isolated. To estimate the sign of the SCE of a torus using Eq. (9), one has to deform it slightly. This generally destroys all symmetries and gives isolated periodic rays, the shortest of which resemble periodic rays of the original torus on its shortest cycles. If the curvature remains sufficiently small on the deformed torus, the number of conjugate points along a primitive orbit continues to vanish. The length of the shortest periodic rays furthermore is a minimum by definition. One thus obtains $\mu_\gamma = \nu_\gamma = \sigma_\gamma = 0$ for the shortest periodic rays of a slightly deformed torus. They all give a negative contribution to the Casimir energy in Eq. (9). Since this sign does not depend on the particular deformation of the torus, one can be confident that

$$\mathcal{E}_c(T_d) < 0 \quad \text{for all } d = 1, 2, \ldots \,. \tag{10}$$

This sign agrees with that of the Casimir energy due to a

massless scalar field satisfying *periodic* boundary conditions on the hypersurface of any d-dimensional parallelepiped obtained by explicit calculation [24,26,27,33].

Equation (9) indicates that the SCE may in principle be of either sign for manifolds without boundaries. A nontrivial example is the Casimir energy of a massless scalar field confined to a spherical shell S_d of dimension d and radius R. Periodic rays follow great circles of radius R on S_d. They again are not isolated and the $(d-1)$ dimensional cross section of a bundle of initially parallel geodesics is reduced to a point at two antipodes. Any starting point on a geodesic of S_d also is a conjugate point of order $d-1$, i.e., it is self-conjugate [36]. One can avoid the associated complications by slightly deforming S_d in a generic fashion. The primitive periodic rays of the deformed sphere should still resemble the original geodesics on S_d, but the starting point generally no longer is self-conjugate. A pencil of rays emanating from a point on a geodesic also generally no longer meets in a single focal point. For a sufficiently small deformation, focal points of the sphere will have been resolved into a series of $d-1$ closely spaced conjugate points of first order. On the *shortest* primitive periodic ray, there generically are only $d-1$ conjugate points, the next group of $d-1$ conjugate points occurring just *after* completion of a full revolution. The reason is that the curvature along the shortest primitive ray in general is slightly too *small* to lead to more than one intersection²—as for a periodic ray about the waist of a slightly elongated ellipsoid. The *shortest* periodic rays furthermore are of minimal length and therefore are stable and $\nu_\gamma = 0$ in this case. The total phase retardation of the shortest primitive rays thus is $\sigma_\gamma \pi/2 = \mu_\gamma \pi/2 = (d-1)\pi/2$. Their contribution to the Casimir energy of a (deformed) sphere in Eq. (9) vanishes in even dimensions and is of alternating sign in odd dimensions. Assuming that the contribution from the shortest periodic rays dominate in Eq. (9), one thus expects that

$$\mathcal{E}_c(S_d) \propto -\cos[\pi(d-1)/2] \begin{cases} <0, & \text{for } d = 1 \bmod 4, \\ \sim 0, & \text{for even } d, \\ >0, & \text{for } d = 3 \bmod 4. \end{cases} \tag{11}$$

The explicit calculations for $d \leq 4$ of Appendix A confirm this not very intuitive pattern for the sign of the Casimir energy of a massless scalar field on low-dimensional spheres. The same pattern has been observed in the explicit calculation of the Casimir energy due to a massless scalar on odd and even dimensional spheres using zeta function regularization [34]. The relation to optical properties of the spheres, however, does not seem to have been noted previously. A little surprisingly, the Casimir energy vanishes *exactly* for S_2,

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²The geodesic distance $\eta(s)$ between two nearby geodesics satisfies the linear second order equation $d^2\eta(s)/ds^2 = -\kappa(s)\eta(s)$, where s is the arc length along the ray and $\kappa(s)$ is the Gaussian curvature at s. If $\kappa(s) < 4\pi^2/L_\gamma^2$ two geodesics at most meet once. This condition generally holds for the *shortest* primitive ray of a slightly deformed sphere.

S_4, and, in fact, any S_{even}. Since our somewhat crude estimation only takes contributions from the shortest primitive rays into account, it cannot in itself predict a vanishing Casimir energy. When the contribution from the shortest periodic rays vanishes, determining the overall sign of the Casimir energy becomes much more involved. To conclude that the Casimir energy vanishes one has to show that it is positive for some (small) deformations of the manifold and negative for others. Although this can be shown for S_{even}, the argument is no longer "simple" and will not be pursued any further here. Below I consider another method of estimating the sign of the SCE of integrable systems that unambiguously confirms Eq. (11) for spheres.

It is amusing to consider further stretching the sphere to an elongated cigarlike shape and eventually a long cylinder with end caps. The subtractions have to remain constant during this deformation. In two and three dimensions, it is sufficient to keep the total surface area, respectively, volume, constant during the deformation. In higher dimensions, certain moments of the curvature and other global characteristics also must not change. As the curvature decreases, the d -1 conjugate points on the shortest (and thus stable) periodic rays move beyond the end of the primitive orbit when the sphere is stretched sufficiently. $\sigma_\gamma \to 0$ for these orbits and they eventually dominate and lead to a negative Casimir energy for a (slightly deformed) long cylindrical d-dimensional surface (for essentially the same reason as the short rays on a torus). Since the length of the shortest periodic rays on a very elongated cylinder is much less than on a sphere of the same d-dimensional volume, the Casimir energy furthermore increases in magnitude during this deformation.[3] A free massless scalar field thus would tend to collapse S_d to a filament. This lack of stability of the Casimir energy with respect to certain deformations could be important for the self-consistent dimensional reduction [37] of some Kaluza-Klein models.

III. INTEGRABLE SYSTEMS

A quantitative comparison with the Casimir energy obtained by other methods generally is possible only for integrable systems for which the true spectrum is explicitly or implicitly [38] known. The semiclassical approximation is exact for some integrable systems. The spectrum of the Laplace-Beltrami operator in fact is reproduced semiclassically for manifolds of any semisimple Lie group [39]. This includes the tori and some of the spheres discussed previously. In the examples studied in Appendix A the SCE in fact coincides with that obtained by zeta function regularization when the analytic continuation is unique. When poles arise and zeta function regularization gives ambiguous answers [8,21,22], the associated subtraction in the spectral density is not universal either. Universal subtractions on the other hand may include terms in $\rho_0(E)$ that depend on global characteristics of the curvature [10,28] [as, for example, the Casimir energy on S_4 in Appendix A].

Since the integral in Eq. (4) had to be performed explicitly to arrive at Eq. (9), all our estimates of the sign of the SCE rely on the dispersion relation $p(E) = E/c$ of a *massless* particle. The sign of the SCE in general will differ for other dispersion relations, as for instance for the spectrum of the Laplace-Beltrami operator without curvature correction. The latter is not the spectrum of a massless scalar [20,39,40] (see also Appendix A). Using zeta function regularization, the Casimir energy in this case is negative for all $S_d, d \leq 4$ [41], but the Casimir energy of S_3 is ambiguous due to the subtraction of a pole contribution.

The explicit calculations of Appendix A strongly suggest that the SCE of a massless scalar on S_d vanishes *exactly* for even dimension d: the integrands in this case are polynomials in $(ER)^2$ only. The subtraction $\rho_0(E)$ for S_{even} otherwise would not be universal. The integral over the energy of the response function (after Wick rotation) has vanishing imaginary part and there is no contribution to the Casimir energy from *any* periodic ray on S_{even}.

Although this supports the previous estimate of the sign of the SCE for low-dimensional spheres, Eq. (9) requires periodic rays that are isolated and thus cannot be directly applied to integrable systems. Let us therefore obtain an expression for the SCE that can be used to determine its sign in the (rather special) case of an integrable system.

Action-angle variables are the canonical phase-space coordinates of integrable systems. The SCE for a d-dimensional integrable system is found by applying Poisson's formula [20,35]

$$\mathcal{E}_c(\text{integrable system}) = \frac{1}{2\hbar^d} {\sum_{\mathbf{m}}}' \int H(\mathbf{I}) e^{2\pi i \mathbf{m} \cdot [\mathbf{I}/\hbar - \beta/4]} d\mathbf{I}.$$

(12)

Here $H(\mathbf{I})$ is the classical Hamiltonian expressed in terms of the actions $\mathbf{I} = (I_1, I_2, \ldots, I_d)$ and $\mathbf{m}$ is a d-dimensional vector of integers. The primed sum in Eq. (12) extends over all such vectors except $\mathbf{m} = \mathbf{0} = (0, 0, \ldots, 0)$. The omitted $\mathbf{m} = \mathbf{0}$ term in the sum in Eq. (12) corresponds to the subtracted "classical" spectral density

$$\rho_0(E) = \int \delta(E - H(\mathbf{I})) \frac{d\mathbf{I}}{\hbar^d}.$$

(13)

Equation (12) expresses the SCE of an integrable system as a sum over classical periodic trajectories on the invariant tori. The classical action of a trajectory with winding numbers $\mathbf{m}$ about each of the cycles of the invariant torus is $S(\mathbf{m}) = 2\pi \mathbf{m} \cdot \mathbf{I}$. The correction to the classical action proportional to $\hbar$ is linear in the winding numbers $\mathbf{m}$. It is a topological quantity that determines how periodic orbits on the invariant torus are projected onto physical coordinate space [42]. The vector β of Keller-Maslov [43] indices gives the phase loss from caustics on a periodic orbit (see below). Note that β is a geometrical quantity that does not depend on $\mathbf{I}$.

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[3]It would be erroneous to compare the Casimir energy densities of a cylindrical surface and a spherical one of the same radius when the subtractions are not the same.

The integrals over the actions of Eq. (12) are evaluated semiclassically at stationary points $\bar{\mathbf{I}}(E,\mathbf{m})$ of the classical action on the energy surface—the vector $\mathbf{m}$ is normal to the energy surface $H(\mathbf{I})=E$ at $\mathbf{I}=\bar{\mathbf{I}}(E,\mathbf{m})$. For given energy E, the $d-1$ integrations along the (compact) energy surface are performed semiclassically by choosing a local frame of actions at $\bar{\mathbf{I}}$ for which one axis, say that of I_1, is in the direction of $\mathbf{m}$ and all others are tangent to the energy surface. Care must be taken with zero modes of the matrix of second derivatives

$$H_{ij} = \partial^2 H/\partial\bar{I}_i\partial\bar{I}_j, \quad i,j = 2, \ldots, d. \quad (14)$$

The integral over the ν_0-dimensional subspace of the zero modes of H_{ij} is stationary only when the corresponding frequencies vanish. The sum over $\mathbf{m}$ is thereby reduced to one over the $(d-\nu_0)$-dimensional vectors $\mathbf{n}$ that are orthogonal to the zero modes. Denoting the ν_0-dimensional volume of this classical moduli space by $V_{\nu_0}(E,\mathbf{n})$, the semiclassical result for the $d-1$ integrations along the energy surface in Eq. (12) is

$\mathcal{E}_c(\text{integrable system})$

$$= \sum_{\mathbf{n}}' \int_0^\infty \frac{EdE}{2\omega\hbar^d} \frac{V_{\nu_0}(E,\mathbf{n})e^{i\pi(\nu_- - \nu_+)/4}}{\sqrt{|\mathbf{n}/\hbar\omega|^{d-1-\nu_0}|\det' H_{ij}|}} e^{2\pi i\mathbf{n}\cdot[\bar{\mathbf{I}}/\hbar - \beta/4]}. \quad (15)$$

The primed determinant here means that the ν_0 zero-modes have been omitted in its calculation. ν_+ and ν_- are the number of positive and negative eigenvalues of the matrix H_{ij}, $\nu_+ + \nu_- + \nu_0 = d-1$. The frequency $\omega = |\omega| = |\nabla_I H|_{I=\bar{I}(E,\mathbf{m})}$ for a *massless* particle in fact does not depend on its energy.

The energy dependence of the integrand in Eq. (15) is made explicit by noting that the classical action of a massless field is $S(E,\mathbf{n}) = |2\pi\mathbf{n}\cdot\bar{\mathbf{I}}| = EL_\mathbf{n}/c$ where $L_\mathbf{n} > 0$ is the length of the periodic ray. For dimensional reasons, $\det' H_{ij}(E)$ scales as $(E/\omega^2)^{1-d+\nu_0}$ and $V_{\nu_0}(E)$ scales as $(E/\omega)^{\nu_0}$. The integration over the energy E in Eq. (15) then gives the SCE of an integrable system without boundaries in the form

$\mathcal{E}_c(\text{integrable system})$

$$= -\hbar c \sum_{\mathbf{n}}' \left(\frac{c}{L_\mathbf{n}}\right)^{d+1} A_{\hat{\mathbf{n}}} \cos[\pi(\beta - \nu_0 - \nu_-)_\mathbf{n}/2]. \quad (16)$$

To perform the integral in Eq. (15), the integer

$$\tilde{\sigma}_\mathbf{n} = (\beta - \nu_0 - \nu_-)_\mathbf{n}, \quad (17)$$

must not depend on E. Since geodesics do not depend on the energy of a ray, the number of non-positive eigenvalues of H_{ij} does not depend on E for massless particles. The Keller-Maslov index $\beta_\mathbf{n} = \mathbf{n}\cdot\beta$ depends only on $\mathbf{n}$ by construction. Note that any *constant* angle variable of a closed geodesic implies that the Hamiltonian does not depend on the conjugate action. The dimension $0 \le \nu_0 \le d-1$ of the space of zero modes of H_{ij} can thus often be found by inspection.

The amplitude $A_{\hat{\mathbf{n}}}$ in Eq. (16) is positive by construction and all dependence on the scale $|\mathbf{n}|$ was absorbed in the length $L_\mathbf{n}$ of a class of periodic rays. $A_{\hat{\mathbf{n}}}$ thus is a function of the dimensions of the integrable system (such as the volume of the space) that do not scale with the length of the periodic orbit. The leading contribution to the Casimir energy of an integrable system in Eq. (16) generally is from the shortest rays that contribute to Eq. (16). The sign of this contribution depends on the Keller-Maslov index $\beta_\mathbf{n}$ of the rays as well as the number of non-positive eigenvalues of H_{ij}. The sign of the contribution from a particular class $\mathbf{n}$ of periodic rays of an integrable system is again given by

$$-\cos(\tilde{\sigma}_\mathbf{n}\pi/2), \quad (18)$$

with $\tilde{\sigma}_\mathbf{n}$ defined by Eq. (17). The SCE of Eq. (16) for an integrable system thus is remarkably similar to the one in Eq. (9) for a system with only isolated periodic rays. It is tempting to identify $\tilde{\sigma}_\mathbf{n}$ in Eq. (17) with σ_γ of Eq. (7). However, while both σ_γ and $\tilde{\sigma}_\mathbf{n}$ are topological quantities, σ_γ in Gutzwiller's expression for the contribution to the response function of isolated rays is a topological property of the ray [32], whereas $\tilde{\sigma}_\mathbf{n}$ in Eq. (17) is a property of a whole continuous family $\mathbf{n}$ of periodic rays. The two integers depend differently on the winding number of the periodic rays. One nevertheless can argue that they coincide for the *shortest* (and thus for primitive) periodic rays.

The Keller-Maslov index $\beta_\mathbf{n}$ of an integrable system is given by the number of caustics a periodic ray encounters (for manifolds without boundary). These caustics are created by the family of rays it is a member of. $\beta_\mathbf{n}$ thus is a topological property of the family of rays that generally does *not* equal μ_γ, the number of conjugate points on a single ray of that family. For one, $\beta_\mathbf{n}$ does not depend on the starting point of the ray and is a well-defined integer even when this point happens to be self-conjugate (as in the case of spheres). $\beta_\mathbf{n}$ is proportional to the number of times the periodic rays wind about the closed geodesic. The number of positive eigenvalues of H_{ij}, on the other hand, is a statement about the curvature at a particular point on the energy surface determined by the direction of its normal $\hat{\mathbf{n}}$. Neither this point nor the energy surface change with the magnitude of $\mathbf{n}$. $(\nu_0 + \nu_-) = (d-1-\nu_+)$ thus *does not* depend on the winding number of the periodic ray. For σ_γ to coincide with $\tilde{\sigma}_\mathbf{n}$ for *all* periodic rays γ as the integrable system is slightly deformed, the number of nonpositive eigenvalues of H_{ij} would have to vanish (as is the case for a torus). However, the sign of the Casimir energy in general does not change under small deformations, if the phase of the *shortest* periodic rays remains the same. This observation gives the desired connection between σ_γ and $\tilde{\sigma}_\mathbf{n}$: the two have to coincide for the *shortest* periodic rays of an integrable system and its (sufficiently small) deformation. Below we will see that this is the case in the previous example of spheres and tori.

A. Covering spaces

The previous arguments imply that the sign of the Casimir energy can often be inferred from the Keller-Maslov index β_γ of periodic rays of integrable systems. (γ here is some

representative of the class **n**.) It thus is important to have a reliable and transparent determination of this index. Keller [42] gives a geometrical construction that generalizes to manifolds with boundaries. Let me recall some points of the construction that are relevant here.

A solution $S(\vec{q},t)$ of the Hamilton-Jacobi equation $H(\vec{q},\vec{p}=\vec{\nabla}S,t)=E$ for constant energy E generally is multiply valued and so may be the momentum $\vec{p}=\vec{\nabla}S$ itself. At any point $\vec{x}$ the momentum $\vec{p}=\vec{\nabla}S$ of a solution only has a finite number of branches, say m of them. One constructs an m-sheeted covering space on which $\vec{p}=\vec{\nabla}S$ is a single-valued function by associating each of the branches of $\vec{\nabla}S$ with a separate sheet. Any two different sheets i and j are joined together on submanifolds on which the momenta coincide $\vec{p}|_i=\vec{p}|_j$. Such a submanifold generically is a caustic of the rays or a boundary in the original space. (If $\vec{\nabla}S$ is defined on only part of x space then only this part is covered.) The advantage of Keller's construction is that $\vec{p}=\vec{\nabla}S$ becomes a single-valued function on the covering space. The rays of a family **n** of periodic rays do not intersect on the covering space.

The semiclassical Casimir energy of the integrable system is obtained by considering the periodic rays on this covering space. The phase is retarded by $m\pi/2$ whenever a ray crosses a caustic of mth order in passing from one sheet to another. The positive integer m is the number of dimensions by which the cross section of a tube of nearby trajectories is reduced at the caustic. Keller's construction does not of itself provide the order of a caustic. The latter must be inferred from the behavior of a bundle of nearby rays.

An example is the construction of this covering space for geodesic motion on a d-dimensional sphere S_d. The $SO(d+1)$ symmetry of S_d implies that geodesics lie in a hyperplane of the $\mathbf{R}_{d+1}$ it can be embedded in. The vector orthogonal to this hyperplane makes a certain angle θ with the vertical which is the inclination of the geodesic. It can be chosen as one of the angles of the action-angle variables and geodesics of S_d thus fall into distinct classes characterized by their inclination. A family of periodic rays of the same inclination covers an annulus of S_d that is bounded by two S_{d-1} hypersurfaces. These two $(d-1)$-dimensional caustics are around the polar regions of S_d. Every closed geodesic in the family of solutions with given inclination touches the "upper" and "lower" caustic once. One constructs the covering space by joining two sheets of this annulus of S_d at the two caustics. A periodic ray of the given inclination passes from one sheet to the other every time it crosses one of these caustics. In this covering space a family of rays of given inclination does not intersect. The number of times a periodic ray winds about the annulus also is the number of times the periodic ray passes through *both* caustics. These caustics are of order $d-1$, because the cross section of a $(d-1)$-dimensional tube of geodesics vanishes when they intersect a given periodic ray at the caustic. The phase loss of a periodic ray that winds about the sphere n times and thus crosses $2n$ caustics is $2n(d-1)\pi/2$ and therefore $\beta_\gamma=2n(d-1)$ for any family of periodic rays. The Hamiltonian depends on the action conjugate to the angle describing the

motion on a great circle only (the magnitude of angular momentum in the two-dimensional case). Therefore $\nu_0=d-1$ and $\nu_-=0$ in this case leading to $\tilde{\sigma}=(2n-1)(d-1)$. For even dimensions d, $\tilde{\sigma}$ thus is odd for all n and the contribution to the SCE of *all* periodic rays vanishes according to Eq. (18). This agrees with our previous estimation of the signs in Eq. (11) based on Eq. (9) and proves that the SCE of even-dimensional spheres indeed vanishes. Appendix A presents explicit results for $0<d\leq 4$. The Casimir energy of a massless scalar on S_d has previously been calculated [34] in zeta function regularization and the dependence of the sign on d also agrees with Eq. (11).

Note that the covering space for rays on a d-dimensional torus is trivial since the momentum is single valued. The Hamiltonian in this case furthermore depends on the modulus of the momentum in each direction and H_{ij} is positive definite. $\nu_0+\nu_-=0$ for a massless particle on the torus and $\tilde{\sigma}_\mathbf{n}=0$ for any periodic ray. The Casimir energy of a torus therefore is always negative, as was already found by deforming it and using Eq. (9).

IV. MANIFOLDS WITH BOUNDARIES

Estimates of the sign of the SCE are more difficult for manifolds with a boundary on which the scalar field satisfies some conditions. I here consider Dirichlet and Neumann boundary conditions only. Since classical rays reflect specularly at a boundary, the basic strategy is to glue copies of the original manifold at the boundaries and consider the resulting covering manifold without boundary. At a boundary $\vec{p}=\vec{\nabla}S$ is discontinuous and there are (at least) two values for the momentum at any point near a boundary. By constructing the covering manifold for which $\vec{p}=\vec{\nabla}S$ is single valued, one thus can treat a boundary in much the same way as a caustic. The phase retardation at a boundary depends on the boundary condition. The phase loss is π for Dirichlet and 0 for Neumann boundary conditions. This ensures the correct behavior of semiclassical Green functions near the boundary.

However, an additional correction to the semiclassical Casimir energy arises from periodic rays of the covering manifold that lie (entirely) within the boundary. The sign and magnitude of this additional contribution can be essential in determining the sign of the semiclassical Casimir energy and its dependence on the boundary condition. Let us examine some simple examples.

A. Semiclassical Casimir energy of a d-dimensional half-sphere

Consider first the semiclassical Casimir energy of a half-sphere in d dimensions. Space in this case is just S_d cut in half at the equatorial S_{d-1}. Classical periodic rays lie on two halves of great circles of the original S_d with the *same* inclination that intersect on the equatorial S_{d-1} of the sphere. The momentum at any point on the half-sphere annulus covered by a family of rays with fixed inclination therefore can take up to four values and one needs a four sheeted covering of this space to make $\vec{\nabla}S$ single valued. This covering space is constructed in two steps. One first doubles the half-sphere

and joins the two sheets at the equators to form an S_d. On this boundaryless doublecovering of the half sphere, geodesics are again great-circles and one again introduces two coverings for each annulus of S_d. Note that this last operation *doubles* the equatorial (boundary) S_{d-1}. One of these "equators" is where the upper and lower parts of the *inner* annulus join, the other is where the corresponding parts of the *outer* annulus join. This doubling of the boundary of the half sphere cannot be avoided if the momentum on the covering space is to be single valued.

The periodic rays of this four-sheeted covering of the d-dimensional half-sphere evidently are those already found for the two-sheeted covering space of S_d. The only difference is that the phase of a ray may be retarded by π at every crossing of an "equator" (for Dirichlet boundary conditions). Since a periodic ray, however, crosses the "equators" an even number of times, one is tempted to conclude that the semiclassical Casimir energy of the half sphere is just half the semiclassical Casimir energy of S_d, irrespective of whether Neumann or Dirichlet boundary conditions have been imposed.

However, this argument ignores classical periodic rays of the covering space that lie entirely *within* the boundary. The contribution of such rays in general depends on the imposed boundary condition. Since the field vanishes on a boundary with Dirichlet's condition, classical periodic rays that lie entirely within the boundary should not contribute to Green's function and the spectral density. Their contribution to the spectral density of the manifold without boundary has to be subtracted in this case.

Often, as in the case of a sphere, symmetry arguments can be invoked to relate the Casimir energy for Dirichlet and Neumann boundary conditions. Due to symmetry under reflections about the equatorial plane, eigenfunctions of the Hamiltonian can be chosen to satisfy either Neumann or Dirichlet boundary conditions at the equator of S_d. The sum of the Casimir energies on the half sphere $\mathcal{E}(S_2/2;N)$ and $\mathcal{E}(S_2/2;D)$ for Neumann and Dirichlet boundary conditions, respectively, therefore is the Casimir energy of the full sphere

$$\mathcal{E}_c(S_d/2;N) + \mathcal{E}_c(S_d/2;D) = \mathcal{E}_c(S_d). \quad (19)$$

The difference $\mathcal{E}_c(S_d/2;N) - \mathcal{E}_c(S_d/2;D)$ is due to contributions to the Casimir energy from the boundary, i.e., due to periodic rays on the equator. The magnitude of this contribution in general is difficult to obtain without explicit calculation. It is *not* simply related to the Casimir energy of a $(d-1)$-dimensional sphere, because $\mathcal{E}_c(S_{d-1})$ does not include fluctuations transverse to the equator. One nevertheless can infer the sign of this difference, that is whether Neumann or Dirichlet boundary conditions lower the Casimir energy. The point is that families of periodic rays in the vicinity of the equator of S_d are similar to those on S_{d-1} except that they nevertheless pass two caustics of order $d-1$—rather than of order $d-2$ as for S_{d-1}—in every revolution. This gives an additional phase loss of π for every revolution no matter how close to the equatorial hyperplane the periodic rays are. Since the sign of the contribution of a periodic ray to the

Casimir energy of S_{d-1} is given by $\tilde{\sigma}(S_{d-1})=(2n-1)(d-2)$, the additional phase loss of $n\pi$ on S_d changes the index to $\tilde{\sigma}(\text{equator } S_d)=(2n-1)(d-2)+2n$. From this one obtains that

$$\mathcal{E}_c(S_d/2;N) - \mathcal{E}_c(S_d/2;D)$$
$$\sim \mathcal{E}_c(\text{equator}) \begin{cases} <0, & \text{for } d=0 \bmod 4, \\ =0, & \text{for odd } d, \\ >0, & \text{for } d=2 \bmod 4. \end{cases} \quad (20)$$

The explicit calculation of the SCE of a two-dimensional half shell given in Eq. (A14) of Appendix A agrees with Eq. (20). It also is well known that the Casimir energy is the same for Dirichlet or Neumann boundary conditions at the endpoints of a half circle (i.e., interval). (Since there is no curvature for $d=1$ this also coincides with Ref. [41].) The Casimir energy of a massless scalar on half an Einstein universe has also been calculated explicitly [44] and was found to be just half of the Casimir energy for the full Einstein universe irrespective of the boundary conditions [which corresponds to $d=3$ in Eq. (20)]. More general orbifold factors of two- and three-spheres were considered in Ref. [45]. The fact that the Casimir energy of a massless scalar on all odd-dimensional half-spheres does not depend on whether Neumann or Dirichlet boundary conditions are imposed has only recently [46] been verified. Note that the fate of the implicitly subtracted infinite vacuum energy proportional to the surface area of the equator of the halved Einstein universe was ignored in Ref. [44]. Fulling has pointed out [6] that some (infinite) changes in the vacuum energy of the universe could be absorbed in the cosmological constant. They also may be cancelled by similar (infinite) contributions from other fields as in the case of supersymmetry. The following example of a d-dimensional parallelepiped shows that the semiclassical evaluation of the Casimir energy of spaces with boundaries that are not smooth and intersect on lower-dimensional manifolds can be even more involved.

B. Semiclassical Casimir energy of a d-dimensional parallelepiped

The Casimir energy of a massless scalar field confined to the interior of a parallelepiped with dimensions $l_1 \times l_2 \times \cdots \times l_d$ has previously been obtained by Ambjörn and Wolfram [27]. They considered Neumann, Dirichlet as well as periodic and electromagnetic boundary conditions on the surface of the parallelepiped. I now give a more geometrical interpretation of some of their results in terms of periodic rays of the covering space.

It suffices to consider the case where all the l_i of the parallelepiped are finite. The result of Ref. [27]. for a parallelepiped with some sides that are much longer than all others is found by taking the appropriate limits. Lacking a more concise notation, the SCE of a parallelepiped with dimensions $l_1 \times \cdots \times l_d$ will be denoted by

$$\mathcal{E}_c(l_1, \ldots, l_N; l_{N+1}, \ldots, l_{N+D}; l_{N+D+1}, \ldots, l_d). \quad (21)$$

Here Neumann boundary conditions are satisfied on $0 \le N \le d$ pairs of parallel hypersurfaces that are distances $l_1, \ldots, l_N$ apart, Dirichlet boundary conditions are satisfied on

$0 \le D \le d-N$ pairs of parallel hypersurfaces that are distances $l_{N+1}, \ldots, l_{N+D}$ apart, and periodic boundary conditions are assumed to hold on the remaining $0 \le d-N-D$ pairs of parallel surfaces.

The symmetry of a parallelepiped implies that eigenfunctions satisfying periodic boundary conditions on a pair of faces, are even or odd under reflection of the parallelepiped about these faces. This leads to the following relation between the Casimir energies of scalar fields satisfying different boundary conditions on the surfaces of d-dimensional parallelepipeds,

$$
\begin{aligned}
& \mathcal{E}_c(l_2, \ldots, l_N; l_{N+1}, \ldots, l_{N+D}; 2l_1, l_{N+D+1}, \ldots, l_d) \\
& = \mathcal{E}_c(l_1, \ldots, l_N; l_{N+1}, \ldots, l_{N+D}; l_{N+D+1}, \ldots, l_d) \\
& \quad + \mathcal{E}_c(l_2, \ldots, l_N; l_1, l_{N+1}, \ldots, l_{N+D}; l_{N+D+1}, \ldots, l_d).
\end{aligned}
\tag{22}
$$

Equation (22) is a relation between the Casimir energies of a scalar field on parallelepipeds where the periodic boundary conditions on a pair of parallel surfaces are replaced by Dirichlet or Neumann boundary conditions *and* the distance between the two surfaces is *halved*. The total volume of the manifolds on the left- and right-hand sides of this equation thus are the same. The subtracted terms of the spectral densities are the same as well, and the Casimir energies indeed are equal.

The spectrum of a parallelepiped differs only in a zero frequency mode for Neumann and Dirichlet conditions on a set of parallel surfaces. This frequency does not depend on the separation of the two surfaces. One therefore also has that

$$
\begin{aligned}
& \mathcal{E}_c(l_2, \ldots, l_N; l_{N+1}, \ldots, l_{N+D}; l_{N+D+1}, \ldots, l_d) \\
& = \mathcal{E}_c(l_1, \ldots, l_N; l_{N+1}, \ldots, l_{N+D}; l_{N+D+1}, \ldots, l_d) \\
& \quad - \mathcal{E}_c(l_2, \ldots, l_N; l_1, l_{N+1}, \ldots, l_{N+D}; l_{N+D+1}, \ldots, l_d).
\end{aligned}
\tag{23}
$$

Note that the subtractions in the spectral density proportional to the d-dimensional volume cancel on the right hand side and the leading remaining subtraction is proportional to the volume of the $(d-1)$-dimensional parallelepiped on the left-hand side.

Combining Eqs. (22) and (23) one obtains the following recursive relations for the SCE of a parallelepiped:

$$
\begin{aligned}
& 2\mathcal{E}_c(l_1, \ldots, l_N; l_{N+1}, \ldots, l_{N+D}; l_{N+D+1}, \ldots, l_d) \\
& = \mathcal{E}_c(l_2, \ldots, l_N; l_{N+1}, \ldots, l_{N+D}; 2l_1, l_{N+D+1}, \ldots, l_d) \\
& \quad + \mathcal{E}_c(l_2, \ldots, l_N; l_{N+1}, \ldots, l_{N+D}; l_{N+D+1}, \ldots, l_d),
\end{aligned}
$$

$$
\begin{aligned}
& 2\mathcal{E}_c(l_2, \ldots, l_N; l_1, l_{N+1}, \ldots, l_{N+D}; l_{N+D+1}, \ldots, l_d) \\
& = \mathcal{E}_c(l_2, \ldots, l_N; l_{N+1}, \ldots, l_{N+D}; 2l_1, l_{N+D+1}, \ldots, l_d) \\
& \quad - \mathcal{E}_c(l_2, \ldots, l_N; l_{N+1}, \ldots, l_{N+D}; l_{N+D+1}, \ldots, l_d).
\end{aligned}
\tag{24}
$$

Equation (24) expresses the Casimir energy of a parallelepiped in terms of Casimir energies of parallelepipeds with nonperiodic boundary conditions on fewer sets of parallel plates. Repeated application of these relations thus gives the Casimir energy of a parallelepiped with Neumann, Dirichlet, and periodic boundary conditions in terms of the Casimir energies of tori only. Thus the Casimir energy of a three-dimensional parallelepiped with Dirichlet boundary conditions on all six faces may be decomposed as

$$
\begin{aligned}
& 8\mathcal{E}_c(; l_1, l_2, l_3;) \\
& = 4\mathcal{E}_c(; l_2, l_3; 2l_1) - 4\mathcal{E}_c(; l_2, l_3;) = 2\mathcal{E}_c(; l_3; 2l_1, 2l_2) \\
& \quad - 2\mathcal{E}_c(; l_3; 2l_1) - 2\mathcal{E}_c(; l_3; 2l_2) + 2\mathcal{E}_c(; l_3;) \\[6pt]
& = \mathcal{E}_c(;; 2l_1, 2l_2, 2l_3) - \mathcal{E}_c(;; 2l_1, 2l_2) - \mathcal{E}_c(;; 2l_1, 2l_3) \\
& \quad - \mathcal{E}_c(;; 2l_2, 2l_3) + \mathcal{E}_c(;; 2l_1) + \mathcal{E}_c(;; 2l_2) + \mathcal{E}_c(;; 2l_3).
\end{aligned}
\tag{25}
$$

The corrections proportional to the Casimir energies of lower-dimensional tori are due to periodic rays of the covering space that lie on the boundaries of the original parallelepiped. Since the subtraction for a torus is the Weyl-contribution proportional to its "volume," the subtractions for a general parallelepiped include Weyl-terms proportional to the "area" of its hypersurfaces, "lengths" of the intersections of its hypersurfaces, etc. The Casimir energy of a general parallelepiped thus can be compared with that of other systems of the same volume, area of the boundary (with the same boundary conditions), length of intersections of hypersurfaces, etc. The simplest class of spaces that satisfy all these conditions are d-dimensional generalizations of Power's box [23] with a fixed number of orthogonal but movable walls.[4]

Let us turn to the construction of the appropriate covering space for a parallelepiped. The momentum is single valued only when periodic boundary conditions hold on all pairs of hypersurfaces. $N+D=0$ and there is no need to introduce additional sheets. The result is the same as for the d-dimensional torus in Sec. II A: the Casimir energy in this case is negative for any dimension of the torus and any lengths, l_i, of its cycles.

For $N+D>0$, the momentum is not single valued. Each pair of faces with nonperiodic boundary conditions requires a double covering since the component of momentum that is perpendicular to these surfaces can have either sign. One recursively constructs this covering space as follows.

Consider first the pair of faces with coordinate $x_1=0$ and $x_1=l_1$. The boundary condition on this pair is not periodic and the first component of momentum (for rays of fixed energy) therefore is double valued. It is single valued on a covering space obtained by joining a second sheet of the original parallelepiped to the first at the boundaries $x_1=0$ and

---

[4]Power's original construction of a box with just one movable wall for the Casimir force between two parallel plates has been reexamined in Ref. [47]. Since the surface areas of the two parallelepipeds depend linearly on their volume, considering a box with just one movable wall does not isolate the surface dependence of the Casimir energy of a parallelepiped—one has to consider more than one movable wall to isolate this contribution.

$x_1 = l_1$ to form a cylinderlike covering space. Periodic rays that reflect from the $x_1 = 0$ and $x_1 = l_1$ faces of the original parallelepiped pass smoothly through these borders from one sheet to the other in the covering space. Although one must keep track of the position of the original boundaries at $x_1 = 0$ and $x_1 = l_1$, the problem of constructing a covering space on which momentum is unique has been reduced to that of a parallelepiped with only $D+N-1$ pairs of hypersurfaces on which nonperiodic boundary conditions hold. Note that the first dimension of this covering space is now twice that of the original parallelepiped.

If $N+D>1$, the procedure is repeated with the pair of faces at $x_2 = 0$ and $x_2 = l_2$ of this double cover of the original parallelepiped. This results in a covering parallelepiped with nonperiodic conditions in just $N+D-2$ dimensions. The length of this covering parallelepiped in the second dimension is again twice that of the original parallelepiped. The process ends with the pair of faces at $x_{D+N} = 0$ and $x_{D+N} = l_{D+N}$.

The covering space of a general d-dimensional parallelepiped thus is a d-dimensional torus with cycles $2l_1, \ldots, 2l_{D+N}, l_{D+N+1}, \ldots, l_d$. The original parallelepiped is covered 2^{D+N} times. The $2(D+N)(d-1)$-dimensional hypersurfaces with nonperiodic boundary conditions of the original parallelepiped now are $(d-1)$-dimensional interfaces between sheets of this covering space at $x_1 = 0$, $x_1 = l_1$, $x_2 = 0, \ldots, x_{D+N} = 0$, and $x_{D+N} = l_{D+N}$.

Since a general periodic ray crosses pairs of boundary surfaces, $\beta_\gamma = 0$ for periodic and Neumann as well as for Dirichlet boundary conditions on opposing pairs of hypersurfaces.[5] On this toroidal covering space there are periodic rays that lie entirely within the $(d-1)$-dimensional hypersurfaces that are projected onto the boundaries of the original parallelepiped. The contribution to the Casimir energy due to these rays depends on the imposed boundary conditions. The hypersurfaces of a parallelepiped in addition intersect on lower-dimensional hyperedges that also contain periodic rays of the covering space.

The corrections due to rays on these lower-dimensional tori are clearly visible in Eqs. (24) and (25). They are in one-to-one correspondence with the boundary surfaces, edges, etc., of the parallelepiped. The contribution of rays on the lower-dimensional surfaces has to be subtracted for Dirichlet boundary conditions because the field vanishes on the surface of the parallelepiped in this case. That the lower-dimensional correction in Eq. (24) is added for Neumann conditions then follows from the reflection symmetry of the parallelepiped, which implies Eq. (22).

The sign of the Casimir energy of a parallelepiped in general therefore is determined by the boundary conditions on its surfaces. One can argue that the Casimir energy due to periodic rays of a boundary surface is always negative when this surface is embedded in a higher-dimensional space of vanishing curvature: there are no caustics to contend with and the energy surface has no zero modes.

Whether the Casimir energy due to periodic rays on such a boundary surface has to be subtracted or added depends on the imposed condition. The sign of the overall Casimir energy of a parallelepiped therefore depends on the relative magnitude of boundary contributions with opposite sign. Which sign prevails in general will depend on the actual dimensions of the parallelepiped. The previous analysis nevertheless allows for a few general statements about the sign of the Casimir energy of a parallelepiped.

Since contributions from periodic rays on lower-dimensional boundary surfaces are always negative, the sign of the SCE of a parallelepiped with only Neumann and periodic boundary conditions is negative in any dimension.

Replacing Dirichlet by Neumann boundary conditions on a pair of parallel surfaces decreases the Casimir energy of the parallelepiped.

If one dimension of the parallelepiped is much smaller than all others, as for two parallel infinite hyperplanes, the difference in Casimir energy for Neumann and Dirichlet boundary conditions tends to vanish (since all contributions to the Casimir energy due to rays on lower-dimensional surfaces become negligible).

V. CONCLUSION

By defining the SCE through the required subtraction $\rho_0(E)$ in the spectral density one describes a class of systems whose vacuum energies can be compared. By definition the subtracted spectral density $\tilde{\rho}(E)$ of Eq. (3) gives a finite Casimir energy. It is approximated semiclassically by contributions due to classical periodic rays. For a massless scalar field on d-dimensional spheres and tori as well as on related spaces with boundary such as half spheres and parallelepipeds, this SCE coincides with other definitions whenever the subtractions in the spectral density are the same for a nontrivial class of systems (see Appendix A).

(One may argue that certain systems, such as wedges formed by two semi-infinite planes joined at the common edge [3], do not have classical periodic rays and that this semiclassical approach may thus be rather limited in scope. However, the implicit subtraction is not universal in this case, and depends on the opening angle of the wedge in a nonlinear fashion [14] that prohibits comparisons between systems with wedges of different opening angle. The extracted finite Casimir energy of a particular wedge in this case is of little physical significance and for instance does *not* determine the torque between the two plates of the wedge.)

The geometrical description of the SCE in terms of periodic rays gives insights into qualitative features of this part of the vacuum energy that otherwise are rather mysterious. The need for an explanation of the sign of Casimir energies was very nicely formulated by Sucher: "Understanding the signs is a sign of understanding" [48]. I have presented some evidence that the sign of the SCE depends critically on op-

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[5]$\beta_\gamma = $even$\geq 2$ if the scalar field satisfies Neumann's condition on one, but Dirichlet's condition on the other of a pair of parallel hypersurfaces. Such asymmetric boundary conditions on a pair of parallel surfaces can give a change in sign of the semiclassical Casimir energy [15,48]. It is due to a phase loss of π for some dominant primitive periodic rays and can be explained along the lines used for manifolds without boundary [15].

tical properties of "important" (short) periodic rays.

The semiclassical contribution to the Casimir energy due to an isolated periodic ray has a definite sign [see Eq. (9)]. The contribution due to a class of periodic rays of an integrable system also is of a definite sign [see Eq. (16)]. For isolated periodic rays, the sign is determined by the winding number σ_γ of their stable and unstable manifolds [32], whereas it essentially is given by the Keller-Maslov index $\beta_\mathbf{n}$ of a class of periodic rays in integrable systems. [When the Hessian of Eq. (14) is not positive definite, the sign of the contribution of a class of rays is given by Eq. (17).]

The semiclassical expressions of Eqs. (9) and (16) suggest that the Casimir energy very often is dominated by the shortest periodic rays or class of periodic rays. Arguing that the contribution of the shortest periodic rays is continuous under small deformations of the manifold, one can infer the sign of the Casimir energy of integrable systems in two ways: either by direct computation of the index in Eq. (17), or by slight deformation of the integrable system and computation of the index of Eq. (7) for the shortest periodic rays. Continuity under deformations also explains the pattern of signs for the Casimir energies of a massless scalar on spheres and on tori of various dimensions as well as changes in sign when spheres are strongly deformed. Sign changes of the SCE in such cases are accompanied by a change in the number of conjugate points of the shortest rays.

The sign of the SCE is harder to determine for spaces with boundary. The shortest periodic rays again dominate, but the sign (and magnitude) of contributions due to periodic rays that lie within the boundary depends on the boundary conditions. Such boundary rays can be among the shortest periodic rays and in some cases dominate the Casimir energy.

Only d-dimensional half spheres and general parallelepipeds with Neumann and Dirichlet boundary conditions were considered. However, the general arguments remain valid in more realistic situations with, for instance, electromagnetic fields. The sign of the Casimir energy of spherical and cylindrical cavities with idealized metallic boundary conditions can apparently be understood in terms of the phases of the shortest periodic rays [49].

Determining the sign of a SCE in this sense is reduced to a problem of geometrical optics. For certain simple manifolds, such as tori and spheres, one finds that *all* contributions due to periodic rays are of the same sign. The sign of the SCE is unambiguous in this case. The fact that classical dynamics of periodic trajectories seems to determine the sign of the Casimir energy raises the intriguing question whether the sign is a topological characteristic of the phase space. In general, and in particular for manifolds with boundaries, periodic rays of comparable length contribute to the SCE with opposite sign. It then depends on the boundary conditions and metric characteristics of the space whether the SCE is positive or negative.

APPENDIX A: CASIMIR ENERGIES AND CURVATURE CORRECTIONS: SPHERES AND HALF SHELLS

I here explicitly compute the Casimir energies due to a scalar field for low-dimensional spheres S_1, S_2, S_3, and S_4 as well as for two-dimensional half-shells with Neumann and Dirichlet boundary conditions. Of special interest are the associated subtractions in the spectral density that render the Casimir energies finite. As explained in the main text, these subtractions determine classes of systems with finite vacuum energy differences. The finite Casimir energy in the examined cases is given by contributions due to periodic rays whenever the subtractions are universal.

1. The circle

The Casimir energy of a scalar field on S_1 probably is the most transparent example. The curvature vanishes and the energy spectrum of a massless field on S_1 clearly is $E_l = l\hbar c/R$ for integer $l \geq 0$. For $l > 0$ the energy eigenvalues are twofold degenerate. The vacuum energy of a massless scalar on S_1 thus is formally given by

$$E_\text{vac}(S_1) = \frac{\hbar c}{2R} \sum_{l=1}^\infty 2l. \tag{A1}$$

One may regularize this divergent expression in many ways. One of the more popular is by analytic continuation in the exponent s of the (for $s > 1$ manifestly convergent) sum $\zeta(s) = \Sigma_1^\infty l^{-s}$. This method, known as zeta function regularization, has been claimed [41] to be not just the most elegant, but also the only rigorous mathematical definition of the sum in Eq. (A1). That may be true but gives appreciably little insight into the physical significance of the finite Casimir energy one obtains, which for a circle is $-(\hbar c)/(12R)$.

Let us subtract from the spectral density $\rho_{S_1}(E; S_1) = \Sigma_{l=0}^\infty 2\delta(E - \hbar cl/R)$ the smooth Weyl density $\rho_0(E) = (2\pi R)/(\pi\hbar c)$. The SCE of a circle then is *defined* to be

$$\mathcal{E}_c(S_1) = \frac{1}{2} \int_0^\infty [\rho(E; S_1) - \rho_0(E)] E\, dE$$

$$= \frac{1}{2} \int_0^\infty \left[\sum_{l=0}^\infty 2\delta(E - \hbar cl/R) - \frac{2R}{\hbar c} \right] E\, dE$$

$$= -\frac{1}{\pi} \text{Im} \frac{2\pi R}{i\hbar c} \sum_{n=1}^\infty \int_0^\infty E\, dE\, e^{in2\pi RE/(\hbar c)}$$

$$= -\frac{\hbar c}{2\pi^2 R} \sum_{n=1}^\infty \frac{1}{n^2}$$

$$= -\frac{\hbar c}{12R}. \tag{A2}$$

The second line of Eq. (A2) expresses the subtracted spectral density in terms of the semiclassical contribution due to periodic rays of length $2\pi Rn, n > 0$. The final answer coincides with that of zeta function regularization, but one now explicitly *knows* the implicit subtraction in the spectral density required to obtain it. The subtracted Weyl contribution to the spectral density is proportional to the circumference of the circle. The difference in Casimir energies of two circles thus is *not* the difference in their vacuum energy. It in fact would cost an infinite amount of energy to change the radius of the circle by a finite amount.

The subtraction nevertheless is universal in the sense that the (so defined) finite Casimir energy reproduces all derivatives of the vacuum energy with respect to the radius of second and higher order. The difference in total Casimir energy, for instance, gives the energy required to change the *relative* radii of disjoint circles while keeping their sum constant.[6]

2. The two-sphere

The two-dimensional sphere S_2 is a prototypical manifold with positive curvature. The Casimir energy of a scalar field on S_2 depends on how the scalar couples to the curvature. It is known [6,20,39,40], and we will verify, that this coupling to the curvature is of a particular strength for a massless field. To compare with Ref. [41], let us for the moment ignore the coupling to the curvature.

The eigenfunctions of the Laplace-Beltrami operator on a two-sphere are the spherical harmonics with $(2l+1)$-fold degenerate eigenvalues. Without coupling to the gaussian curvature of S_2, the energy eigenvalues of a scalar field are $E_l = \hbar c \sqrt{l(l+1)}/R$. The vacuum energy of a two-sphere of radius R in this case is formally given by the divergent sum

$$E_{\text{vac}}(S_2) = \frac{\hbar c}{2R} \sum_{l=0}^{\infty} (2l+1)\sqrt{l(l+1)}. \quad (A3)$$

Zeta function regularization (ZR) of Eq. (A3) gives [41] the *negative* value

$$\mathcal{E}_{\text{ZR}}(S_2) = \lim_{s \to 1/2} \frac{\hbar c}{2R} \sum_{l=0}^{\infty} (2l+1)[l(l+1)]^s = -0.132548 \frac{\hbar c}{R} \quad (A4)$$

for this Casimir energy. A physically perhaps more transparent treatment of the sum in Eq. (A3) by heat kernel regularization (HK) gives

$$\mathcal{E}_{\text{HK}}(S_2) = \lim_{\varepsilon \to 0_+} \frac{\hbar c}{2R} \left\{ \sum_{l=0}^{\infty} (2l+1)\sqrt{l(l+1)} e^{-\varepsilon l(l+1)} - \frac{\sqrt{\pi}}{2\varepsilon^{3/2}} \right\}$$

$$= -0.132548 \frac{\hbar c}{R}. \quad (A5)$$

Subtracting the integral over $l \in [0, \infty]$ from the sum over

angular momenta thus leads to the same finite result for the Casimir energy as zeta function regularization. However, no obvious *physical* principle apart from finiteness of the result seems to dictate the particular subtraction in Eq. (A5). (For a justification and the relation to subtractions in other regularization schemes see Ref. [10].)

A straightforward physical interpretation of the subtraction is possible in the semiclassical treatment: one again subtracts a smooth "classical" part of the spectral density. $\rho_0(E)$ should not depend on the detailed shape of the surface. For a two-dimensional manifold without boundaries, one may subtract the classical Weyl contribution to the spectral density proportional to the area A of the manifold, $\rho_0(E) = AE/[2\pi(\hbar c)^2]$. The subtracted spectral density is

$$\tilde{\rho}(E) = \rho(E) - \rho_0(E)$$

$$= \left[\sum_{l=0}^{\infty} (2l+1)\delta(E - \hbar c \sqrt{l(l+1)}/R) \right] - \frac{2R^2 E}{(\hbar c)^2}. \quad (A6)$$

$\tilde{\rho}(E)$ can be expressed in terms of semiclassical contributions from periodic rays [20,50]

$$\tilde{\rho}(E) = -\frac{1}{\pi} \text{Im} \frac{4\pi R^2 E}{i(\hbar c)^2} \sum_{n=1}^{\infty}$$

$$\times \exp\left\{ in\left[\frac{2\pi R}{\hbar c} \sqrt{E^2 + \left(\frac{\hbar c}{2R}\right)^2} - \pi \right] \right\}. \quad (A7)$$

One verifies that Eq. (A7) is equivalent to Eq. (A6) by recognizing that apart from a $n=0$ term, the real part of the sum over n in Eq. (A7) is a periodic δ distribution. The sum in Eq. (A7) on the other hand can be interpreted as due to contributions from classical periodic rays that wind n times about a geodesic of the two-sphere with a total length $L_n = 2\pi Rn$.

The Casimir energy corresponding to the spectral density $\tilde{\rho}(E)$ of Eq. (A7) is

$$\mathcal{E}_c(S_2) = \frac{1}{2} \int_0^{\infty} E\tilde{\rho}(E)dE$$

$$= \frac{\hbar c}{4R} \text{Re } e^{3i\pi/4} \sum_{n=1}^{\infty} \int_0^{\infty} (1+i\xi)\sqrt{\xi(2+i\xi)} e^{-n\pi\xi} d\xi$$

$$= \frac{\hbar c}{4R} \text{Re}(i-1) \int_0^{\infty} \frac{(1+i\xi)\sqrt{\xi(1+i\xi/2)}}{e^{\pi\xi}-1} d\xi$$

$$= -0.132548 \cdots \frac{\hbar c}{R}, \quad (A8)$$

where the last number was obtained by numerical evaluation of the integral.[7] The SCE thus coincides with that of zeta function and heat kernel regularization. The Casimir energy

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[6]The situation is unstable due to the minus sign of the Casimir energy in Eq. (A2). It implies that the energy is minimized by shrinking all but one of the circles to a point.

[7]Substitution of the variable $i\xi(E) = \sqrt{1 + \left(\frac{2RE}{\hbar c}\right)^2} - 1$ and a clockwise rotation of the integration contour by $90°$ gives the integral that converges exponentially and is numerically well behaved.

of a spherical shell is completely described in terms of its classical periodic rays. Moreover, this procedure gives a transparent physical meaning to the subtraction: the spectral density $\tilde{\rho}(E)$ is the difference to the universal Weyl term in density for two-dimensional surfaces of the same area. The Casimir energy thus does *not* give the energy required to change the radius of S_2: the subtracted (divergent) term proportional to the surface area of the sphere is far more important for this energy difference [29]. The Casimir energy on the other hand *does* allow the computation of the energy required to *deform* the sphere into another smooth manifold without boundary of the *same* total area. It thus for instance makes sense to speak of the energy required to change a two-sphere to an oblate or an elongated ellipsoid of the *same* area. The energy required for such deformations is finite.

The effective action of a periodic ray of arc length $L_n = 2\pi Rn$ in Eq. (A7) is

$$S_n = \oint \mathbf{p} \cdot \mathbf{dx} = p(E)L_n, \qquad (A9)$$

with

$$p^2(E) = (E/c)^2 + \hbar^2/(4R^2). \qquad (A10)$$

Equation (A10) is not the dispersion relation one expects for a massless particle. $p^2(E)$ differs from $(E/c)^2$ by a term of order $\hbar^2$ that is proportional to the Gaussian curvature $\kappa = 1/R^2$ of S_2. The curvature of the manifold results in a dispersion relation that would correspond to a tachyon with velocity $v(E) = (dp/dE)^{-1} = c\sqrt{1 + [\hbar c/(2RE)]^2} > c$. This can also be seen by rewriting

$$E_l^2 = \frac{(\hbar c)^2}{R^2} l(l+1) = \left(\frac{\hbar c(l+1/2)}{R}\right)^2 - \left(\frac{\hbar c}{2R}\right)^2. \quad (A11)$$

Interpreting $\hbar(l+1/2)/R$ as the eigenvalue of the momentum operator for a field satisfying anti-periodic boundary conditions on a circle of radius R, this dispersion clearly is tachyonic: the effective mass squared is negative $m^2 = -(\hbar/2cR)^2$. The violation of Huygens' Principle without (appropriate) coupling to the curvature was already emphasized in Ref. [39]. The fact that the effective action in Eq. (A9) does not vanish for $E=0$ is another indication that the spectrum of the Laplace-Beltrami operator on curved spaces is not the spectrum of a massless scalar field.

A particle is massless if its dispersion is $p(E) = E/c$. The generalization of the wave equation for d-dimensional Riemannian manifolds $\mathcal{M}$ with metric g_{ij} and Gaussian curvature κ includes a coupling to the curvature. The appropriate wave equation for a $(d+1)$-dimensional (ultrastatic) spacetime with curvature is $\Delta_d \phi = c^{-2} \partial_t^2 \phi$, where

$$\Delta_d = (1/\sqrt{g}) \frac{\partial}{\partial x^i} \sqrt{g} g^{ij} \frac{\partial}{\partial x^j} - \frac{(d-1)^2}{4} \kappa. \qquad (A12)$$

This curvature correction to the Laplace-Beltrami operator of $\mathcal{M}$ also arises naturally from the measure of the path integral

and is consistent with the required short-time behavior of the Feynman propagator [40]. The particular strength of the coupling to the curvature in Eq. (A12) preserves conformal invariance[8] of the wave equation [6]. For a d-dimensional sphere S_d of radius R, $\kappa = 1/R^2$ and the eigenvalues of $-\Delta_d$ defined in Eq. (A12) are $[l+(d-1)/2]^2/R^2$ for integer $l \geq 0$. The Casimir energy due to a *massless* scalar field on a two-sphere thus is

$$\mathcal{E}_c(S_2; m=0)$$

$$= \int_0^\infty \frac{EdE}{2} \left[\sum_{l=0}^\infty (2l+1)\delta(E - \hbar c(l+1/2)/R) - \frac{2R^2 E}{(\hbar c)^2} \right]$$

$$= -\frac{1}{2\pi} \operatorname{Im} \frac{4\pi R^2}{i(\hbar c)^2} \sum_{n=1}^\infty \int_0^\infty E^2 dE \exp\left[in\left(\frac{2\pi R}{\hbar c}E - \pi\right) \right]$$

$$= \operatorname{Im} \frac{\hbar c}{4\pi^3 R} \sum_{n=1}^\infty \frac{(-1)^n}{n^3} = 0. \qquad (A13)$$

That the Casimir energy of a massless scalar field vanishes for a two-sphere agrees with the arguments presented in Secs. II A and III. Zeta function- and heat kernel regularization also give a vanishing Casimir energy when the coupling of a massless scalar to the curvature is included [34]. Note that the average spectral density of a two-dimensional manifold of the same area again was subtracted to obtain the finite answer of Eq. (A13). Although the second line of Eq. (A13) is a sum over contributions due to periodic rays of the two-sphere, it does not have the form of Eq. (6). The amplitude in particular is proportional to the energy here. As discussed in Sec. III, free motion on a two-sphere is integrable and the corresponding periodic rays are not isolated. As shown in Appendix B, a semiclassical treatment along the lines of Ref. [35] gives the semiclassical response function of Eq. (A13) without reference to the exact quantum mechanical spectrum.

Let us now turn to a manifold with boundary and consider a two-dimensional half-sphere $S_2/2$ with Neumann (N) or Dirichlet (D) boundary conditions on the equatorial circle.[9] Due to the symmetry of the two-sphere upon reflection about its equator, eigenfunctions of the Laplace-Beltrami operator either satisfy Neumann or Dirichlet conditions on the equator and thus $\mathcal{E}(S_2/2; N) + \mathcal{E}(S_2/2; D) = \mathcal{E}(S_2) = 0$. Since one more mode satisfies Neumann's boundary condition than Dirichlet's for every energy eigenvalue, one has for the Casimir energy of a half-sphere

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$$\mathcal{E}_c(S_2/2; {}^N_D) = \pm \int_0^\infty \frac{EdE}{4} \left[\sum_{l=0}^\infty \delta(E - \hbar c(l+1/2)/R) - \frac{R}{\hbar c} \right]$$

$$= \mp \frac{1}{\pi} \operatorname{Im} \frac{2\pi R}{i\hbar c} \sum_{n=1}^\infty \int_0^\infty \frac{EdE}{4}$$

$$\times \exp\left[in\left(\frac{2\pi R}{\hbar c} E - \pi \right) \right]$$

$$= \mp \frac{\hbar c}{8\pi^2 R} \sum_{n=1}^\infty \frac{(-1)^n}{n^2} = \pm \frac{\hbar c}{96R}. \qquad (A14)$$

This again agrees with zeta function regularization for this case. Note that an additional universal subtraction is necessary. It is a Weyl contribution to the spectral density proportional to the *length* of the boundary. Due to the subtraction of terms proportional to the overal area of the surface and length of its boundary, the Casimir energy of a massless scalar on the half-sphere can be compared to the Casimir energy on another smooth two-dimensional manifold of the *same area* and with a smooth boundary of the *same length* only.

Note that the necessary subtraction for a half-sphere with boundary is *not* universal if the curvature correction to the Laplace-Beltrami operator in Eq. (A13) is ignored. In this case one obtains the following difference in Casimir energies

$$\mathcal{E}(S_2/2;N) - \mathcal{E}(S_2/2;D)$$

$$= \int_0^\infty \frac{EdE}{2} \left[\sum_{l=0}^\infty \delta(E - \hbar c\sqrt{l(l+1)}/R) - \frac{R}{\hbar c} \frac{E}{\sqrt{E^2 + \left(\frac{\hbar c}{2R}\right)^2}} \right]$$

$$= -\frac{1}{\pi} \operatorname{Im} \frac{2\pi R}{i\hbar c} \sum_{n=1}^\infty \int_0^\infty \frac{EdE}{2}$$

$$\times \frac{E}{\sqrt{E^2 + \left(\frac{\hbar c}{2R}\right)^2}} e^{in((2\pi R/\hbar c)\sqrt{E^2 + (\hbar c/2R)^2} - \pi)}$$

$$= \frac{\hbar c}{4R} \operatorname{Re}(i-1) \int_0^\infty d\xi \frac{\sqrt{\xi(1+i\xi/2)}}{e^{\pi\xi} - 1}$$

$$= -0.110687 \cdots \frac{\hbar c}{R}. \qquad (A15)$$

The subtracted spectral density $\rho_0(E)$ in Eq. (A15) is again proportional to the length of the boundary, but it now depends on the Gaussian curvature $\kappa = 1/R^2$ as well. Expanding for small values of the curvature $E/\sqrt{E^2 + \left(\frac{\hbar c}{2R}\right)^2} \sim 1 - \kappa(\hbar c)^2/(8E^2)$ shows the presence of a (logarithmic) divergent contribution to the vacuum energy that is proportional to $1/R$. No derivative of the vacuum energy with respect to the radius of the half-shell is finite and the subtracted (finite) Casimir energy of Eq. (A15) is of no physical significance.

In zeta function regularization the logarithmic divergence manifests itself as a pole at $s = -1/2$ whose subtraction is ambiguous [21,22]. Taking only the principal value of this

pole contribution to the Casimir energy gives a finite value [41] of $-0.166080\hbar c/R$ for the above difference of the Casimir energies of half-shells with Neumann and Dirichlet boundary conditions. Since the implicit subtraction is not universal, it should not surprise that the value ascribed by this prescription differs from the one in Eq. (A15)—obtained by prescribing a particular subtraction of the spectral density. The finite part of the Casimir energy of this system in fact is quite arbitrary since its vacuum energy cannot be compared to that of any other system anyway. The appearance of a pole in zeta function regularization thus indicates that the associated implicit subtraction in the spectral density is not universal.

If one is of the opinion that a Casimir energy due to a physical field should enjoy some degree of universality, small deformations of a smooth manifold should not require an infinite amount of energy. The lack of universality of the Casimir energy on a half-shell in this sense is yet another indication that the spectrum of the Laplace-Beltrami operator is not that of a physical particle. The argument perhaps is more convincing if one notes that even the Casimir energy of a scalar on S_3 (a manifold without boundary that until recently could have served as a model for the spatial part of our universe) is not universal [41] without curvature correction.

3. S_3 and S_4

For comparison with the results of Ref. [41] and in support of the arguments for the sign of the Casimir energy of a massless scalar field on d-dimensional spheres of Secs. II A and III, I here state the universal Casimir energies of a massless scalar field on $(d=3)$- and $(d=4)$-dimensional spheres. The appropriate coupling to the curvature for a massless scalar of Eq. (A12) has been used and the results therefore differ from those without curvature correction in [41]. The difference is not minor: the Casimir energy of a scalar on S_3 in particular is not universal and devoid of physical implications if this coupling to the curvature is ignored. In zeta function regularization this is indicated by the presence of a pole [21,22,41].

The Casimir energies due to a scalar on S_3 and S_4 can be obtained in much the same manner as for the two-sphere—the main difference is the degeneracy of the spectrum of the Laplace-Beltrami operator in Eq. (A12). It is $(l+1)^2$ for S_3 and $(l+1)(l+2)(2l+3)/6$ for S_4. As can be seen from Eq. (A12), the spectrum of a massless field with the appropriate coupling to the curvature is $E_l = \hbar c(l+1)/R$ on S_3 and $E_l = \hbar c(l+3/2)/R$ on S_4. One then obtains

$$\mathcal{E}_c(S_3; m=0) = \frac{\hbar c}{240R} = +0.0041666 \cdots \frac{\hbar c}{R},$$

$$\mathcal{E}_c(S_4; m=0) = 0. \qquad (A16)$$

These results again are in agreement with those of zeta function regularization [34] for a conformally coupled scalar. There are no logarithmic divergent subtractions and no poles appear in zeta function regularization with the appropriate

coupling to the curvature. For S_4 the subtraction of the Weyl contribution proportional to the four-volume of S_4 is not sufficient to obtain a finite Casimir energy. An additional smooth spectral density proportional to R^2 must be subtracted and

$$\rho_0(E;S_4) = \frac{\pi R^4 E^3}{3(\hbar c)^4}\left[1 - \left(\frac{\hbar c}{2RE}\right)^2\right]. \qquad (A17)$$

The additional term in the smooth part of the spectral density is proportional to the *integrated* local curvature of the manifold (see Refs. [10,28,52,53]). One thus can compare the (vanishing) Casimir energy of a massless scalar field on S_4 with that of other four-dimensional manifolds without boundary of the same four-volume and average curvature.

APPENDIX B: SEMICLASSICAL DERIVATION OF THE SPECTRAL DENSITY OF A SCALAR ON A TWO-SPHERE

For completeness I here give the semiclassical calculation of the Casimir energy $\mathcal{E}_c(S_2)$ of Eq. (A13) for a massless scalar particle on a two-sphere without reference to the quantum-mechanical spectrum. Although a somewhat trivial example of the more general asymptotic expansion for elliptic self-adjoint pseudodifferential operators on Riemannian manifolds [54,55], this calculation does illustrate some of the basic ideas at an elementary level and is an example of the general procedure of Sec. III.

The system has two constants of motion in involution and is thus integrable. One of these can be taken to be the square or magnitude of the momentum $|p|$, the other the angular momentum l_z. The corresponding action variables for a two-sphere of radius R are $I_1 = R|p| \geq 0$ and $I_2 = l_z$ with $|I_2| \leq I_1$. The Hamiltonian for a free massless particle on the two-sphere is $H(I_1, I_2) = c|p| = cI_1/R$. Following Refs. [20,35], the oscillating part of the semiclassical spectral density is

$$\tilde{\rho}(E;S_2) = \frac{1}{\hbar^2}\sum_{n,m=-\infty}^{\infty}{}' \int_0^{\infty} dI_1 \int_{-I_1}^{I_1} dI_2 \delta[E - H(I_1, I_2)]$$
$$\times e^{2\pi i[(nI_1 + mI_2)/\hbar - n/2]}. \qquad (B1)$$

The integration domain of the action variables in Eq. (B1) is due to classical considerations only and the primed sum in Eq. (B1) here implies that the (classical) contribution with $m = n = 0$ is to be omitted. Note that the $m = n = 0$ term is just the $\rho_0(E)$ that is subtracted from the full spectral density. The spectral density evidently is given in terms of contributions from periodic trajectories that wind (n, m) times about the (in this case two-dimensional) invariant tori of constant energy. The phase retardation by $n\pi$ of the family of trajectories that wind n-times about the whole two-sphere is related to the uniqueness of the quantum mechanical wave function [42] and might not be expected classically. Keller's construction

described in Sec. III A shows that this phase loss is due to the two caustics of first order formed by families of periodic rays of the same inclination.[10] Inserting Eq. (B1) in Eq. (4) and performing the integral over E one obtains

$$\mathcal{E}_c(S_2) = \frac{1}{2\hbar^2}\sum_{n,m=-\infty}^{\infty}{}' \int_0^{\infty} dI_1 \int_{-I_1}^{I_1} dI_2 H(I_1, I_2) e^{2\pi i[(nI_1 + mI_2)/\hbar - n/2]}. \qquad (B2)$$

Although a direct consequence of Eq. (B1), it perhaps is remarkable that the SCE of an integrable system is expressible in terms of the Fourier transform of the classical Hamiltonian with respect to action variables. Since the Hamiltonian $H(I_1, I_2) = cI_1/R$ does not depend on I_2 and one is interested in the term of order $\hbar$ only, one can perform the integral over I_2 in saddle point approximation. This gives

$$\int_{-I_1}^{I_1} dI_2 e^{2\pi i m I_2/\hbar} = \hbar \int_{-I_1/\hbar}^{I_1/\hbar} dx e^{2\pi i m x} = 2I_1 \delta_{m0} + O(\hbar). \qquad (B3)$$

The only sizable contribution to the I_2 integral thus is from the $m = 0$ term of the sum. Inserting Eq. (B3) in Eq. (B2) one finally obtains for the SCE due to a massless scalar on S_2,

$$\mathcal{E}_c(S_2) = \frac{c}{R\hbar^2}\sum_{n=-\infty}^{\infty}{}'(-1)^n \int_0^{\infty} dI_1 I_1^2 e^{2\pi i n I_1/\hbar}$$
$$= \frac{-i\hbar c}{(4\pi^3)R}\sum_{n=-\infty}^{\infty}{}'\frac{(-1)^n}{n^3} = 0. \qquad (B4)$$

The integral over I_1 in Eq. (B4) coincides with that over the energy in Eq. (A13); recall that $I_1 = ER/c$. Note that higher moments of the semiclassical spectral density do not vanish. One in particular finds that the second moment of the semiclassical spectral density is negative

$$\langle E^2 \rangle_c(S_2) = 12\left(\frac{\hbar c}{4\pi^2 R}\right)^2 \sum_{n=-\infty}^{\infty}{}'\frac{(-1)^n}{n^4} = -\frac{7}{480}\left(\frac{\hbar c}{R}\right)^2. \qquad (B5)$$

The contribution of periodic rays to the spectral density of a massless scalar thus cannot be positive semi-definite for S_2. Upon subtracting the "classical" Weyl contribution to the spectral density there in fact is no reason why the remainder must be positive. The negative sign in Eq. (B5) can be traced to the phase loss of π for every revolution of a periodic ray and thus ultimately to the Keller-Maslov index of a class of periodic rays.

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[10]For fixed energy the inclination of a periodic orbit is determined by $l_z = I_2$ on S_2

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