Relativistic corrections in a three-boson system of equal masses

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Three-body systems of scalar bosons are covariantly described in the framework of relativistic constraint dynamics. With help of a change of variables followed by a change of wave function, two redundant degrees of freedom get eliminated and the mass-shell constraints can be reduced to a three-dimensional eigenvalue problem. In general, the reduced equation obtained by this procedure involves the spectral parameter in a nonconventional manner, but for three equal masses a drastic simplification arises at the first post-Galilean order: the reduced wave equation becomes a conventional eigenvalue problem that we treat perturbatively, computing a first-order correction beyond the nonrelativistic limit. The harmonic interaction is displayed as a toy model.

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I. INTRODUCTION, BASIC EQUATIONS

A relativistic system of mutually interacting particles can be described, in a manifestly covariant way, by mass-shell constraints. These constraints determine the evolution of a wave function which depends on four-dimensional arguments [1–3]. The price paid for covariance is the presence of redundant degrees of freedom, just like in the Bethe-Salpeter (BS) approach.

For two-body systems, the extra degree of freedom is trivially factorized out. Moreover, in that case, the contact with the BS equation was established [4].

In contrast, for three or more particles it is difficult to find an interaction term such that the mass-shell constraints are compatible among themselves, respect Poincaré invariance, reproduce free-body motion when this term vanishes, and allow for eliminating the redundant degrees of freedom.

In this paper we focus on the case of three spinless particles. We thus consider three Klein-Gordon equations coupled by a mutual interaction which should be either derived from the underlying field theory (QCD, for instance) or motivated by phenomenological considerations. Our basic equations are

$$2K_a\Phi \equiv (p_a^2 + 2W)\Phi = m_a^2 c^2 \Phi, \quad a,b = 1,2,3, \qquad (1)$$

for a wave function with three four-dimensional arguments (either q_a or p_b according to the representation used). The relativistic "potential" W must be Poincaré invariant and chosen such that the equations above are mutually compatible.

Remark. In principle it seems that one could also consider more general equations involving three distinct relativistic potentials W_1, W_2, and W_3. But even if we leave aside the problem of mutual compatibility which would become more complicated, in this more general case there is no evidence that the superfluous degrees of freedom could be eliminated at all. It is therefore natural to focus on the simple class of models characterized by $W_a = W$. This choice is reminiscent of what is currently done in the two-body case, where using the same interaction function in both wave equations is general enough to accommodate most realistic situations [5,6].

For three particles, the assumption of a single interaction function in Eqs. (1) will be justified *a posteriori* by its efficiency in the task of reducing the degrees of freedom. Of course in a future work, a justification by a contact with field theory would be desirable.

It should be emphasized that having $W_a = W$ by no means forbids us from taking into account differences in the couplings that concern each particle. In most systems of practical interest W is a sum of three terms; each one of these three terms, although it is not strictly binary (due to the three-particle forces automatically included), still carries some two-body input (the three-body forces being of higher order).

This will be seen for instance in the covariant harmonic potential of Sec. IV, where the potential given by Eq. (51) includes three distinct spring constants permitting us to implement a different interaction law inside each cluster; for electromagnetic interactions, different charges could be handled in a similar way.

Poincaré algebra is realized in the same manner as for the free-particle case—say, $P = p_1 + p_2 + p_3$ and $M = \Sigma q_a \wedge p_a$. It is convenient to introduce relative variables with the "heliocentric" notation: relative-particle indices are $A,B = 2,3$. We define the four-vectors

$$z_A = q_1 - q_A, \quad y_B = \frac{P}{3} - p_B. \qquad (2)$$

Their transverse parts are $\tilde{z}_A, \tilde{y}_B$. The tilde denotes the projection orthogonal to P—for instance, $\tilde{z}_A = z_A - (z_A \cdot P)P/P^2$, etc.

With help of the identity

$$3\sum p^2 \equiv P^2 + D + 6P^2\Xi, \qquad (3)$$

where

$$D = 6(\tilde{y}_2^2 + \tilde{y}_3^2 + \tilde{y}_2 \cdot \tilde{y}_3), \qquad (4)$$

$$\Xi = (P^2)^{-2}[(y_2 \cdot P)^2 + (y_3 \cdot P)^2 + (y_2 \cdot P)(y_3 \cdot P)], \qquad (5)$$

the sum of Eqs. (1) yields a dynamical equation involving the potential. On the other hand, the differences of Eqs. (1) take on the purely kinematic form

$$(p_1 - p_A)(p_1 + p_A)\Phi = 2\nu_A c^2 \Phi, \tag{6}$$

where the half squared-mass differences are $\nu_A = \frac{1}{2}(m_1^2 - m_A^2)$. For the sake of compatibility we require that W commute with both products $(p_1 - p_A)(p_1 + p_A)$.

In order to achieve the elimination of two degrees of freedom, we have proposed [7] a quadratic change of variables in momentum space—say, $p_a \mapsto p_b'$, or equivalently,

$$P, y_A \mapsto P, y_A'.$$

This transformation can be characterized as a redefinition of the relative energies such that

$$(p_1 + p_A) \cdot (p_1 - p_A) = P \cdot (p_1' - p_A') \tag{7}$$

and by the requirement that it leave P, $\tilde{y}_2$, and $\tilde{y}_3$ unchanged—that is,

$$P' = P, \quad \tilde{y}_A' = \tilde{y}_A. \tag{8}$$

Clearly Eq. (7) determines in closed form the longitudinal pieces of p_2', p_3' (y_2', y_3') in terms of all the primitive variables p_a (P, y_A) [8].

Of course, we define the new relative momenta y_A' as linearly related with the p_a''s through a formula similar to Eqs. (2): namely,

$$y_A' = \frac{P}{3} - p_A'.$$

Note that our transformation preserves Poincaré invariance; as a result, the generators of spacetime displacements have the usual form also in terms of p_a'.

Naturally, this procedure gives rise to new configuration variables z_A'. In general, the new variables z_A', y_B' are referred to as *reducible*.

It is noteworthy that instead of Φ, we can equivalently use a new wave function

$$\Psi = |J|^{1/2}\Phi, \quad J = \frac{D(p_1, p_2, p_3)}{D(p_1', p_2', p_3')}. \tag{9}$$

Accordingly, the operators K and W are mapped to H and V, respectively,

$$H = |J|^{1/2} K |J|^{-1/2}, \quad V = |J|^{1/2} W |J|^{-1/2}. \tag{10}$$

In contrast to z_2, z_3, the operators z_A' are "formally Hermitian" (i.e., symmetric) with respect to the Hilbert space $L^2(\mathbf{R}^{12}, d^{12}p')$. In other words, setting

$$d^{12}p = d^4p_1\, d^4p_2\, d^4p_3, \quad d^{12}p' = d^4p_1'\, d^4p_2'\, d^4p_3',$$

we have

$$\int (z_A' \Upsilon)^* \Omega d^{12}p' = \int \Upsilon^* z_A' \Omega d^{12}p' \tag{11}$$

whenever Υ and Ω are square integrable in terms of the volume element $d^{12}p'$. Owing to Eqs. (10), H and V have the same property [9].

The results of Ref. [7] are as follows.

(a) Provided the three masses are not too much different one from another, Eqs. (7) and (8) can be inverted in closed analytic form.

Indeed our model is reliable insofar as, in the no-interaction limit obtained by putting the potential equal to zero, one recovers the free motion of independent particles. The discussion of this point in Ref. [7] led us to require

$$|\nu_2 + \nu_3| < \frac{1}{24}\sum m_a^2, \quad |\nu_2 - \nu_3| < \frac{1}{8}\sum m_a^2. \tag{12}$$

For instance, if only two masses are equal—say, $m_2 = m_3$—their value is allowed to deviate from m_1 by an amount of almost 6%.

(b) Conditions (7) amount to redefining the relative energies in such a way that the *new* relative energies can be eliminated.

(c) The compatibility conditions can be satisfied easily in terms of our *new* variables. Actually, in view of the compatibility requirement, a closed form of the interaction is available only in terms of the *new variables*.

A typical example of a potential satisfying compatibility and Poincaré invariance would be of the form

$$V = f((\tilde{z}_2')^2, (\tilde{z}_3')^2, \tilde{z}_2' \cdot \tilde{z}_3', P^2) \tag{13}$$

since any function of $\tilde{z}_B'$, $\tilde{y}_C'$, and P commutes with $y_A' \cdot P$. In this equation all the arguments of f are mutually commuting [10].

This situation is in favor of using Ψ and the new variables, as we shall do hereafter.

By our transformations, the difference equations (6) become

$$y_A' \cdot P\Psi = \left(\frac{4}{3}\nu_A - \frac{2}{3}\nu_B\right)c^2\Psi, \quad A \neq B, \tag{14}$$

and the dynamical equation of motion is mapped to

$$\left(3\sum m^2c^2 - P^2\right)\Psi = (D + 6P^2\Xi + 18V)\Psi. \tag{15}$$

In order to handle this equation we need to express Ξ in terms of the new variables.

Lengthy but elementary manipulations reported in [7] show that

$$\Xi = \xi^2 + \eta^2 + \eta\xi, \tag{16}$$

where ξ, η are determined by the system

$$\frac{2}{3}\xi + \frac{1}{3}\eta + \xi\eta + \frac{\eta^2}{2} = u, \tag{17}$$

$$\frac{2}{3}\eta + \frac{1}{3}\xi + \xi\eta + \frac{\xi^2}{2} = v, \tag{18}$$

u, v being determined as follows:

$$P^2 u = y_2' \cdot P + \frac{1}{2}y_3' \cdot P - \left(\tilde{y}_2 \cdot \tilde{y}_3 + \frac{1}{2}\tilde{y}_3^2\right), \tag{19}$$

$$P^2 v = y_3' \cdot P + \frac{1}{2}y_2' \cdot P - \left(\tilde{y}_2 \cdot \tilde{y}_3 + \frac{1}{2}\tilde{y}_2^2\right). \tag{20}$$

II. THREE-DIMENSIONAL REDUCTION

Now the dependence of Ψ on the *new* relative energies is easily factorized out, provided we assume a sharp linear momentum—say,

$$P^\alpha \Psi = k^\alpha \Psi, \quad k^2 = M^2 c^2, \tag{21}$$

for some constant timelike vector k. Let the caret denote the projection orthogonal to k. For instance, the transverse piece of z with respect to k is $\hat{z} = z - [(z\cdot k)/k^2]k$, etc. In the rest frame we have $\hat{y}_A^2 = -\mathbf{y}_A^2$, $\hat{z}_A^2 = -\mathbf{z}_A^2$, etc.

We make this convention that, in any operator F depending on the dynamical variables, the underline indicates that we replace $y_A' \cdot P$ by $\left(\frac{4}{3}\nu_A - \frac{2}{3}\nu_B\right)c^2$ and P^α by k^α, hence P^2 by M^2c^2. Let us write this symbolically

$$\underline{F} = \mathcal{S}\left(y_A' \cdot P = \left(\frac{4}{3}\nu_A - \frac{2}{3}\nu_B\right)c^2,\ P^\alpha = k^\alpha,\ F\right).$$

It is clear that F reduces to $\underline{F}$ on the "mass-momentum shell" (defined as the subspace of solutions to the mass-shell constraints which are eigenstates of total linear momentum).

Note that $\underline{\Xi}$ depends only on $\hat{y}_2, \hat{y}_3$.

Equation (15) yields the *reduced equation*

$$\left(3\sum m^2 - M^2\right)c^2\psi = [6(\hat{y}_2^2 + \hat{y}_3^2 + \hat{y}_2\cdot\hat{y}_3) + 18\underline{V} + 6M^2c^2\underline{\Xi}]\psi \tag{22}$$

for a reduced wave function ψ which depends on three-dimensional arguments only (say, $\hat{y}_2, \hat{y}_3$ in the momentum representation).

III. EQUAL MASSES

Fortunately, *in the case of three equal masses*—say, $m_a = m$—we have this further simplification that $\nu_A = 0$, which finally renders $\underline{u}, \underline{v}$ of the order of $1/c^2$. More precisely, Eqs. (19) and (20) entail

$$M^2c^2\underline{u} = -\left(\hat{y}_2\cdot\hat{y}_3 + \frac{1}{2}\hat{y}_3^2\right), \tag{23}$$

$$M^2c^2\underline{v} = -\left(\hat{y}_2\cdot\hat{y}_3 + \frac{1}{2}\hat{y}_2^2\right), \tag{24}$$

hence $\underline{u}$ and $\underline{v}$ to be inserted into the reduced version of the system (17) and (18). Solving for $\underline{\xi}, \underline{\eta}$ we obtain

$$\underline{\xi} = 2\underline{u} - \underline{v} + O(1/c^4), \tag{25}$$

$$\underline{\eta} = 2\underline{v} - \underline{u} + O(1/c^4). \tag{26}$$

Inserting this into Eq. (16) we get

$$\underline{\Xi} = 3(\underline{u}^2 + \underline{v}^2 - \underline{u}\cdot\underline{v}) + O(1/c^6), \tag{27}$$

correcting a misprint in the higher-order term of Eq. (100) of Ref. [7]. Since the leading term in $\underline{\Xi}$ is $O(1/c^4)$, we have defined Γ by setting

$$M^4c^4\underline{\Xi} = \Gamma = \Gamma_{(0)} + \frac{1}{c^2}\Gamma_{(1)} + \cdots, \tag{28}$$

where $\Gamma_{(0)}, \Gamma_{(1)}, \ldots$ remain finite when $c \to \infty$. In view of Eq. (27) it is clear that

$$M^4c^4\underline{\Xi} = 3M^4c^4(\underline{u}^2 + \underline{v}^2 - \underline{u}\cdot\underline{v}) + O(1/c^2).$$

We compute respectively $\underline{u}^2, \underline{v}^2$ and hence

$$\Gamma_{(0)} = \frac{3}{4}\{(\hat{y}_2^2)^2 + (\hat{y}_3^2)^2 + 4(\hat{y}_2\cdot\hat{y}_3)^2 + 2(\hat{y}_2^2 + \hat{y}_3^2)(\hat{y}_2\cdot\hat{y}_3) - \hat{y}_2^2\hat{y}_3^2\}, \tag{29}$$

an expression valid only for three equal masses (this formula was given in Ref. [7] without proof). Note that Γ is a positive operator and would survive in the absence of interaction.

For three equal masses, Eq. (22) takes on the form

$$(9m^2 - M^2)c^2\psi = 6(\hat{y}_2^2 + \hat{y}_3^2 + \hat{y}_2\cdot\hat{y}_3)\psi + 18\underline{V}\psi + \frac{6}{M^2c^2}\Gamma\psi. \tag{30}$$

Defining

$$6\lambda = (M^2 - 9m^2)c^2 \tag{31}$$

and using the rest frame (where $\hat{y}_A\cdot\hat{y}_B = -\mathbf{y}_A\cdot\mathbf{y}_B$) we can write

$$\lambda\psi = (\mathbf{y}_2^2 + \mathbf{y}_3^2 + \mathbf{y}_2\cdot\mathbf{y}_3)\psi - 3\underline{V}\psi - \frac{\Gamma}{M^2c^2}\psi. \tag{32}$$

Naturally $\Gamma_{(0)}$ admits an expression identical to Eq. (29) in terms of $\mathbf{y}_2, \mathbf{y}_3$.

In spite of being three dimensional, the reduced equation (32), as it stands, is more problematic than an ordinary eigenvalue problem. Even if the interaction does not depend on the total energy (that is, V does not depend on P^2), the term $P^2\Xi$ in Eq. (15), which has no counterpart in two-body systems and yields $M^2c^2\underline{\Xi}$ in Eq. (22), brings out some energy dependence. It follows that Eq. (32) is not a conventional eigenvalue equation: through Eq. (31) the operator to be diagonalized depends on its own eigenvalue. This complication is by no means a drawback special to our model. As emphasized in [11] it plagues most relativistic wave equations; the mathematical theory of this situation is rather involved, but fortunately this difficulty can be more easily handled in a perturbation scheme, *provided the unperturbed equation is not energy dependent*.

In the rest of this paper we focus on the first relativistic corrections. Therefore we solve Eq. (32) after expansion in powers of $1/c^2$, taking Eq. (31) into account—say, $M^2 = 9m^2 + 6\lambda/c^2$. In principle, the exact analytic expression for Γ is known and is itself a series in $1/c^2$. In fact knowledge of $\Gamma_{(0)}$ is sufficient for our purpose. Assuming that λ remains finite in the nonrelativistic limit, we select these solutions that are in some sense "close to" the nonrelativistic Schrödinger equation obtained by dropping $1/c^2$ in Eq. (32).

This development is justified insofar as the velocity of light can be considered as large with respect to some velocity formed with help of the physical parameters defining the

system. Practically, the constituent masses and the coupling constants involved in the interaction term must be combined as to form a quantity having the dimension of speed. In principle one should check that this "characteristic velocity" actually has something to do with the average velocities of the constituent particles in the slow-motion approximation.

The legitimacy and limitations of this procedure vary according to the analytic shape of the interaction term and must be discussed in each specific case.

A. Post-Galilean approximation

Let us start expanding in powers of $1/c^2$. Using the rest frame and assuming that

$$\psi = \psi_{(0)} + \frac{1}{c^2}\psi_{(1)} + \cdots, \quad \underline{V} = \underline{V}_{(0)} + \frac{1}{c^2}\underline{V}_{(1)} + \cdots, \quad (33)$$

the *zeroth-order* approximation to Eq. (32) yields the nonrelativistic limit

$$\lambda_0\psi_{(0)} - (\mathbf{y}_2^2 + \mathbf{y}_3^2 + \mathbf{y}_2 \cdot \mathbf{y}_3)\psi_{(0)} + 3\underline{V}_{(0)}\psi_{(0)} = 0. \quad (34)$$

Setting

$$E_{(0)} = \frac{\lambda_{(0)}}{m}, \quad U = -\frac{3}{m}\underline{V}_{(0)}, \quad (35)$$

Eq. (34) can be rewritten as

$$E_{(0)}\psi_{(0)} = \frac{1}{m}(\mathbf{y}_2^2 + \mathbf{y}_3^2 + \mathbf{y}_2 \cdot \mathbf{y}_3)\psi_{(0)} + U\psi_{(0)}, \quad (36)$$

which is similar to the Schrödinger equation of a nonrelativistic problem with three equal masses (*except perhaps* for complications resulting from a possible dependence of V on P^2). Indeed we consider equal masses, thus $m=2m_0$ where m_0 is the reduced mass of either of particles 2, 3, with respect to particle 1. The first operator in the right-hand side is nothing but the kinetic energy for a nonrelativistic system of three masses m, when the center-of-mass motion has been separated.

At the *first order* in $1/c^2$ we can, in the last term of Eq. (32), replace Γ which depends on M^2, by $\Gamma_{(0)}$, which does not. In view of Eq. (31), in this last term, we can also replace M^2 by $9m^2$. Hence

$$\lambda\psi = \left(\mathbf{y}_2^2 + \mathbf{y}_3^2 + \mathbf{y}_2 \cdot \mathbf{y}_3 - 3\underline{V} - \frac{\Gamma_{(0)}}{9m^2c^2}\right)\psi, \quad (37)$$

with $\Gamma_{(0)}$ biquadratic in $\mathbf{y}$. Inasmuch as V is not energy dependent, the above equation still has the structure of a nonrelativistic eigenvalue problem and can be solved by treating the last term as a perturbation.

More care is needed for most realistic potentials, for which V depends on P^2, and hence $\underline{V}$ depends on M^2c^2. Fortunately, in several cases, this dependence is of higher order, so that it can be accounted for by addition of an extra perturbation term, as follows. Assuming that $\underline{V}$ is as in Eqs. (33) we have

$$\lambda\psi = (\mathbf{y}_2^2 + \mathbf{y}_3^2 + \mathbf{y}_2 \cdot \mathbf{y}_3)\psi - 3\underline{V}_{(0)}\psi - \frac{1}{c^2}\left(\frac{\Gamma_{(0)}}{9m^2} + 3\underline{V}_{(1)}\right)\psi. \quad (38)$$

Since we do not go beyond first order, let us write $\lambda = \lambda_{(0)} + \frac{1}{c^2}\lambda_{(1)}$.

For any *nondegenerate* level λ, we have

$$\lambda_{(1)} = -\left\langle \frac{\Gamma_{(0)}}{9m^2} + 3\underline{V}_{(1)} \right\rangle, \quad (39)$$

where the expectation value must be calculated in the unperturbed eigenstate $\psi_{(0)}$.

Binding energy

Now we are in a position to calculate, at first post-Galilean order, the binding energy of a bound state. This quantity is usually defined through the (linear) mass defect [12]. So let us evaluate $M-\Sigma m = M-3m$. Taylor expansion of Eq. (31) yields

$$(M-3m)c^2 = \frac{\lambda}{m} - \frac{\lambda^2}{6m^3c^2} + O(1/c^4), \quad (40)$$

$$(M-3m)c^2 = \frac{\lambda_{(0)}}{m} + \frac{1}{c^2}\left(\frac{\lambda_{(1)}}{m} - \frac{\lambda_{(0)}^2}{6m^3}\right) + O(1/c^4), \quad (41)$$

which yields the first correction to binding energy.

B. Jacobi's coordinates

Equation (38) amounts to a nonrelativistic problem, formulated in terms of the canonically conjugate variables $\mathbf{z}'_A, \mathbf{y}'_B$. Before we turn to the harmonic interaction it is convenient to introduce *Jacobi's coordinates* that have the virtue of simplifying the expression of the kinetic energy. So we perform a *linear change* from $\mathbf{z}'_A, \mathbf{y}'_B$ to $\mathbf{R}_A, \mathbf{\Pi}_B$, as follows.

For *three equal masses*, the Jacobi coordinates $\mathbf{R}_2, \mathbf{R}_3$ associated with $\mathbf{q}'_2, \mathbf{q}'_3$, are defined by the formulas [13]

$$\mathbf{R}_2 = \mathbf{q}'_2 - \mathbf{q}'_3, \quad \mathbf{R}_3 = \frac{1}{\sqrt{3}}(2\mathbf{q}'_1 - \mathbf{q}'_2 - \mathbf{q}'_3), \quad (42)$$

in other words,

$$\mathbf{R}_2 = -\mathbf{z}'_2 + \mathbf{z}'_3, \quad \mathbf{R}_3 = \frac{1}{\sqrt{3}}(\mathbf{z}'_2 + \mathbf{z}'_3). \quad (43)$$

Inverting Eqs. (43) yields

$$\mathbf{z}'_2 = \frac{1}{2}(\sqrt{3}\mathbf{R}_3 - \mathbf{R}_2), \quad \mathbf{z}'_3 = \frac{1}{2}(\sqrt{3}\mathbf{R}_3 + \mathbf{R}_2). \quad (44)$$

Since Eqs. (43) are a linear transformation, it is easy to determine conjugate momenta—say, $\mathbf{\Pi}_2, \mathbf{\Pi}_3$—such that $[\mathbf{R}_2, \mathbf{\Pi}_2] = [\mathbf{R}_3, \mathbf{\Pi}_3] = i\delta$ and $[\mathbf{R}_2, \mathbf{\Pi}_3] = [\mathbf{R}_3, \mathbf{\Pi}_2] = \mathbf{0}$, etc. We find

$$\mathbf{\Pi}_2 = -\frac{1}{2}\mathbf{y}_2 + \frac{1}{2}\mathbf{y}_3, \quad \mathbf{\Pi}_3 = \frac{\sqrt{3}}{2}(\mathbf{y}_2 + \mathbf{y}_3), \quad (45)$$

hence the inverse formulas

$$\mathbf{y}_2 = -\mathbf{\Pi}_2 + \frac{1}{\sqrt{3}}\mathbf{\Pi}_3, \quad \mathbf{y}_3 = \mathbf{\Pi}_2 + \frac{1}{\sqrt{3}}\mathbf{\Pi}_3. \tag{46}$$

In Eq. (36) kinetic energy was expressed in terms of the heliocentric coordinates. But with help of Eqs. (46) we can write

$$\mathbf{y}_2^2 + \mathbf{y}_3^2 + \mathbf{y}_2 \cdot \mathbf{y}_3 = \mathbf{\Pi}_2^2 + \mathbf{\Pi}_3^2. \tag{47}$$

Now Eq. (36) may be rewritten in terms of the Jacobi coordinates. For the *total* kinetic energy we have

$$\sum \frac{\mathbf{p}^2}{2m} = \frac{\mathbf{P}^2}{6m} + \frac{1}{m}(\mathbf{\Pi}_2^2 + \mathbf{\Pi}_3^2). \tag{48}$$

In order to compute the first relativistic corrections we need to evaluate also $\Gamma_{(0)}$ in terms of $\mathbf{\Pi}_2, \mathbf{\Pi}_3$. So we must insert Eqs. (46) into Eq. (29). To this end we can write

$$\frac{4}{3}\Gamma_{(0)} = A^2 + B^2 + 4C^2 + 2(A+B)C - AB, \tag{49}$$

with the notation

$$A = (\mathbf{y}_2)^2, \quad B = (\mathbf{y}_3)^2, \quad C = \mathbf{y}_2 \cdot \mathbf{y}_3,$$

$$\mathcal{A} = (\mathbf{\Pi}_2)^2, \quad \mathcal{B} = (\mathbf{\Pi}_3)^2, \quad \mathcal{C} = \mathbf{\Pi}_2 \cdot \mathbf{\Pi}_3.$$

From Eqs. (46) we get

$$A = \mathbf{\Pi}_2^2 - \frac{2}{\sqrt{3}}\mathbf{\Pi}_2 \cdot \mathbf{\Pi}_3 + \frac{1}{3}\mathbf{\Pi}_3^2,$$

$$B = \mathbf{\Pi}_2^2 + \frac{2}{\sqrt{3}}\mathbf{\Pi}_2 \cdot \mathbf{\Pi}_3 + \frac{1}{3}\mathbf{\Pi}_3^2,$$

$$C = \frac{1}{3}\mathbf{\Pi}_3^2 - \mathbf{\Pi}_2^2,$$

in other words,

$$A = \mathcal{A} - \frac{2}{\sqrt{3}}\mathcal{C} + \frac{1}{3}\mathcal{B},$$

$$B = \mathcal{A} + \frac{2}{\sqrt{3}}\mathcal{C} + \frac{1}{3}\mathcal{B},$$

$$C = \frac{1}{3}\mathcal{B} - \mathcal{A}.$$

Inserting this into Eq. (29) we get

$$\frac{4}{3}\Gamma_{(0)} = (\mathcal{A}+\mathcal{B})^2 + 4\mathcal{C}^2,$$

$$\frac{4}{3}\Gamma_{(0)} = (\mathbf{\Pi}_2^2)^2 + (\mathbf{\Pi}_3^2)^2 + 2\mathbf{\Pi}_2^2\mathbf{\Pi}_3^2 + 4(\mathbf{\Pi}_2 \cdot \mathbf{\Pi}_3)^2. \tag{50}$$

IV. COVARIANT HARMONIC POTENTIAL

In order to test the formalism, it is natural to consider first a toy model: namely, the harmonic oscillator.

Harmonic interactions are implemented through the potential

$$V = \kappa_{12}\widetilde{(q_1' - q_2')^2} + \kappa_{23}\widetilde{(q_2' - q_3')^2} + \kappa_{13}\widetilde{(q_1' - q_3')^2}, \tag{51}$$

where κ_{ab} are positive coupling constants.

If, for the sake of simplicity, we assume that all these constants are equal, we obtain the version

$$V = \kappa \sum_{a<b} \widetilde{(q_a' - q_b')^2} = 2\kappa\{(\vec{z}_2')^2 + (\vec{z}_3')^2 - \vec{z}_2' \cdot \vec{z}_3'\}, \tag{52}$$

where κ is a positive constant. The identity

$$\sum_{a<b} \widetilde{(q_a' - q_b')^2} \equiv 2\{(\vec{z}_2')^2 + (\vec{z}_3')^2 - \vec{z}_2' \cdot \vec{z}_3'\} \tag{53}$$

reads, after reduction to the rest frame,

$$\sum (\mathbf{q}_a' - \mathbf{q}_b')^2 = 2[\mathbf{z}_2'^2 + \mathbf{z}_3'^2 - \mathbf{z}_2' \cdot \mathbf{z}_3'] = \mathbf{z}_2'^2 + \mathbf{z}_3'^2 + (\mathbf{z}_2' - \mathbf{z}_3')^2. \tag{54}$$

With help of Eqs. (44) we have $\mathbf{z}_2' - \mathbf{z}_3' = -\mathbf{R}_2$ and

$$\mathbf{z}_2'^2 = \frac{1}{4}(3\mathbf{R}_3^2 - 2\sqrt{3}\mathbf{R}_3 \cdot \mathbf{R}_2 + \mathbf{R}_2^2),$$

$$\mathbf{z}_3'^2 = \frac{1}{4}(3\mathbf{R}_3^2 + 2\sqrt{3}\mathbf{R}_3 \cdot \mathbf{R}_2 + \mathbf{R}_2^2).$$

Note that

$$2[\mathbf{z}_2'^2 + \mathbf{z}_3'^2 - \mathbf{z}_2' \cdot \mathbf{z}_3'] = \frac{3}{2}(\mathbf{R}_2^2 + \mathbf{R}_3^2)$$

for all choice of units. Hence we obtain

$$\sum (\mathbf{q}_a' - \mathbf{q}_b')^2 = \frac{3}{2}(\mathbf{R}_2^2 + \mathbf{R}_3^2), \tag{55}$$

which exhibits the O_6 invariance of our potential U.

Finally Eq. (37) takes on the form

$$\lambda\psi = (\mathbf{\Pi}_2^2 + \mathbf{\Pi}_3^2)\psi + \frac{9}{2}\kappa(\mathbf{R}_2^2 + \mathbf{R}_3^2)\psi - \left(\frac{\Gamma_{(0)}}{9m^2c^2}\right)\psi. \tag{56}$$

For the moment, let us consider the zeroth-order approximation and divide by m. We obtain the Schrödinger equation of a nonrelativistic three-body oscillator with equal masses, written in Jacobi coordinates $\mathbf{R}_2, \mathbf{R}_3, \mathbf{\Pi}_2, \mathbf{\Pi}_3$ (the SU_6 invariance of the nonrelativistic limit would become manifest if we were to choose an appropriate unit of length).

In order to make the contact with textbook notations [13], we may define

$$K = \frac{6\kappa}{m}.$$

The nonrelativistic potential is

$$U = -\frac{3}{m}\underline{V} = \frac{K}{2}\sum (\mathbf{q}_a' - \mathbf{q}_b')^2.$$

At the zeroth order the ground-state wave function is a Gaussian, as well in the coordinate as in the momentum rep-

resentation. It is better to choose the latter, where the operator Γ is multiplicative. Then the unperturbed ground state is

$$\psi_{(0)} = \phi = \text{const} \times \exp\left\{ -\frac{1}{3\sqrt{2\kappa}}(\mathbf{\Pi}_2^2 + \mathbf{\Pi}_3^2) \right\}. \quad (57)$$

In order to check the validity of expanding in powers of $1/c$ we observe that the quantity $\frac{1}{m}\sqrt{3K/m}$ has the dimension of a *squared* velocity. The velocity obtained by taking its square root is a characteristic of the system and should be reasonably small with respect to the speed of light.

Since V does not depend on P^2, it follows that $\underline{V}$ does not depend on M^2; thus, Eq. (36) is an eigenvalue problem in the conventional sense.

With the notations (35) we have, for the nth level of the unperturbed harmonic oscillator,

$$E_{(0)} = \sqrt{3K/m}(3+n), \quad n = 0,1,2,\ldots,\infty. \quad (58)$$

It is convenient to set

$$\omega = \sqrt{3K/m} = \frac{3}{m}\sqrt{2\kappa}. \quad (59)$$

Indeed, for the ground state we have, in particular,

$$E_{(0)} = 3\omega, \quad \lambda_{(0)} = 9\sqrt{2\kappa} = 3m\omega. \quad (60)$$

Let us now consider the first post-Galilean contribution; first-order perturbation theory applies as usual.

We focus on the ground state; in order to compute the first correction, we need to evaluate the expectation value of $\Gamma_{(0)}$ in the state $\psi_{(0)}$. At this stage we observe that $\Gamma_{(0)}$ is a homogeneous function of fourth degree in the six-dimensional vector $X=(\mathbf{\Pi}_2, \mathbf{\Pi}_3)$. It follows that, with obvious notations, α being any constant,

$$\langle \Gamma_{(0)} \rangle = \alpha^{-4} \int \mathrm{e}^{-X^2} \Gamma_{(0)}(X) d^6X, \quad (61)$$

provided that $\alpha^2 = \frac{2}{3}(2\kappa)^{-1/2}$, which corresponds to $\psi_{(0)} = \text{const} \times \mathrm{e}^{-\alpha^2 X^2/2}$. Therefore it is sufficient to carry out the calculation in the case where $\alpha=1$, so let us provisionally choose the unit of length such that $\kappa = 2/9$.

It is convenient to note that $\phi = \phi_2\phi_3$ introducing the normalized functions

$$\phi_A = \pi^{-3/4} \exp\left(-\frac{1}{2}\mathbf{\Pi}_A^2 \right).$$

As an operator $\mathbf{\Pi}_A$ does not affect ϕ_B when $B \neq A$. Moreover, ϕ_2, ϕ_3 are normalized to unity, so we have that

$$\langle (\mathbf{\Pi}_A^2)^2 \rangle = \langle \phi_A, (\mathbf{\Pi}_A^2)^2 \phi_A \rangle.$$

For instance, we obtain

$$\langle (\mathbf{\Pi}_2^2)^2 \rangle = \frac{15}{4} \quad (62)$$

and in the same way

$$\langle (\mathbf{\Pi}_3^2)^2 \rangle = \frac{15}{4}. \quad (63)$$

Further we have

$$\langle \phi_2\phi_3, \mathbf{\Pi}_2^2\mathbf{\Pi}_3^2 \phi_2\phi_3 \rangle = \langle \phi_2, \mathbf{\Pi}_2^2 \phi_2 \rangle \langle \phi_3, \mathbf{\Pi}_3^2 \phi_3 \rangle,$$

but we compute easily

$$\langle \phi_2, \mathbf{\Pi}_2^2 \phi_2 \rangle = \langle \phi_3, \mathbf{\Pi}_3^2 \phi_3 \rangle = \frac{3}{2}$$

and thus

$$\langle \mathbf{\Pi}_2^2 \mathbf{\Pi}_3^2 \rangle = \frac{9}{4}. \quad (64)$$

Finally, if the Π_A^j are the coordinates of the three-vector $\mathbf{\Pi}_A$, we have that

$$(\mathbf{\Pi}_2 \cdot \mathbf{\Pi}_3)^2 = (\Pi_2^1\Pi_3^1 + \Pi_2^2\Pi_3^2 + \Pi_2^3\Pi_3^3)^2. \quad (65)$$

For the sixfold integral

$$\langle (\mathbf{\Pi}_2 \cdot \mathbf{\Pi}_3)^2 \rangle = \pi^{-3} \int (\mathbf{\Pi}_2 \cdot \mathbf{\Pi}_3)^2 \mathrm{e}^{-\mathbf{\Pi}_2^2 - \mathbf{\Pi}_3^2} d^3\mathbf{\Pi}_2 d^3\mathbf{\Pi}_3, \quad (66)$$

we find

$$\langle (\mathbf{\Pi}_2 \cdot \mathbf{\Pi}_3)^2 \rangle = \frac{3}{4}. \quad (67)$$

Linear combination of all these results yields, according to Eqs. (50),

$$\langle \Gamma_{(0)} \rangle = \frac{45}{4} = 11 + \frac{1}{4}.$$

In view of Eq. (61) we now revert to an arbitrary unit of length and write

$$\langle \Gamma_{(0)} \rangle = \frac{405}{8}\kappa. \quad (68)$$

Applying this result to Eq. (39) where $\underline{V}_{(1)}$ is supposed to vanish,

$$\lambda_{(1)} = -\frac{45}{8}\frac{\kappa}{m^2}. \quad (69)$$

It is interesting to evaluate the relative importance of this correction. For this purpose consider the quantity

$$\frac{\Delta\lambda}{\lambda} = \frac{1}{c^2}\frac{\lambda_{(1)}}{\lambda_{(0)}}.$$

In view of Eqs. (59), (60), and (69) we finally obtain

$$\frac{\Delta\lambda}{\lambda} = -\frac{5}{16}\frac{\sqrt{2\kappa}}{m^2c^2} = -\frac{5}{48}\frac{\omega}{mc^2}. \quad (70)$$

Recall that ω/m is the square of the characteristic velocity.

Now inserting $\lambda_{(0)}$ and $\lambda_{(1)}$, respectively, given by Eqs. (60) and (69), into Eq. (31) we obtain, up to $O(1/c^6)$,

$$\frac{M^2}{m^2} - 9 = \frac{54\sqrt{2\kappa}}{m^2c^2} - \frac{135\kappa}{4m^4c^4} \tag{71}$$

or, equivalently,

$$\frac{M^2}{m^2} - 9 = 18\frac{\omega}{mc^2} - \frac{15}{8}\left(\frac{\omega}{mc^2}\right)^2. \tag{72}$$

In principle these formulas permit us to calculate M at the first post-Galilean order when κ and m are given.

But in practice one may be interested in a naive model of baryon. In this case it is natural to fix M (e.g., the proton mass) and adjust m and κ in agreement with Eq. (71). The most simple possibility is to choose first the ratio M/m within reasonable limits discussed below, then extract κ from Eq. (71) or alternatively extract $\frac{\omega}{mc^2}$ from Eq. (72). In this procedure the choice of M/m must allow for a reasonable value of the characteristic velocity. More precisely, $\frac{\omega}{mc^2}$ must be small enough in order to justify our first-order treatment. It is clear that the more M/m exceeds 3, the more our system is rapid.

Example. If $\frac{M}{m}=3.03$, solving Eq. (72) yields

$$\frac{\omega}{mc^2} = 0.0100$$

so that the critical velocity is about 10% of the velocity of light. If M is the proton mass ($Mc^2=920$ MeV), we find $mc^2=303.6$ MeV for the constituent quark mass.

V. CONCLUSION

Our basic equations involve a unique interaction term and are tailored for allowing elimination of the redundant variables implied by manifest covariance. In most relevant cases, the interaction term looks as if it were made of two-body contributions. In fact, the two-body nature of these contributions *is not* exact, because the transformation from original coordinates to the reducible ones somehow mixes the individual variables. For instance, the reducible relative variable $\tilde{z}'_A$ does not exactly match the cluster $(1A)$ and so on. This situation can be interpreted as due to genuine three-body forces that we have automatically introduced in order to ensure the mutual compatibility of the constraints and the possibility of a three-dimensional (3D) reduction. In general, this reduction gives rise to a nonconventional eigenvalue problem.

Starting from manifestly covariant basic equations offers several advantages:

Conceptually we realize that a general description of the system must exist even before we assign a sharp value to P. And before we impose a sharp value to P, there is no rest frame available yet (P being just an operator) which seems to discard a three-dimensional formulation *at this stage*.

Our approach yields a Schrödinger-like equation only *at the end*; the reduced wave equation (32) contains a post-Galilean correction which would hardly be derived from an *a priori* 3D theory.

Another motivation in favor of constraint dynamics is the fact that the contact with quantum field theory is more easy in a covariant framework. Actually, this contact has been thoroughly established in the two-body case [14] where constraint dynamics inspired an improved way of summing Feynman's diagrams (see the "constraint diagrams" exhibited by Jallouli and Sazdjian [6]). Of course, further work is still needed in order to determine if the mass-shell constraints of a three-body model also can suggest similar simplifications in the three-body case.

We performed a systematic expansion in powers of $1/c^2$. At least for three particles with equal masses, the nonrelativistic limit has familiar features and the first post-Galilean formulation is tractable: at this order the reduced wave equation is similar to a nonrelativistic equation modified by an overall perturbation of kinematic origin, supplemented by an additional term which stems from the possible energy dependence of the interaction potential.

Within this framework it is possible to compute for instance the first relativistic correction to the binding energy of three given (equal) masses bound by a given interaction. Or alternatively, in a simple naive model like the harmonic oscillator, one may determine the free parameters (constituent mass and/or coupling constant) in order to fit a fixed value of the ground-state mass.

Although we focused on the first post-Galilean corrections, let us stress that our formalism is ready for use at higher orders, with help of Eq. (32) where Γ, or equivalently Ξ, must be suitably expanded.

Future work is needed, however, for concrete applications: we plan to implement spin, consider the case of unequal masses, and improve the contact with other approaches [15]. In particular, it may be interesting to revisit the BS equation in terms of the reducible variables employed here.

Finally, our approach seems to be more specially designed for confined systems. When nonconfining forces are present, the occurrence of scattering states may raise the question of cluster separability which is not addressed here. But even so, our picture may provide, if not a complete theory, at least a reasonable effective *model* valid in the sector of bound states.

[1] A first (nonexhaustive) account of modern relativistic dynamics is available in *Relativistic Action-at-a-distance, Classical and quantum aspects*, Lecture Notes in Physics, Vol. 162, edited by J. Llosa (Springer-Verlag, New York, 1982), and references therein.

[2] Ph. Droz-Vincent, Rep. Math. Phys. **8**, 79 (1975); Phys. Rev. D **19**, 702 (1979).

[3] L. Bel, in *Differential Geometry and Relativity*, edited by M. Cahen and M. Flato (Reidel, Dordrecht, 1976), p. 197; Phys. Rev. D **28**, 1308 (1983); H. Leutwyler and J. Stern, Ann. Phys.

(N.Y.) **112**, 94 (1978); Phys. Lett. **73B**, 75 (1978); H. W. Crater and P. Van Alstine, *ibid.* **100B**, 166 (1981); I. T. Todorov (unpublished); V. V. Molotkov and I. T. Todorov, Commun. Math. Phys. **79**, 111 (1981); I. T. Todorov, in Ref. [1].

[4] H. Sazdjian, Phys. Lett. **156B**, 381 (1985), J. Math. Phys. **28**, 2618 (1987).

[5] H. Sazdjian, Phys. Rev. D **33**, 3425 (1986).

[6] H. Jallouli and H. Sazdjian, Ann. Phys. (N.Y.) **253**, 376 (1997).

[7] Ph. Droz-Vincent, Int. J. Theor. Phys. **42**, 1809 (2003).

[8] This formula should not be confused with a different transformation of the momenta attempted earlier in H. Sazdjian, Phys. Lett. B **208**, 470 (1988).

[9] The *off-shell* Hilbert spaces $L^2(\mathbf{R}^{12},d^{12}p)$ and $L^2(\mathbf{R}^{12},d^{12}p')$ have no direct physical meaning, but they allow for representing the Poincaré algebra and give a rigorous status to the operators involved in the wave equations. The wave function Φ $[\Psi]$ belongs only to the *rigged Hilbert space* constructed by taking $L^2(\mathbf{R}^{12},d^{12}p)$ $[L^2(\mathbf{R}^{12},d^{12}p')]$ as Hilbert space.

[10] A straightforward generalization would include a dependence on the square of the pseudovector $\varepsilon^{\alpha\mu\nu\rho}\tilde{z}'_{2\mu}\tilde{z}'_{3\nu}P_\rho$. The complete list of the dynamical variables that are translation invariant and commute with $y'_2\cdot P$ and $y'_3\cdot P$ is given by the vectors $\tilde{z}'_2,\tilde{z}'_3,y'_2,y'_3,P$. Lorentz invariance is respected when V is any function of the pairwise scalar products of these vectors, but in general, going beyond the form (13) of V may require a factor ordering of its analytic expression.

[11] V. A. Rizov, H. Sazdjian, and I. T. Todorov, Ann. Phys. (N.Y.) **165**, 59 (1985).

[12] Ph. Droz-Vincent, Few-Body Syst. **31**, 165 (2002). In that article the binding energy was defined with the opposite sign and the coefficients in $\Gamma(0)$ were not correct.

[13] See, for instance, J.-M. Richard, Phys. Rep. **212**, 1 (1992), Sec. 3.3. With respect to Eqs. (3.17) and (3.18) of that article, we exchange particle 1 with particle 3.

[14] Owing to a bridge between constraint relativistic dynamics and the quasipotential approach, the contact with quantum field theory can be traced back to I. T. Todorov, Phys. Rev. D **3**, 2351 (1971); in *Properties of Fundamental Interactions*, edited by A. Zichichi (Editrice Compositori, Bologna, 1973), Vol. 9, Pt. C, p. 951. For an exposition in terms of constraints see Sazdjian [4] and H. W. Crater, R. L. Becker, C. Y. Wong, and P. Van Alstine, Phys. Rev. D **46**, 5117 (1992).

[15] It may be interesting to compare the present scheme with recent efforts made for reducing the three-body BS equation: namely, J. Bijtebier, Nucl. Phys. A **696**, 581 (2001); e-print hep-th/9912099.