Superradiance from hydrodynamic vortices: A numerical study

F. Federici,[1,*] C. Cherubini,[2] S. Succi,[3] and M. P. Tosi[1]
[1]*NEST-INFM and Classe di Scienze, Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy*
[2]*Facolta' di Ingegneria, Università Campus Biomedico di Roma, Via Longoni 83, 00155 Roma, Italy*
[3]*Istituto per le Applicazioni del Calcolo, CNR, Viale del Policlinico 137, 00161 Roma, Italy*
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The scattering of sound-wave perturbations from vortex excitations in hydrodynamic systems with typical Bose-Einstein-condensate (BEC) parameters is investigated by numerical integration of the associated Klein-Gordon equation. The simulations indicate that at sufficiently high angular speeds, in the perturbative limit where back-reaction effects can be neglected, sound wave packets can extract a sizable fraction of the vortex energy through a mechanism of superradiant scattering. It is conjectured that this superradiant regime may be detectable in BEC experiments.

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Recent years have witnessed a growing interest in pursuing analog models of gravitational physics in condensed matter systems. The rationale for such models traces back to a seminal observation by Unruh [1], who noted a close analogy between sound-wave propagation in an inhomogeneous background flow and field propagation in curved space-time. The analogy goes on by observing that, much as superfluid hydrodynamics is a large-scale effective theory of microscopic superfluids, field theory on a curved space-time might also be regarded as a large-scale limit of a possible microscopic formulation of quantum gravity. The crucial point is that, whereas microscopic theories of quantum gravity are still largely a matter of speculation, the microscopic theory of superfluids is well developed. It can thus be hoped that the wide body of knowledge available for the latter can be brought to the benefit of the former [2]. For instance, assessing the mechanisms of sound radiation from "terrestrial black holes" beyond the hydrodynamic picture may in principle offer new insights into the microscopic origin of cosmic black hole radiance, the Hawking effect, and other cosmological phenomena.

A key step along this long-term program is the study of scattering and radiance phenomena from black holes whose background space-time can be associated with fluid excitations such as vortices. A model of fluid flow which seems particularly well suited to pursue the "analog gravity" program is the so-called draining-bathtub geometry [3], namely, a three-dimensional flow with a sink (vortex) at the origin. The flow field induced by the vortex is associated with an acoustic metric with two crucial ingredients of rotating black hole physics: an event horizon and an ergosphere. The former is a spatial surface which allows only one-way propagation of physical signals (from the outside into the vortex). A rotating black hole is characterized by an additional surface outside the event horizon which can be intuitively thought of as the sphere where the rotational velocity of the surrounding space is dragged along with the velocity of light. Within this sphere the dragging is greater than the speed of light; thus no observer or particle can maintain itself in a nonrotating orbit, but is forced to become corotated. The region outside the event horizon but inside the sphere where the rotational velocity is the speed of light is called the ergosphere. Particles falling within the ergosphere are forced to rotate faster and thereby gain energy. Because they are still outside the event horizon, they may escape the black hole. The net process is that the rotating black hole emits energetic particles at the cost of its own total energy. The possibility of extracting spin energy from a rotating black hole was first proposed by Penrose and is thus called the Penrose process [8]. Therefore, in the presence of an ergosphere, part of the the vortex energy can be extracted via the mechanism of superradiance.

Such a phenomenon was first studied by Zel'dovich [4] with regard to the generation of waves by a rotating body and was then analyzed as stimulated emission in black hole radiance [5–7]. Superresonance is an acoustic-wave version of the Penrose process, whereby a plane-wave solution of a scalar massless field in the black hole background is scattered from the ergosphere with an amplification at the expenses of the rotational energy of the black hole. Such a process has been shown to occur in a certain class of analog (2+1)-dimensional rotating black holes [1]. Later studies [9,10] have discussed the frequency dependence of the amplification factor in superresonant scattering of acoustic perturbations from a rotating acoustic black hole by deriving the reflection coefficient as a function of the frequency σ_0 of the incoming monochromatic wave. It is found that in the range $0<\sigma_0<m\Omega$ the reflection coefficient is greater than unity, with m being the azimuthal wave number and Ω the angular frequency of the acoustic horizon (see also [11] for the derivation of the same frequency range derived from thermodynamic considerations).

The main purpose of this paper is to present a quantitative investigation of superradiant scattering from sonic black holes associated with vortex configurations characterized by parameters typical of Bose-Einstein condensates (BECs). Although superradiant scattering from hydrodynamic vortices has been discussed in the recent literature [12,13], we believe that this is the first quantitative assessment of such a phe-

*Electronic address: fr.federici@sns.it

nomenon using specific BEC-like parameters.

In the limit of zero temperature, gaseous Bose-Einstein condensates are well described by the Gross-Pitaevskii equation (GPE)

$$i\hbar\partial_t\Phi = \left(-\frac{\hbar^2}{2M}\nabla^2 + V_{\mathrm{ext}} + \frac{4\pi\hbar^2 a_s}{M}|\Phi|^2\right)\Phi, \tag{1}$$

where $\Phi(\vec{r},t)$ is the wave function of the condensate normalized to the total number of bosons N, a_s being the s-wave scattering length and M the mass of the atoms. If we now use the Madelung representation $\Phi(\vec{r},t)=\sqrt{\rho(\vec{r},t)}e^{iM\theta(\vec{r},t)/\hbar}$ [14] in Eq. (1), where $\rho(\vec{r},t)=|\Phi(\vec{r},t)|^2$ is the condensate density, in the Thomas-Fermi approximation, which holds for large N, we neglect the quantum pressure term in the kinetic energy of the condensate and the GPE takes a hydrodynamic form: the imaginary part is a continuity equation for an irrotational fluid flow of velocity $\vec{v}(\vec{r},t)=\vec{\nabla}\theta(\vec{r},t)$ and density $\rho(\vec{r},t)$, and the real part is a Hamilton-Jacobi equation whose gradient leads to the Euler equation. As is well known, the GPE is equivalent to irrotational inviscid hydrodynamics [15].

Low-frequency perturbations around the stationary state are essentially sound waves (zero sound) and obey the Bogoliubov set of differential equations for the density perturbation $\rho^{(1)}$ and the phase perturbation $\theta^{(1)}$ in terms of the local speed of sound $c(\vec{r})=\sqrt{4\pi\hbar^2 a_s\rho(\vec{r})/M^2}$. These equations, within the limit of validity of the hydrodynamic approximation can be reduced to a single second-order equation for the phase perturbation [2]. This differential equation for $\theta^{(1)}\equiv\Psi$ has the form of a relativistic Klein-Gordon equation $\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Psi)=0$, with $g=\det g_{\mu\nu}$ in a curved spacetime whose metric $g_{\mu\nu}$ is determined by the local speed of sound c and the background stationary velocity $\vec{v}$.

It should be noted that the linearization suppresses the quantum nature of the GPE so that, within the linear perturbation theory, the circulation of vortices is not quantized as in BEC systems. The calculations reported below are aimed at examining what fraction of the energy can be extracted through superradiant scattering of a sound wave from a vortex described by BEC-like parameters. As discussed further below, the full nonlinear GPE will have to be used for a quantitative assessment of the extraction of energy quanta from BEC vortices.

For a single vortex with a drain at $r=0$ and angular velocity Ω in the draining-bathtub model, the velocity field of the flow is

$$\vec{v} = \vec{\nabla}\theta(r,\phi) = (-ca\hat{r} + \Omega a^2\hat{\phi})/r, \tag{2}$$

where $\hat{r}$ and $\hat{\phi}$ denote unit vectors in polar coordinates, a is the radius of the event horizon, and the background density ρ_0 of the fluid and the speed of sound c are taken as constant throughout the flow. Although such a configuration is not easily achieved experimentally, as it requires an output coupling mechanism through which matter is continuously coupled out from the origin of the vortex line, such mechanisms have been devised (see for example [16,17]). The acoustic metric associated with this configuration is

$$ds^2 = -\{c^2 - [(a^2c^2 + a^4\Omega^2)/r^2]\}dt^2 + (2ca/r)dt\,dr - 2\Omega a^2 dt\,d\phi + dr^2 + r^2 d\phi^2 + dz^2. \tag{3}$$

It is readily checked that this metric has an ergosphere whose radius is $r_{\mathrm{erg}}=a\sqrt{1+\Omega^2a^2/c^2}$. The growth of the ergosphere with increasing Ω allows an increasing extraction of energy from the vortex in superradiance conditions.

Linear perturbations of the velocity potential Ψ satisfy the massless Klein-Gordon scalar wave equation on this background, i.e.,

$$\left[-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{2a}{cr}\frac{\partial^2}{\partial t\,\partial r} - \frac{2a^2\Omega}{c^2r^2}\frac{\partial^2}{\partial t\,\partial\phi} + \left(1-\frac{a^2}{r^2}\right)\frac{\partial^2}{\partial r^2} + \frac{2a^3\Omega}{cr^3}\frac{\partial^2}{\partial r\,\partial\phi} + \frac{c^2r^2 - a^4\Omega^2}{c^2r^4}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} + \frac{r^2+a^2}{r^3}\frac{\partial}{\partial r} - \frac{2a^3\Omega}{cr^4}\frac{\partial}{\partial\phi}\right]\Psi = 0. \tag{4}$$

Equation (4) is solved by means of numerical methods developed in [18] for the integration of massless scalar-field perturbations on a rotating Kerr black hole background. For this purpose, Eq. (4) is conveniently recast into a system of first-order (strongly) hyperbolic equations through the definition of two conjugate fields $\Xi_i=\partial\Psi/\partial x^i$ and $\Pi=-(1/\alpha)\times(\partial\Psi/\partial t-\beta^i\Xi_i)$, where $\beta^i=(acr^{-1},-a^2\Omega r^{-2},0)$ and $\alpha=c$ are the space and time shifts of the acoustic metric. By setting $\Psi=\psi_1(r,t)e^{im\phi}e^{ikz}$, $\Pi=\pi_1(r,t)e^{im\phi}e^{ikz}$, $\Xi_1=\xi_1(r,t)e^{im\phi}e^{ikz}$, $\Xi_2=im\Psi$, and $\Xi_3=ik\Psi$, where (k,m) are the axial and azimuthal wave numbers, the hyperbolic system reads as follows:

$$\partial_t\pi_1 + c\partial_r(\xi_1 - a\pi_1/r) = (ac - ima^2\Omega)\pi_1/r^2 + c(k^2 + m^2/r^2)\psi_1 - c\xi_1/r,$$

$$\partial_t\psi_1 - c\partial_r(a\psi_1/r) = (ac - ima^2\Omega)\psi_1/r^2 - c\pi_1,$$

$$\partial_t\xi_1 + c\partial_r(\pi_1 - a\xi_1/r) = 2ima^2\Omega\psi_1/r^3 - ima^2\Omega\xi_1/r^2. \tag{5}$$

The set of Eqs. (5) is augmented with the constraint $|C|\equiv|\partial_r\psi_1 - \xi_1|=0$, which is used to monitor the quality of the numerical results. One-way inward propagation from the horizon is accounted for by an ingoing-radiation boundary condition, imposed through an excision technique. Details of the numerical procedure will be given in a forthcoming publication [19].

Following the standard prescription for scattering processes in Kerr black holes [20], the initial condition is chosen as a Gaussian pulse centered at $r=r_0$ and modulated by a monochromatic wave,

$$\psi_1(r,0) = A\exp\left[-(r - r_0 + ct)^2/b^2 - i\sigma(r - r_0 + ct)/c\right]\Big|_{t=0}. \tag{6}$$

The corresponding power spectrum is a Gaussian distribution $P(\omega)=P_{\max}\exp[-(\omega-\sigma)^2b^2/4c^2]$, centered at frequency σ with spectral width $1/b$. The superradiant regime is $0<\sigma/\Omega<m$ for $m\geqslant 1$.

As already remarked, the main purpose of our calculations is to assess the amount of energy that can be extracted

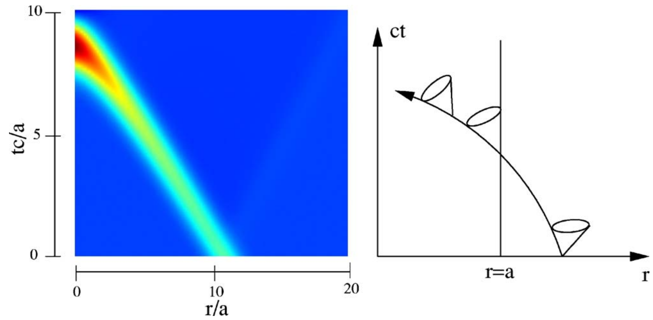

FIG. 1. (Color online) A density map of the real part of ψ_1 in the r-t plane (for the sake of clarity, the case of a quasilocalized wave packet at $m=0$ is being shown). Due to the curved background, the wave-packet trajectory bends toward the sonic horizon at $r=a$. The light cone becomes parallel to the $r=a$ axis, since no signal can escape from the horizon.

from a sonic hole as a function of its angular velocity Ω. We use as reference a set of parameters relevant to a BEC of ^{87}Rb atoms [21], with vortex core radius $a_v \simeq 0.2$ μm and angular speed $\Omega_v \simeq 18$ kHz. It is interesting to note that the inverse transversal time of the BEC vortex, $c/a_v \sim 15$ kHz, is very close to the corresponding value for a cosmic black hole of radius $a \sim 10$ km. Such a quantitative match stems from the very low speed of sound in BEC's, of the order of a few mm/s.

In the following we take $c=\hbar/(\sqrt{2}M\xi)$ and $a=\xi$, where $\xi=(8\pi\rho a_s)^{-1/2}$ is the healing length. The integration of Eqs. (5) is performed in the space-time domain $r\in[0,150]$ and $t\in[0,150]$, in units of $a=1$ and $a/c=1$ in space and time. The angular frequency is analyzed in the range $0.14<\Omega a/c<14$, corresponding to a frequency range $1.8<\Omega<18$ kHz and a density range $5\times10^{13}<\rho_0<5\times10^{14}$ cm^{-3}. The initial Gaussian pulse is centered at $r_0=50a$ and $\sigma=0.5\Omega$, with amplitude $A=0.3c$ and variance $b=10a$. We perform our study for $m=1$ and $k=0.02/a$, corresponding to a condensate axial extent $H=0.9$ mm. Violations of the constraint $C(t)=0$ are monitored over the entire space domain and are found to be consistently below 10^{-6} for all sets of parameters under investigation.

Figure 1 shows a density map of the real part of ψ_1 in the range $r/a\in[1,20]$ and $tc/a\in[0,10]$. The initial Gaussian pulse moves toward the vortex horizon placed at $r=a$ and its trajectory is bent by the potential outside the horizon. The bending of the trajectory, with light cones heading toward the horizon, is consistent with similar findings in numerical relativity [7,22] as shown in the diagram in Fig. 1.

In Fig. 2 we show a typical time evolution of the energy of wave packet $E_p(t)=(\rho_0 M/2)\int_0^{2\pi}d\phi\int_0^H dz\int_a^{143a}v_1^2 r dr$ with $v_1=\nabla\theta^{(1)}$ (top curves), normalized to its initial value $E_p(0)$, as well as the (independently calculated) rate $F(t)$ of change of the energy (bottom curves), normalized to its initial value $F(0)$, for $\sigma=0.7c/a$ and $\Omega=1.4c/a$, within the superradiant regime ($m=1$, solid lines) and outside it ($m=0$, dashed lines). $F(t)$ includes the net flux across the surfaces at $r=a$ and $143a$ as well as a term due to the bulk compressibility,

$$F(t)=\frac{dE_p(t)}{dt}=\int \vec{v}_1\cdot\frac{d\vec{v}_1}{dt}dV$$
$$=-\frac{1}{2}\left(\int v_1^2\vec{v}\cdot\hat{n}\,dS-\int v_1^2\vec{\nabla}\cdot\vec{v}_1\,dV\right). \qquad (7)$$

In the nonsuperradiant case, the energy of the scattered wave packet goes asymptotically to zero, indicating that all the energy of the impinging wave packet is lost to the vortex sink. In the superradiant case instead, the energy of the back-scattered wave packet exceeds its initial value, indicating extraction of energy from the ergosphere at the expense of the rotational energy of the vortex. Consistently with this picture, the energy flux for the superradiant (nonsuperradiant) case lies above (below) its background value during the scattering event, approximately in the range $35<tc/a<55$. The energy gained via superradiance is by no means small, as it is seen to exceed in this case 20% of the initial value $E_p(0)$.

It is now of great interest to examine the dependence of the superradiant energy gain on the angular speed of the vortex, so as to possibly identify an optimal value at which such energy gain can be maximized. In Fig. 3 we show the time evolution of the energy gain $E_p(t)/E_p(0)$ for a series of

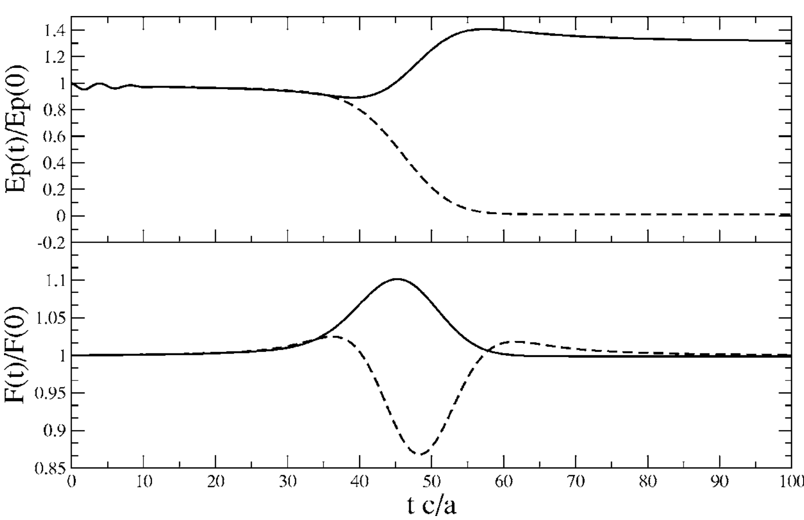

FIG. 2. Time evolution of the energy gain (top) and corresponding fluxes (bottom) of the wave packet for $\sigma=0.7c/a$ and $\Omega=1.4c/a$. The solid lines correspond to the superradiant case ($m=1$) and the dashed lines to the nonsuperradiant one ($m=0$).

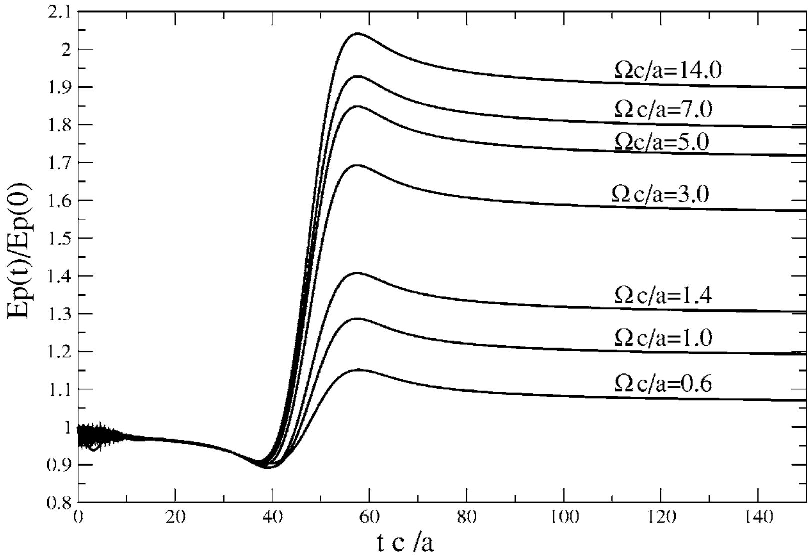

FIG. 3. Time evolution of the energy gain for different values of Ω all in the superradiant regime. Here $m=1$, $b=10a$, $r_0=50a$, and $\sigma=0.5\Omega$.

values of Ω in the superradiant range $\Omega a/c \in [0.6, 14]$, as can be experimentally achieved by varying the density of the condensate. A sharp increase of the energy gain is observed in the region $\Omega > c/a$. This is plausible, since $\Omega > c/a$ marks a transition from the regime where the radius of the ergosphere remains within a factor of 2 of the sonic horizon, to the regime where it grows linearly with Ω, thereby creating a sizable ergospheric shell that energy can be extracted from.

Since vortices are quantized metastable structures, one is naturally led to ask whether part (excited states) or all (ground state) of their energy can be extracted by the sonic wave packet. The latter instance corresponds to the break-even condition, that is, $E_p(\infty) - E_p(0) = E_b$ where E_b denotes the energy of the background vortex. In Fig. 4 we show the background energy E_b (dashed line) and the total energy gain $\Delta E_p = [E_p(\infty) - E_p(0)]$ for three values of σ/Ω in the superradiant range (solid lines), as functions of Ω. In the perturbative regime (for $\Omega \leq 3c/a$, say) the efficiency of energy extraction from the vortex grows much faster with Ω than the quadratic increase of the background energy. This is especially true at large values of the ratio σ/Ω. Although substantial values of $\Delta E_p/E_b$ are—by definition—beyond the scope of the perturbative Klein-Gordon description used throughout this work, it appears that nonlinear effects may primarily determine the way in which the energy extraction behaves as it becomes comparable to the background energy. The possibility that substantial superradiance efficiencies, as they emerge from the Klein-Gordon analysis, may persist even in the nonperturbative quantum regime described by the GPE cannot be ruled out. Even though the condition $\Delta E_p = E_b$ may remain out of reach for a single wave packet, one may still conjecture that a train of wave packets could reach the goal. It would be interesting to test this conjecture by numerical and experimental means.

As a further development of the present model, it will also be interesting to consider a vortex with no drain [12,23] and to apply our analysis to superradiant scattering from a giant vortex [24–27]. Such vortices have been found to have up to 60 quanta of circulation and can therefore be well approximated within the classical limit.

In summary, numerical simulations of the Klein-Gordon

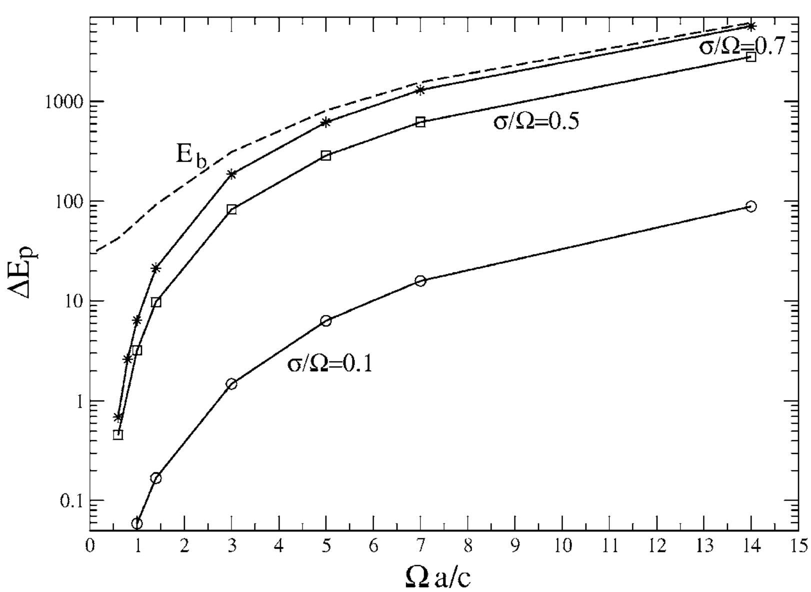

FIG. 4. ΔE_p is plotted in units of $\rho_0 M c^2 a^2 H/2$ on a logarithmic scale as a function of $\Omega a/c$ for three values of σ/Ω (solid lines) and compared with the background vortex energy E_b on the same scale and in the same units (dashed line). It may be noticed that the scaling unit of the energy used here corresponds to the kinetic energy of a background vortex in the limit when $\Omega a/c$ goes to zero.

equation for sonic perturbations impinging on a hydrodynamic vortex with parameter typical of BECs suggest the possibility that, under typical conditions of BEC experiments, a significant fraction of the vortex energy may be extracted via the mechanism of superradiance. Since the present Klein-Gordon analysis is necessarily restricted to a perturbative regime in which back-reaction effects are neglected, it would be interesting to test the realizability of this scenario both via the numerical solution of the Gross-Pitaevski equation and by actual experiments on rotating Bose-Einstein [28] condensates.

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