

Realization of an n -qubit controlled- U gate with superconducting quantum interference devices or atoms in cavity QED

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We propose an approach to realize an n -qubit controlled- U gate with superconducting quantum interference devices (SQUIDs) in cavity QED. In this approach, the two lowest levels of a SQUID represent the two logical states of a qubit while a higher-energy intermediate level serves the gate manipulation. Our method operates essentially by creating a single photon through one of the control SQUIDs and then performing an arbitrary unitary transformation on the target SQUID with the assistance of the cavity photon. In addition, we show that the method can be applied to implement an n -qubit controlled- U gate with atomic qubits in cavity QED.

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I. INTRODUCTION

Superconducting devices such as Cooper pair boxes, Josephson junctions, and superconducting quantum interference devices (SQUIDs) have attracted much attention in the quantum information community. Because they are relatively easy to scale up and have been demonstrated to have relatively long decoherence times [1–7], they have been considered as promising candidates for physical implementation of quantum computing. It is known that the building blocks of quantum computers are single-qubit logic gates and two-qubit logic gates [8]. In the past few years, for SQUID systems, many methods for realizing a single-qubit arbitrary rotation gate and a two-qubit controlled-NOT (or controlled-phase shift) gate have been presented [9–16].

Multiqubit controlled gates play an important role in constructing quantum computational networks and realizing quantum error correction protocols and implementing quantum algorithms. Recently, much attention is paid to physical realization of multiqubit controlled gates [17–19]. It is known that when using the conventional gate-decomposition protocols to construct a multiqubit controlled gate [20–22], the procedure usually becomes complicated as the number of qubits increases. Therefore, it is important to find a more efficient way to implement multiqubit controlled gates.

In this paper we focus on how to realize an n -qubit controlled- U gate with n SQUIDs (1, 2, ..., n) based on cavity QED. Recently, it has been predicted that the strong coupling limit of cavity QED, which is difficult to achieve with atoms in a microwave cavity, can readily be realized with superconducting charge qubits [23,24], superconducting flux qubits [25], or semiconducting quantum dots [26]. And more recently, the strong coupling cavity QED has been experimentally demonstrated with superconducting charge qubits and flux qubits [27,28], and semiconductor quantum dots embedded in a microcavity [29–31]. The controlled gate considered in this paper is shown in Fig. 1, which is defined as follows: (a) it contains $n-1$ control qubits and one target qubit, (b) it leaves the state of the target qubit unchanged if not all the control qubits are in the state $|1\rangle$. (c) However, when all control qubits are in the state $|1\rangle$, an arbitrary unitary transformation U is performed on the target qubit.

To implement the general multiqubit controlled gate described above, three levels $|0\rangle$, $|1\rangle$, and $|2\rangle$ of each SQUID will be employed (Fig. 2). For each SQUID, the two lowest levels $|0\rangle$ and $|1\rangle$ represent two logical states of a qubit while the higher-energy level $|2\rangle$ is used to facilitate coherent control and manipulation of quantum states of the qubit. The method presented here operates essentially by (a) creating a single photon through one of the control SQUIDs, (b) performing a general U on the target SQUID with the aid of the cavity photon as follows [32]:

$$U = e^{i\alpha}R_z(\beta)R_y(\gamma)R_z(\delta), \quad (1)$$

and (c) finally performing operations to have the cavity mode return to its original vacuum state. In Eq. (1), α, β, γ , and δ are arbitrary real numbers, R_y and R_z represent rotations along y and z axes on a Bloch sphere, which are described by matrices

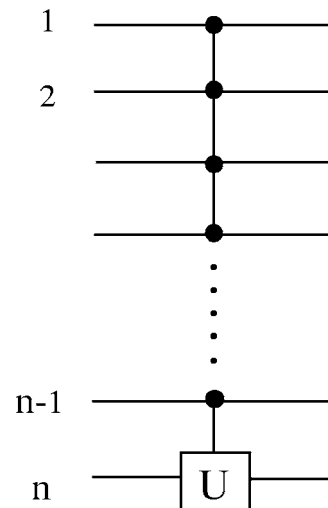


FIG. 1. Schematic circuit of an n -qubit controlled- U gate. A unitary transformation U is performed on the target qubit (qubit n) when the $n-1$ controls on the filled circles (qubits 1, 2, ..., and $n-1$) are all in the state $|1\rangle$.

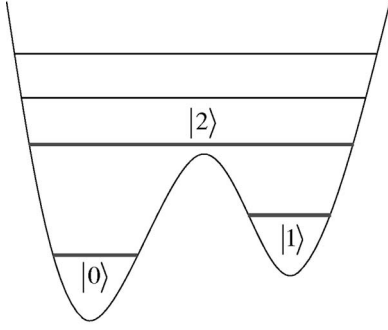


FIG. 2. Level diagram of a SQUID with the Λ -type three lowest levels $|0\rangle$, $|1\rangle$, and $|2\rangle$.

$$R_y(\gamma) = \begin{pmatrix} \cos \frac{\gamma}{2} & -\sin \frac{\gamma}{2} \\ \sin \frac{\gamma}{2} & \cos \frac{\gamma}{2} \end{pmatrix}, \quad (2)$$

$$R_z(\vartheta) = \begin{pmatrix} e^{-i\vartheta/2} & 0 \\ 0 & e^{i\vartheta/2} \end{pmatrix}, \quad \vartheta = \beta, \delta, \quad (3)$$

in a single-qubit computational subspace formed by the two logical states $|0\rangle = (1, 0)^T$ and $|1\rangle = (0, 1)^T$ of the target qubit.

As shown below, this scheme has the following advantages: (i) no auxiliary SQUIDs or measurement is needed during the entire operation, thus the hardware resources are reduced and the operation is simplified; (ii) as tunneling between the qubit levels $|0\rangle$ and $|1\rangle$ is not required during the operation, decay from the level $|1\rangle$ can be made negligibly small during the operation (via prior adjustment of the potential barrier between the qubit levels $|0\rangle$ and $|1\rangle$ [33]) and therefore the storage time of each qubit can be made much longer; (iii) the coupling constants of SQUIDs with the cavity mode could be different, hence neither identical SQUIDs nor exact placement of SQUIDs is needed; and (iv) more importantly, the gate operations are significantly simplified as the number of qubits increases, when compared with the conventional gate-decomposition protocols. In addition, it is interesting to note that the method can readily be extended to obtain an n -qubit controlled- U with atomic qubits in cavity QED.

This paper is outlined as follows. In Sec. II, we review the basic theory of a SQUID coupled to a single-mode cavity or driven by a classical microwave pulse. In Sec. III, we show a way to realize a two-qubit controlled- U gate with two SQUIDs coupled to a cavity. In Sec. IV, we discuss how to extend the method to achieve an n -qubit controlled- U gate with n SQUIDs in cavity QED. In Sec. V, we compare the present method with conventional gate-construction protocols. In Sec. VI, we give a brief discussion on experimental issues for the realization of an n -qubit controlled-rotation gate. In Sec. VII, we further show how to apply our method to implementing an n -qubit controlled- U gate with n atoms using one cavity only. A concluding summary is given in Sec. VIII.

II. BASIC THEORY

The SQUIDs throughout this paper are rf SQUIDs each consisting of a Josephson tunnel junction in a superconducting loop (the typical size of an rf SQUID is on the order of 10–100 μm). The Hamiltonian of an rf SQUID (with junction capacitance C and loop inductance L) has the usual form [5]

$$H_s = \frac{Q^2}{2C} + \frac{(\Phi - \Phi_x)^2}{2L} - E_J \cos\left(2\pi \frac{\Phi}{\Phi_0}\right), \quad (4)$$

where Φ is the magnetic flux threading the ring, Q is the total charge on the capacitor, Φ_x is the external magnetic flux applied to the ring, and $E_J \equiv I_c \Phi_0 / 2\pi$ is the maximum Josephson coupling energy (I_c is the critical current of the junction and $\Phi_0 = h/2e$ is the flux quantum).

A. SQUID-cavity resonant interaction

Consider a SQUID coupled to a single-mode microwave cavity field. The SQUID is biased properly to have a Λ -type configuration formed by three lowest levels, denoted by $|0\rangle$, $|1\rangle$, and $|2\rangle$ with energy eigenvalues E_0 , E_1 , and E_2 , respectively (Fig. 2). The transition frequency between the two levels $|i\rangle$ and $|j\rangle$ is $\nu_{ij} \equiv \omega_{ij} / (2\pi) = |E_i - E_j| / \hbar$ ($i, j \in \{0, 1, 2\}, i \neq j$). Suppose that the coupling of $|0\rangle, |1\rangle$, and $|2\rangle$ with other levels of the SQUID via the cavity is negligible, which can readily be achieved by adjusting the level spacings of the SQUID [33]. We can show that when the cavity mode is resonant with the $|0\rangle \leftrightarrow |2\rangle$ transition while decoupled (highly detuned) from the $|1\rangle \leftrightarrow |2\rangle$ transition and the $|0\rangle \leftrightarrow |1\rangle$ transition of the SQUID, the interaction Hamiltonian in the interaction picture, after the rotating-wave approximation, is described by [12]

$$H_I = \hbar (ga^+|0\rangle\langle 2| + \text{H.c.}), \quad (5)$$

where a^+ and a are the creation and annihilation operators of the cavity mode, and g is the coupling constant between the cavity mode and the $|0\rangle \leftrightarrow |2\rangle$ transition of the SQUID. For a superconducting one-dimensional transmission line standing-wave cavity, g is given by

$$\hbar g(x) = \frac{M_{sc}}{L} \sqrt{\frac{\hbar v_c}{L_0 l}} \langle 0|\Phi|2\rangle \sin\left(\frac{2\pi}{\lambda}x\right), \quad (6)$$

where M_{sc} is the SQUID-cavity mutual inductance, L_0 is the inductance per unit length of the cavity, l is the length of the cavity, $\nu_c \equiv \omega_c / (2\pi)$ is the frequency of the cavity mode with wavelength λ , and x is position of the center of the SQUID in the cavity.

In the case when the cavity is initially in the photon number state $|n\rangle$, the time evolution of the states of the system, under the Hamiltonian (5), is as follows:

$$|0\rangle|n\rangle \rightarrow \cos \sqrt{ngt}|0\rangle|n\rangle - i \sin \sqrt{ngt}|2\rangle|n-1\rangle,$$

$$|2\rangle|n\rangle \rightarrow -i \sin \sqrt{(n+1)gt}|0\rangle|n+1\rangle + \cos \sqrt{(n+1)gt}|2\rangle|n\rangle. \quad (7)$$

SQUIDs may have nonuniform device parameters and/or not be exactly placed in the cavity. Therefore, the coupling

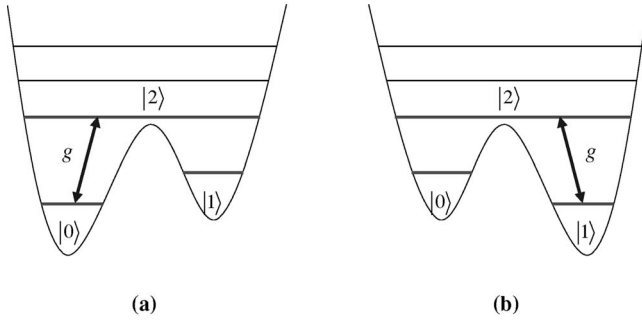


FIG. 3. (a) Resonant interaction of the cavity mode with the $|0\rangle \leftrightarrow |2\rangle$ transition of a SQUID. (b) Resonant interaction of the cavity mode with the $|1\rangle \leftrightarrow |2\rangle$ transition of the same SQUID. In (a) and (b), the coupling constant g is the same. (b), where the ground level is $|1\rangle$ and the first excited level is $|0\rangle$, is obtained by flipping (a). The potential and the level structure shown in (a) and (b) can be obtained with external flux bias of $(0.5 - \delta)\Phi_0$ and $(0.5 + \delta)\Phi_0$, respectively.

strength g may not be identical for different SQUIDs. In the following, we will replace g by g_1, g_2, \dots , and g_n for SQUIDs 1, 2, \dots , and n , respectively.

As shown below, the resonant interaction between the cavity mode and the $|1\rangle \leftrightarrow |2\rangle$ transition of the SQUID is needed. In this case, the time evolution for the states of the system is similar to Eq. (7). One just needs to replace $|0\rangle$ in Eq. (7) by $|1\rangle$. The coupling constant of the cavity mode with the $|1\rangle \leftrightarrow |2\rangle$ transition of the SQUID can be set to be the same as that with the $|0\rangle \leftrightarrow |2\rangle$ transition of the same SQUID by simply changing the external flux bias from $(0.5 - \delta)\Phi_0$ to $(0.5 + \delta)\Phi_0$ (see Fig. 3).

B. SQUID-cavity off-resonant interaction

Consider a system composed of a SQUID and a single-mode cavity. Suppose that the cavity mode is off resonant with the $|0\rangle \leftrightarrow |2\rangle$ transition (i.e., $\Delta = \omega_{20} - \omega_c \gg \hat{g}$) while decoupled from the $|1\rangle \leftrightarrow |2\rangle$ transition and the $|0\rangle \leftrightarrow |1\rangle$ transition of the SQUID [Fig. 4(a)]. Here, Δ is the detuning be-

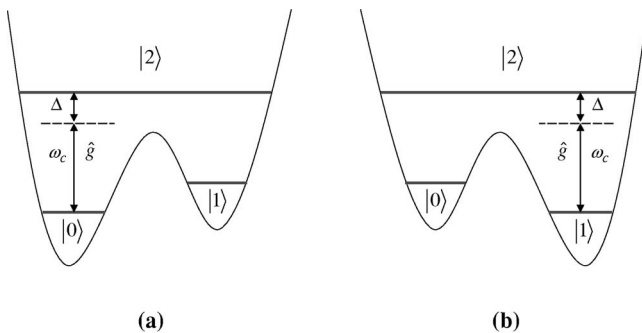


FIG. 4. (a) Off-resonant interaction between the cavity mode and the $|0\rangle \leftrightarrow |2\rangle$ transition of a SQUID. (b) Off-resonant interaction between the cavity mode and the $|1\rangle \leftrightarrow |2\rangle$ transition of the same SQUID. The coupling constant \hat{g} and the detuning Δ in (b) are the same as those in (a), which can be achieved with external flux bias of $(0.5 - \delta)\Phi_0$ and $(0.5 + \delta)\Phi_0$ for (a) and (b), respectively. In (a) $\Delta = \omega_{20} - \omega_c$, while in (b) $\Delta = \omega_{21} - \omega_c$.

tween the cavity-mode frequency and the $|0\rangle \leftrightarrow |2\rangle$ transition frequency of the SQUID, and \hat{g} is the coupling constant of the cavity mode with the $|0\rangle \leftrightarrow |2\rangle$ transition of the SQUID. Under this condition, the SQUID has a negligible probability of making a transition between the ground level $|0\rangle$ and the excited level $|2\rangle$. Therefore, the effective interaction Hamiltonian in the interaction picture can be written as [34]

$$H_e = \hbar \frac{\hat{g}^2}{\Delta} (|2\rangle\langle 2| - |0\rangle\langle 0|) a^\dagger a. \quad (8)$$

From the Hamiltonian (8), it is straightforward to see that if the cavity mode is initially in the photon number state $|n\rangle$, the time evolution of the states of the system is then given by

$$\begin{aligned} |0\rangle|n\rangle &\rightarrow e^{i(\hat{g}^2/\Delta)nt} |0\rangle|n\rangle, \\ |2\rangle|n\rangle &\rightarrow e^{-i(\hat{g}^2/\Delta)nt} |2\rangle|n\rangle. \end{aligned} \quad (9)$$

In the following, we will need to use off-resonant interaction between the cavity mode and the $|1\rangle \leftrightarrow |2\rangle$ transition of the SQUID [Fig. 4(b)]. In this case, the time evolution for the states of the system is similar to Eq. (9). One just needs to replace $|0\rangle$ in Eq. (9) by $|1\rangle$. The detuning is given by $\Delta = \omega_{21} - \omega_c$. \hat{g} is the coupling constant between the cavity mode and the $|1\rangle \leftrightarrow |2\rangle$ transition of the SQUID. As described above, i.e., by changing the external flux bias from $(0.5 - \delta)\Phi_0$ to $(0.5 + \delta)\Phi_0$, both Δ and \hat{g} can be set to be the same as those for the case of the cavity mode being off-resonant with the $|0\rangle \leftrightarrow |2\rangle$ transition of the same SQUID (Fig. 4).

C. SQUID-microwave resonant interaction

In this section, we consider a SQUID driven by a classical microwave pulse with the magnetic component $\mathbf{B}_{\mu w}(\mathbf{r}, t) = \mathbf{B}_{\mu w}(\mathbf{r}) \cos(\omega_{\mu w} t + \phi)$. Here, $\mathbf{B}_{\mu w}(\mathbf{r})$, $\omega_{\mu w}$, and ϕ are the magnetic field amplitude, carrier frequency, and phase of the microwave pulse. Assume that the microwave pulse is resonant with the $|1\rangle \leftrightarrow |2\rangle$ transition but decoupled from the $|0\rangle \leftrightarrow |2\rangle$ transition and the $|0\rangle \leftrightarrow |1\rangle$ transition of the SQUID. The interaction Hamiltonian in the interaction picture is then given by

$$H_I = \frac{\hbar}{2} (\Omega e^{i\phi} |1\rangle\langle 2| + \text{H.c.}), \quad (10)$$

where Ω is the Rabi frequency of the pulse, which has the following form [12]:

$$\Omega(t) = \frac{1}{L\hbar} \langle 1|\Phi|2\rangle \int_S \mathbf{B}_{\mu w}(\mathbf{r}) \cdot d\mathbf{S}. \quad (11)$$

From the Hamiltonian (10), it is easy to find the following state rotation:

$$\begin{aligned} |1\rangle &\rightarrow \cos \frac{\Omega}{2} t |1\rangle - ie^{-i\phi} \sin \frac{\Omega}{2} t |2\rangle, \\ |2\rangle &\rightarrow -ie^{i\phi} \sin \frac{\Omega}{2} t |1\rangle + \cos \frac{\Omega}{2} t |2\rangle. \end{aligned} \quad (12)$$

III. TWO-QUBIT CONTROLLED- U GATES

For two qubits, there are four computational basis states denoted by $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$, respectively. A two-qubit controlled- U gate is described by

$$\begin{aligned} |00\rangle &\rightarrow |00\rangle, \\ |01\rangle &\rightarrow |01\rangle, \\ |10\rangle &\rightarrow |1\rangle U|0\rangle, \\ |11\rangle &\rightarrow |1\rangle U|1\rangle, \end{aligned} \quad (13)$$

which implies that if and only if the control qubit (the first qubit) is in the state $|1\rangle$, a unitary transformation U is performed on the target qubit (the second qubit) and nothing happens otherwise.

Now let us discuss how to obtain the two-qubit controlled- U gate (13) with two SQUIDs 1 and 2 coupled to a microwave cavity. The SQUIDs considered here have the Λ -type three-level configuration as depicted in Fig. 2. The transition between any two levels for each SQUID is initially decoupled from the cavity mode (e.g., via prior adjustment of the level spacings). And the cavity mode is initially in the vacuum state $|0\rangle_c$. To realize the gate (13), it is required to perform the following transformation:

$$\begin{aligned} |0\rangle_1|0\rangle_2 \otimes |0\rangle_c &\rightarrow |0\rangle_1|0\rangle_2 \otimes |0\rangle_c, \\ |0\rangle_1|1\rangle_2 \otimes |0\rangle_c &\rightarrow |0\rangle_1|1\rangle_2 \otimes |0\rangle_c, \\ |1\rangle_1|0\rangle_2 \otimes |0\rangle_c &\rightarrow |1\rangle_1[e^{i\alpha}R_z(\beta)R_y(\gamma)R_z(\delta)|0\rangle_2] \otimes |0\rangle_c, \\ |1\rangle_1|1\rangle_2 \otimes |0\rangle_c &\rightarrow |1\rangle_1[e^{i\alpha}R_z(\beta)R_y(\gamma)R_z(\delta)|1\rangle_2] \otimes |0\rangle_c, \end{aligned} \quad (14)$$

where subscripts 1 and 2 represent SQUID 1 (the control qubit) and SQUID 2 (the target qubit), respectively.

We note that the unitary transformations, involved in the last two lines of Eq. (14), can be realized through the following operation sequence:

first, creating a single photon through SQUID 1 as follows:

$$|1\rangle_1|0\rangle_c \rightarrow |0\rangle_1|1\rangle_c. \quad (15)$$

Second, performing rotations $R_z(\delta)$, $R_y(\gamma)$, and then $R_z(\beta)$ on the states of SQUID 2 with the assistance of the photon, i.e.,

$$R_z(\delta): \begin{cases} |0\rangle_2|1\rangle_c \rightarrow e^{-i\delta}|0\rangle_2|1\rangle_c, \\ |1\rangle_2|1\rangle_c \rightarrow e^{i\delta}|1\rangle_2|1\rangle_c, \end{cases} \quad (16)$$

$$R_y(\gamma): \begin{cases} |0\rangle_2|1\rangle_c \rightarrow \left(\cos\frac{\gamma}{2}|0\rangle_2 + \sin\frac{\gamma}{2}|1\rangle_2\right)|1\rangle_c, \\ |1\rangle_2|1\rangle_c \rightarrow \left(-\sin\frac{\gamma}{2}|0\rangle_2 + \cos\frac{\gamma}{2}|1\rangle_2\right)|1\rangle_c, \end{cases} \quad (17)$$

$$R_z(\beta): \begin{cases} |0\rangle_2|1\rangle_c \rightarrow e^{-i\beta}|0\rangle_2|1\rangle_c, \\ |1\rangle_2|1\rangle_c \rightarrow e^{i\beta}|1\rangle_2|1\rangle_c. \end{cases} \quad (18)$$

Third, performing a phase shift $e^{i\alpha}$ on the states of SQUID 2 with the aid of the photon, i.e.,

$$\begin{aligned} |0\rangle_2|1\rangle_c &\rightarrow e^{i\alpha}|0\rangle_2|1\rangle_c, \\ |1\rangle_2|1\rangle_c &\rightarrow e^{i\alpha}|1\rangle_2|1\rangle_c. \end{aligned} \quad (19)$$

Last, returning SQUID 1 and the cavity mode to their original states, i.e.,

$$|0\rangle_1|1\rangle_c \rightarrow |1\rangle_1|0\rangle_c. \quad (20)$$

In the following, we will list operations required for the realization of the above transformations (15)–(20).

Step (i). Apply a π microwave pulse ($\Omega\tau_{\mu w} = \pi$, where $\tau_{\mu w}$ is the pulse duration) with $\phi = -\pi/2$ to SQUID 1 [Fig. 5(a)]. The pulse is resonant with the $|1\rangle \leftrightarrow |2\rangle$ transition of SQUID 1. After the pulse, the transformation $|1\rangle \rightarrow |2\rangle$ of SQUID 1 is obtained.

Step (ii). Bring the $|0\rangle \leftrightarrow |2\rangle$ transition of SQUID 1 to resonance with the cavity mode for an interaction time $\tau_1 = \pi/(2g_1)$ [Fig. 5(b)], resulting in $|2\rangle_1|0\rangle_c \rightarrow -i|0\rangle_1|1\rangle_c$.

After the operations of step (i) and step (ii), the transformation (15) is obtained as follows:

$$|1\rangle_1|0\rangle_c \xrightarrow{(i)} |2\rangle_1|0\rangle_c \xrightarrow{(ii)} -i|0\rangle_1|1\rangle_c \quad (21)$$

up to a phase factor $-i$, which is inevitable according to Eq. (7) but can be removed by introducing a phase factor i to the transformation (20) (see below).

Step (iii). Adjust the level structure of SQUID 2 and apply a π microwave pulse with $\phi = -\pi/2$ to SQUID 2 [Fig. 5(c)]. The pulse is resonant with the $|1\rangle \leftrightarrow |2\rangle$ transition of SQUID 1, leading to the transformation $|1\rangle \rightarrow |2\rangle$ of SQUID 2.

Step (iv). Adjust the level structure of SQUID 2 to obtain an off-resonant interaction between the cavity mode and the $|0\rangle \leftrightarrow |2\rangle$ transition of SQUID 2 [Fig. 5(d)]. After an interaction time $\tau_2 = \delta\tilde{\Delta}/\tilde{g}^2$, the state $|0\rangle_2|1\rangle_c$ goes to $e^{-i\delta}|0\rangle_2|1\rangle_c$ while the state $|2\rangle_2|1\rangle_c$ changes to $e^{i\delta}|2\rangle_2|1\rangle_c$.

Step (v). Repeat the operation of step (iii) but set $\phi = \pi/2$, leading to the transformation $|2\rangle \rightarrow |1\rangle$ of SQUID 2.

It is easy to see that after the operations of steps (iii)–(v), the transformation (16) is implemented as follows:

$$\begin{aligned} |0\rangle_2|1\rangle_c &\xrightarrow{(iii)} |0\rangle_2|1\rangle_c \xrightarrow{(iv)} e^{-i\delta}|0\rangle_2|1\rangle_c \xrightarrow{(v)} e^{-i\delta}|0\rangle_2|1\rangle_c, \\ |1\rangle_2|1\rangle_c &\xrightarrow{(iii)} |2\rangle_2|1\rangle_c \xrightarrow{(iv)} e^{i\delta}|2\rangle_2|1\rangle_c \xrightarrow{(v)} e^{i\delta}|1\rangle_2|1\rangle_c. \end{aligned} \quad (22)$$

Step (vi). Bring the $|1\rangle \leftrightarrow |2\rangle$ transition of SQUID 2 to resonance with the cavity mode for an interaction time $\tau_3 = \pi/(2g_2)$ [Fig. 5(e)]. As a result, the state $|0\rangle_2|1\rangle_c$ remains unchanged while the state $|1\rangle_2|1\rangle_c$ changes to $-i|2\rangle_2|0\rangle_c$.

Step (vii). Bring the $|0\rangle \leftrightarrow |2\rangle$ transition of SQUID 2 to resonance with the cavity mode for an interaction time $\tau_4 = \gamma/(2g_2)$ [Fig. 5(f)], resulting in $|0\rangle_2|1\rangle_c \rightarrow \cos(\gamma/2)|0\rangle_2|1\rangle_c - i\sin(\gamma/2)|2\rangle_2|0\rangle_c$ and $|2\rangle_2|0\rangle_c$

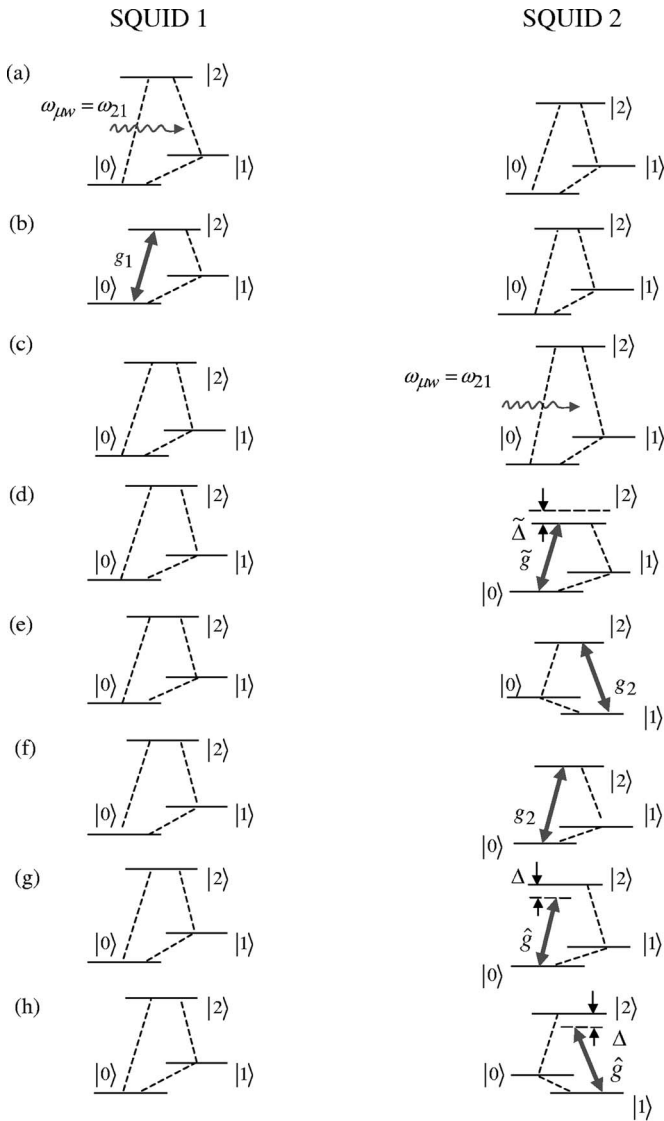


FIG. 5. Change of the level structure (reduced) of SQUIDS (1, 2) during a two-qubit controlled- U gate performance. In (a), (b), (c), (d), (e), (f), (g), and (h), figures from left to right represent the level structures for SQUIDS 1 and 2, respectively; the nonidentical level spacings of the SQUIDS could be caused by nonuniform device parameters. In (a) and (c), the level spacings for the two SQUIDS are set to be much different, such that the irrelevant SQUID is decoupled from the applied pulse. The transition between any two levels linked by a dashed line is decoupled from the cavity mode. \tilde{g} and \hat{g} are the off-resonant coupling constants between the cavity mode and the corresponding two-level transition of SQUID 2. g_1 and g_2 are the SQUID-cavity resonant coupling constants for SQUID 1 and SQUID 2, respectively. The detuning $\Delta = \omega_{20} - \omega_c$ for (g) while $\omega_{21} - \omega_c$ for (h). In addition, $\tilde{\Delta} = \omega_c - \omega_{20}$.

$$\rightarrow -i \sin(\gamma/2) |0\rangle_2 |1\rangle_c + \cos(\gamma/2) |2\rangle_2 |0\rangle_c.$$

Step (viii). Bring the $|1\rangle \leftrightarrow |2\rangle$ transition of SQUID 2 to resonance with the cavity mode for an interaction time $\tau_5 = 3\pi/(2g_2)$ [Fig. 5(e)]. As a result, the states $|0\rangle_2 |1\rangle_c$ remain unchanged while the state $|2\rangle_2 |0\rangle_c$ becomes $i|1\rangle_2 |1\rangle_c$.

One can see that after the operations of steps (vi)–(viii), the transformation (17) is obtained as follows:

$$\begin{aligned} & \stackrel{\text{(vi)}}{|0\rangle_2 |1\rangle_c} \rightarrow |0\rangle_2 |1\rangle_c \\ & \stackrel{\text{(vii)}}{\rightarrow} \cos \frac{\gamma}{2} |0\rangle_2 |1\rangle_c - i \sin \frac{\gamma}{2} |2\rangle_2 |0\rangle_c \\ & \stackrel{\text{(viii)}}{\rightarrow} \left(\cos \frac{\gamma}{2} |0\rangle_2 + \sin \frac{\gamma}{2} |1\rangle_2 \right) |1\rangle_c, \\ & \stackrel{\text{(vi)}}{|1\rangle_2 |1\rangle_c} \rightarrow -i |2\rangle_2 |0\rangle_c \\ & \stackrel{\text{(vii)}}{\rightarrow} -\sin \frac{\gamma}{2} |0\rangle_2 |1\rangle_c - i \cos \frac{\gamma}{2} |2\rangle_2 |0\rangle_c \\ & \stackrel{\text{(viii)}}{\rightarrow} \left(-\sin \frac{\gamma}{2} |0\rangle_2 + \cos \frac{\gamma}{2} |1\rangle_2 \right) |1\rangle_c. \end{aligned} \quad (23)$$

Steps (ix)–(xi). Repeat the operations of steps (iii)–(v) but set the cavity-SQUID off-resonant interaction time as $\tau_6 = \beta \tilde{\Delta} / \tilde{g}^2$, leading to the transformation (18).

Step (xii). Adjust the level structure of SQUID 2 to obtain an off-resonant interaction between the cavity mode and the $|0\rangle \leftrightarrow |2\rangle$ transition of SQUID 2 [Fig. 5(g)]. After an interaction time $\tau_7 = \alpha \Delta / \hat{g}^2$, the state $|0\rangle_2 |1\rangle_c$ goes to $e^{i\alpha} |0\rangle_2 |1\rangle_c$ while the state $|1\rangle_2 |1\rangle_c$ remains unchanged.

Step (xiii). Adjust the level structure of SQUID 2 to achieve an off-resonant interaction between the cavity mode and the $|1\rangle \leftrightarrow |2\rangle$ transition of SQUID 2 [Fig. 5(h)]. After an interaction time $\tau_8 = \alpha \Delta / \hat{g}^2$, the state $|1\rangle_2 |1\rangle_c$ changes to $e^{i\alpha} |1\rangle_2 |1\rangle_c$ while nothing happens to the state $|0\rangle_2 |1\rangle_c$.

It can be seen that after the operations of step (xii) and step (xiii), the transformation (19) is obtained as follows:

$$\begin{aligned} & \stackrel{\text{(xii)}}{|0\rangle_2 |1\rangle_c} \rightarrow e^{i\alpha} |0\rangle_2 |1\rangle_c \rightarrow \stackrel{\text{(xiii)}}{e^{i\alpha} |0\rangle_2 |1\rangle_c}, \\ & \stackrel{\text{(xii)}}{|1\rangle_2 |1\rangle_c} \rightarrow |1\rangle_2 |1\rangle_c \rightarrow \stackrel{\text{(xiii)}}{e^{i\alpha} |1\rangle_2 |1\rangle_c}. \end{aligned} \quad (24)$$

Step (xiv). Bring the $|0\rangle \leftrightarrow |2\rangle$ transition of SQUID 1 to resonance with the cavity mode for an interaction time $\tau_9 = 3\pi/(2g_1)$ [Fig. 5(b)], resulting in $|0\rangle_1 |1\rangle_c \rightarrow i|2\rangle_1 |0\rangle_c$.

Step (xv). Apply a π microwave pulse with $\phi = \pi/2$ to SQUID 1 [Fig. 5(a)]. The pulse is resonant with the $|1\rangle \leftrightarrow |2\rangle$ transition of SQUID 1. After the pulse, the transformation $|2\rangle \rightarrow |1\rangle$ of SQUID 1 is obtained.

One can see that the operations of steps (xiv) and (xv) lead to the following transformation:

$$|0\rangle_1 |1\rangle_c \rightarrow i|2\rangle_1 |0\rangle_c \rightarrow i|1\rangle_1 |0\rangle_c, \quad (25)$$

which is actually the transformation (20) up to a phase factor i .

As detailed above, we have explicitly shown how to realize the transformations (15)–(20), i.e., performing a general U on SQUID 2 (the target) when SQUID 1 (the control) is initially in the state $|1\rangle$. On the other hand, it is noted that the following states of the system:

$$|0\rangle_1 |0\rangle_2 |0\rangle_c, |0\rangle_1 |1\rangle_2 |0\rangle_c \quad (26)$$

remains unchanged during the entire operation. This is because (a) during the operation of step (i), the state $|0\rangle$ of

SQUID 1 was not affected by the applied microwave pulse, since the $|0\rangle \leftrightarrow |2\rangle$ transition and the $|0\rangle \leftrightarrow |1\rangle$ transition of SQUID 1 are decoupled from the pulse; and (b) no photon was emitted to the cavity during the operation of step (ii), when SQUID 1 is initially in the state $|0\rangle$. Hence, it can be concluded that the transformation (14), i.e., the two-qubit controlled- U gate (13) was implemented with two SQUIDS after the above manipulation.

Before closing this section, several issues need to be addressed. The irrelevant SQUIDS in each step of the operation need to be decoupled from the cavity (pulse) during the cavity-SQUID (pulse-SQUID) interaction. The cavity mode needs to be not excited during the pulse-SQUID resonant interaction. In addition, for each SQUID, the coupling of the levels $|0\rangle$, $|1\rangle$, and $|2\rangle$ with the other levels should be negligible. Note that for a SQUID, the level spacings can readily be changed by varying the external flux Φ_x or the critical current I_c (e.g., for variable barrier rf SQUIDS) [33]. Therefore, these conditions can in principle be satisfied by adjusting the level spacings of the SQUIDS.

Imperfect decoupling between the irrelevant SQUIDS and the cavity during the operations using off-resonant interaction could, in principle, result in gate errors. Note that the population of the level $|j\rangle$ of any irrelevant SQUID initially in the state $|i\rangle$, induced due to the coupling between the $|i\rangle \leftrightarrow |j\rangle$ transition and the cavity mode, is on the order of $p_j \approx g_{ij}^2 / (g_{ij}^2 + \Delta_{ij}^2)$, where $i, j \in \{0, 1, 2\}$ and $i \neq j$, g_{ij} is the coupling constant of the cavity mode with the $|i\rangle \leftrightarrow |j\rangle$ transition of the SQUID, and $\Delta_{ij} = \omega_{ij} - \omega_c$ is the detuning of the $|i\rangle \leftrightarrow |j\rangle$ transition from the cavity mode. Therefore, we remark that the coupling between the irrelevant SQUIDS and the cavity can be made negligible as long as the condition $\Delta_{ij} \gg g_{ij}$ and $\Delta_{ij} / g_{ij}^2 \gg \Delta / \tilde{g}^2$, $\tilde{\Delta} / \tilde{g}^2$ can be satisfied. Here, \tilde{g} and \tilde{g} are the off-resonant coupling constants described above, i.e., the coupling constants of the cavity mode with the transition between the corresponding two levels of the SQUID which is involved during the operation using off-resonant interaction. The required decoupling condition can be obtained by adjusting the level spacings of the irrelevant SQUIDS before the off-resonant operations, so that the transition frequency between any two levels of the irrelevant SQUIDS is highly detuned from the cavity-mode frequency. It can be achieved with the available experiment technique because the level spacings of a SQUID can be adjusted rapidly in experiment (~ 1 ns). Since a more quantitative answer to the question of “how well decoupled the irrelevant SQUIDS need to be from the cavity” requires a very lengthy and complex analysis, we will not give a detailed discussion.

IV. n -QUBIT CONTROLLED- U GATES

For n qubits, there are a total number of 2^n computational basis states from $|00 \cdots 0\rangle$ to $|11 \cdots 1\rangle$, which form a set of complete orthogonal bases in a 2^n -dimensional Hilbert space of the n qubits. As discussed in the Introduction, an n -qubit controlled- U gate performs an arbitrary unitary transformation U on the target qubit only when the $n-1$ control qubits are all in the state $|1\rangle$, i.e.,

$$|1\rangle^{\otimes(n-1)}|0\rangle \rightarrow |1\rangle^{\otimes(n-1)}U|0\rangle,$$

$$|1\rangle^{\otimes(n-1)}|1\rangle \rightarrow |1\rangle^{\otimes(n-1)}U|1\rangle, \quad (27)$$

while nothing happens to all other 2^n-2 computational basis states. In Eq. (27), the first $n-1$ qubits represent control qubits while the last qubit acts as a target. In the following, we will discuss how this gate can be achieved with n SQUIDS coupled to a cavity.

The n SQUIDS are labeled by $1, 2, \dots$, and n . The first $n-1$ SQUIDS ($1, 2, \dots, n-1$) represent control qubits while SQUID n is the target qubit. Suppose that all SQUIDS ($1, 2, \dots, n$) are initially decoupled from the cavity (which is in the vacuum state). We find that the n -qubit controlled- U gate described above can be obtained through the following sequence of operations (from right to left):

$$U_1^+ \otimes \left(\prod_{l=n-1}^1 U_{lc}^+ \right) \otimes U_{nc} \otimes \left(\prod_{l=1}^{n-1} U_{lc} \right) \otimes U_1, \quad (28)$$

where $\prod_{l=1}^{n-1} U_{lc} \equiv U_{(n-1)c} \cdots U_{2c} U_{1c}$; U_1 denotes the operation on SQUID 1 represented by matrix

$$U_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (29)$$

in the basis states $|1\rangle_1 = (0, 1)^T$ and $|2\rangle_1 = (1, 0)^T$; U_{lc} is a joint operator on the SQUID l and the cavity mode ($l = 1, 2, \dots, n-1$), represented by the matrix

$$U_{lc} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \quad (30)$$

in the basis states $|0\rangle_l |1\rangle_c = (0, 1)^T$ and $|2\rangle_l |0\rangle_c = (1, 0)^T$; and U_{nc} is a joint operator on SQUID n and the cavity mode, given by

$$U_{nc} = U \otimes I = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta) \otimes I, \quad (31)$$

which performs an arbitrary unitary transformation U on the SQUID n while nothing to the cavity state.

From the description in the previous section, the following can be seen:

(i) U_1 (U_1^+) can be realized by a π microwave pulse ($\Omega \tau_{\mu w} = \pi$, where $\tau_{\mu w}$ is the pulse duration) with $\phi = -\pi/2$ ($\pi/2$) and $\omega_{\mu w} = \omega_{21}$ to SQUID 1.

(ii) U_{lc} corresponds to the operation of bringing the $|0\rangle \leftrightarrow |2\rangle$ transition of SQUID l ($l = 1, 2, \dots, n-1$) to resonance with the cavity mode for an interaction time $\tau_l = \pi / (2g_l)$.

(iii) U_{nc} can be realized via the operations of steps (iii)–(xiii) described in the previous section. For the present case, the qubit involved in the operations is SQUID n instead of SQUID 2.

(iv) U_{lc}^+ corresponds to the operation of bringing the $|0\rangle \leftrightarrow |2\rangle$ transition of SQUID l ($l = 1, 2, \dots, n-1$) to resonance with the cavity mode for an interaction time $\tau_l = 3\pi / (2g_l)$.

To further understand Eq. (28), let us give some explanation on system evolutions after the operations described above. First, the state $|1\rangle$ of SQUID 1 is changed to the ground state $|0\rangle$ and a single photon is emitted to the cavity mode after the joint operation $U_{1c} U_1$. Second, after the joint

operation $\Pi_{l=2}^{n-1} U_{lc}$, the photon is left in the cavity mode when SQUIDS (2, 3, ..., $n-1$) are initially in the computational basis state $|11\dots 1\rangle$; however, it is absorbed by SQUIDS (2, 3, ..., $n-1$) initially in all other computational basis states. Third, after the operation U_{nc} , an arbitrary unitary transformation U is performed on SQUID n with the assistance of the cavity photon; in contrast, nothing happens to the SQUID n if no photon, after the previous operation $\Pi_{l=2}^{n-1} U_{lc}$, is left in the cavity mode. Fourth, after the joint operation $\Pi_{l=2}^{n-1} U_{lc}^+$, the photon, originally absorbed by SQUIDS (2, 3, ..., $n-1$), is emitted back to the cavity mode. Lastly, after the joint operation $U_1^+ U_{1c}^+$, the cavity mode returns to its original vacuum state $|0\rangle_c$ and the state $|0\rangle$ of SQUID 1 is changed back to the initial state $|1\rangle$. It should be

mentioned that when SQUID 1 is initially in the state $|0\rangle$, nothing happens to the whole system during the entire operation, due to the reason discussed in the previous section.

To see it more clearly, let us consider the case of three qubits, i.e., realizing a three-qubit controlled- U gate with three SQUIDS (1,2,3) in cavity QED. For three qubits, there are a total number of eight (2^3) computational basis states, denoted by $|000\rangle, |001\rangle, \dots, |111\rangle$, respectively. The first four basis states $|000\rangle, |001\rangle, |010\rangle$, and $|011\rangle$ of the three SQUIDS remain unchanged during the operations described above. However, if the three SQUIDS are initially in the other four basis states, the states of the whole system after each unitary transformation of Eq. (28) ($n=3$) are as follows:

$$\begin{array}{cccccccc}
 |100\rangle|0\rangle_c & |200\rangle|0\rangle_c & -i|000\rangle|1\rangle_c & -|020\rangle|0\rangle_c & -|020\rangle|0\rangle_c & -i|000\rangle|1\rangle_c & |200\rangle|0\rangle_c & |100\rangle|0\rangle_c \\
 |101\rangle|0\rangle_c & \xrightarrow{U_1} |201\rangle|0\rangle_c & \xrightarrow{U_{1c}} -i|001\rangle|1\rangle_c & \xrightarrow{U_{2c}} -|021\rangle|0\rangle_c & \xrightarrow{U_{3c}} -|021\rangle|0\rangle_c & \xrightarrow{U_{2c}^+} -i|001\rangle|1\rangle_c & \xrightarrow{U_{1c}^+} |201\rangle|0\rangle_c & \xrightarrow{U_1^+} |101\rangle|0\rangle_c \\
 |110\rangle|0\rangle_c & \xrightarrow{U_1} |210\rangle|0\rangle_c & \xrightarrow{U_{1c}} -i|010\rangle|1\rangle_c & \xrightarrow{U_{2c}} -i|010\rangle|1\rangle_c & \xrightarrow{U_{3c}} -i|01(U|0)\rangle|1\rangle_c & \xrightarrow{U_{2c}^+} -i|01(U|0)\rangle|1\rangle_c & \xrightarrow{U_{1c}^+} |21(U|0)\rangle|0\rangle_c & \xrightarrow{U_1^+} |11(U|0)\rangle|0\rangle_c \\
 |111\rangle|0\rangle_c & |211\rangle|0\rangle_c & -i|011\rangle|1\rangle_c & -i|011\rangle|1\rangle_c & -i|01(U|1)\rangle|1\rangle_c & -i|01(U|1)\rangle|1\rangle_c & |21(U|1)\rangle|0\rangle_c & |11(U|1)\rangle|0\rangle_c,
 \end{array} \tag{32}$$

where $|ijk\rangle$ is abbreviation of the state $|i\rangle_1 |j\rangle_2 |k\rangle_3$ of SQUIDS (1, 2, 3) with $i, j, k \in \{0, 1, 2\}$. From Eq. (32), it can be seen that a three-qubit controlled- U gate is achieved with three SQUIDS, where the third SQUID is the target qubit, after the transformations described in Eq. (28).

On a final note, we point out that a single-mode cavity is not necessary since for a multimode cavity one can in principle choose one mode to interact with the SQUIDS while have all other modes well decoupled from the three lowest levels of the SQUIDS. In addition, the method presented here is applicable to a one-, two-, or three-dimensional microwave resonator/cavity as long as the conditions described above are satisfied.

V. COMPARISON WITH CONVENTIONAL GATE CONSTRUCTION

The universality of quantum computation implies that it is possible to generate arbitrary n -qubit gates by using sequences of one-qubit and two-qubit gates only [35–37]. Barenco *et al.* developed methods to design networks for n -qubit controlled gates [20]. They have shown that it requires $2^{n-1}-1$ two-qubit controlled- V and controlled- V^+ gates and $2^{n-1}-2$ two-qubit controlled-NOT gates to accomplish an n -qubit controlled- U gate described above ($n \geq 3$). Here, V satisfies $V^{2^{n-2}} = U$. Namely, at least 2^n-3 steps of operations are required, assuming that realizing any two-qubit controlled gate requires one-step operation only.

Since the work of Barenco *et al.*, much attention has been paid to optimal implementation of quantum gates. Recently, Möttönen *et al.* have considered a generic elementary gate

sequence for realizing a general multiqubit gate [21]. More recently, Bergholm *et al.* have presented quantum circuits with uniformly controlled one-qubit gates [22]. According to their results, $2^{n-1}-1$ two-qubit controlled-NOT gates, 2^{n-1} one-qubit gates, and a single diagonal n -qubit gate are needed to construct the above n -qubit controlled- U gate. Hence, at least 2^n steps of operation are required, provided that any two-qubit gate, one-qubit gate, or an n -qubit diagonal gate can be implemented using one-step operation only.

It is interesting to note that when compared with the use of the conventional gate-decomposition protocols [20,22], our method significantly reduces the number of operations needed to implement the above n -qubit controlled- U gate. As shown above, our method only needs $2n+11$ steps of operations. The advantage of this method starts at $n=5$ and becomes more dramatic as n increases. It should be mentioned that the comparison presented here is a conservative case because it was assumed above that a two-qubit controlled- V/V^+ /NOT or an n -qubit diagonal gate can be realized with one-step operation only.

VI. DISCUSSION

In this section, we give a brief discussion on relevant experimental issues for the implementation of n -qubit controlled- U gates. Without loss of generality, let us consider n identical SQUIDS (1, 2, ..., n) at locations where the \mathbf{B}_c fields are the same (e.g., antinodes of the cavity field). Thus, we have $g_l = g$ ($l=1, 2, \dots, n$). The total operation time is given by

TABLE I. Parameters for a SQUID-cavity. β_L is the SQUIDs potential shape parameter, R is the SQUIDs effective damping resistance, and S is the surface bounded by the loop of the SQUID with width a and length b . γ_2^{-1} (γ_1^{-1}) is the energy relaxation time of the level $|2\rangle$ ($|1\rangle$). ν_{20} (ν_{21}) is the $|0\rangle \leftrightarrow |2\rangle$ ($|1\rangle \leftrightarrow |2\rangle$) transition frequency. $\phi_{ij} \equiv \langle i | \Phi | j \rangle / \Phi_0$ is the magnetic dipole coupling matrix element between levels $|i\rangle$ and $|j\rangle$ ($i = 1, 2; j = 0, 1$). l is the length of the quasi-one-dimensional CPW cavity, λ is the wavelength of the cavity mode with frequency ν_c , d is the gap between the center conductor and the adjacent ground plane, w is the width of the center conductor, t is the width of each ground plane, L_0 is the inductance per unit length of the waveguide, and ϵ_e is the effective relative dielectric constant.

SQUID	$C = 135$ fF $L = 240$ pH	$\beta_L = 1.13$	$\Phi_x = 0.4991\Phi_0$	$R = 20M\Omega$	$S = 200 \times 100 \mu\text{m}^2$
	$\nu_{20} \sim 11.4$ GHz $\nu_{21} \sim 5.8$ GHz	$\phi_{10} = 6.0 \times 10^{-3}$	$\phi_{20} = 3.2 \times 10^{-2}$	$\phi_{21} = 2.6 \times 10^{-2}$	$\gamma_2^{-1} \sim 3.2 \mu\text{s}$ $\gamma_1^{-1} \sim 0.16$ ms
Cavity	$\nu_c = 11.4$ GHz $t \gg d$	$\lambda \sim 10.5$ mm $\epsilon_e \sim 6.3$	$l = 2.5\lambda$ $L_0 \sim 0.65$ pH/ μm	$d \sim 45 \mu\text{m}$ $Q \sim 6 \times 10^4$	$w \sim 20 \mu\text{m}$ $\kappa^{-1} \sim 0.8 \mu\text{s}$

$$\tau = [2n + \gamma/(2\pi)]\tau_c^{(1)} + 2\alpha\tau_c^{(2)} + (\beta + \delta)\tau_c^{(3)} + (2n + 9)\tau_a + 4\tau_{\mu w}, \quad (33)$$

where $\tau_c^{(1)} = \pi/g$, $\tau_c^{(2)} = \Delta/\hat{g}^2$, $\tau_c^{(3)} = \tilde{\Delta}/\tilde{g}^2$, and τ_a is the typical time required for adjusting the level spacings of a single SQUID. For the method to work, τ should be much shorter than the energy relaxation time γ_2^{-1} of the level $|2\rangle$, and the lifetime of the cavity mode $\kappa^{-1} = Q/2\pi\nu_c$, where Q is the (loaded) quality factor of the cavity. In addition, direct coupling between SQUIDs needs to be negligible since this interaction is not intended.

These requirements can in principle be realized, since one can (i) reduce $\tau_c^{(1)}$ by increasing the coupling constant g , (ii) shorten τ_a by rapid adjustment of the level spacings of the SQUIDs, (iii) increase κ^{-1} by employing a high- Q cavity so that the cavity dissipation is negligible during the operation, and (iv) design SQUIDs and control (readout) circuitry so that the energy relaxation time γ_2^{-1} of the level $|2\rangle$ is sufficiently long. In addition, it is noted that direct interaction between SQUIDs can be made negligible as long as the following condition is satisfied:

$$M_{ss} \ll M_{sc}, \quad (34)$$

where M_{ss} is the mutual inductance between two adjacent SQUIDs and M_{sc} is the mutual inductance between each SQUID and the cavity.

For the sake of definitiveness, let us consider the experimental feasibility of realizing a five-qubit controlled- U gate using SQUIDs with the parameters listed in Table I. A five-qubit controlled- U gate performs the following transformation:

$$\begin{aligned} |1111\rangle|0\rangle &\rightarrow |1111\rangle U(\alpha, \beta, \gamma, \delta)|0\rangle, \\ |1111\rangle|1\rangle &\rightarrow |1111\rangle U(\alpha, \beta, \gamma, \delta)|1\rangle, \end{aligned} \quad (35)$$

when the four control qubits are in the state $|1111\rangle$ while nothing otherwise. Here, $\gamma/2 \in [0, 2\pi]$, and $\alpha, \beta/2, \delta/2 \in [-\pi, \pi]$. (The positive or negative value for α , β , and δ can be achieved by setting blue or red detuning.) Note that SQUIDs with the parameters in Table I are readily available at the present time [2,3,38] and have the desired three-level

structure as depicted in Fig. 2. For a superconducting one-dimensional standing-wave CPW (coplanar waveguide) cavity with the parameters listed in Table I and SQUIDs placed along the cavity axis (Fig. 6), one has $M_{sc} \sim 10^2$ pH. When each SQUID is located at one of the antinodes of the cavity mode (Fig. 6), a simple calculation gives $g \sim 5.8 \times 10^9$ s $^{-1}$, resulting in $\tau_c^{(1)} \sim 0.5$ ns. On the other hand, as a rough estimate, we assume $\hat{g} \sim \tilde{g} \sim 0.5g$, $\Delta \sim 10\hat{g}$, and $\tilde{\Delta} \sim 10\tilde{g}$, which can be readily achieved by adjusting the level spacings. As a result, we have $\tau_c^{(2)} \sim \tau_c^{(3)} \sim 3.4$ ns. With the choice of $\tau_{\mu w} \sim \tau_a \sim \tau_c^{(1)}$, one has $\tau \sim 81.1$ ns for $\gamma/2 = 2\pi$ and $\alpha, \beta/2, \gamma/2, \delta/2 = \pm\pi$ (a case requiring the longest operation time), which is much shorter than $\gamma_2^{-1} \sim 3.2 \mu\text{s}$ and $\kappa^{-1} \sim 0.8 \mu\text{s}$ for a cavity with $Q \sim 6 \times 10^4$. Note that superconducting CPW resonators with a quality factor of $Q > 10^6$, patterned into a thin superconducting film deposited on the surface of a silicon chip, has been experimentally demonstrated [39] (also see Refs. [23,26,27] regarding its application for loaded superconducting qubits or semiconductor qubits).

For a cavity with $\nu_c = 11.4$ GHz, the wavelength of the cavity mode is $\lambda \sim 10.5$ mm. For each SQUID being placed at an antinode of the \mathbf{B}_c field (Fig. 6), one has $D \sim 5.25$ mm, where D is the distance between the two nearest SQUIDs. A simple numerical calculation gives $M_{ss} \sim 0.1$ aH, which is much smaller than M_{sc} . Hence, the condition of negligible direct coupling between SQUIDs is very well satisfied.

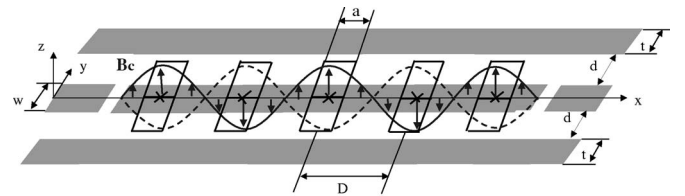


FIG. 6. Sketch of the setup for five SQUIDs (1, 2, 3, 4, 5) and a standing-wave quasi-one-dimensional CPW cavity (not drawn to scale). Each SQUID is placed in the plane of the resonator between the two lateral ground planes (i.e., the x - y plane) and at an antinode of the \mathbf{B}_c field. The two curved lines represent the standing-wave \mathbf{B}_c field, which is in the z direction.

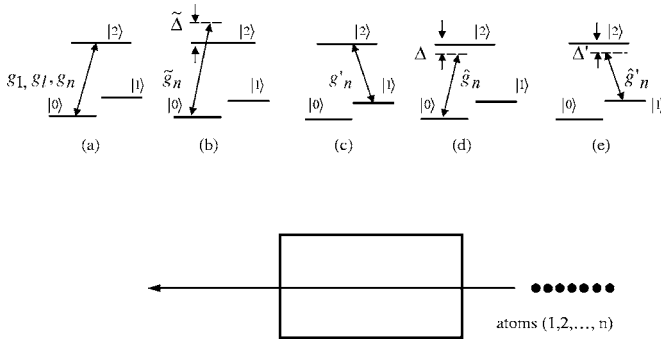


FIG. 7. Sketch of the setup for the realization of a controlled- U gate with three-level atoms and a cavity. (a) Resonant interaction of the cavity mode with the $|0\rangle \leftrightarrow |2\rangle$ transition of atom 1, atom l ($l = 2, 3, \dots, n-1$), or atom n . (c) Resonant interaction of the cavity mode with the $|1\rangle \leftrightarrow |2\rangle$ transition of atom n . (b), (d) Off-resonant interaction of the cavity mode with the $|0\rangle \leftrightarrow |2\rangle$ transition of atom n for different detuning setting. (e) Off-resonant interaction of the cavity mode with the $|1\rangle \leftrightarrow |2\rangle$ transition of atom n . g_1, g_l, g_n are the resonant coupling constants between the cavity mode and the $|0\rangle \leftrightarrow |2\rangle$ transition of atom 1, atom l ($l = 2, 3, \dots, n-1$), and atom n , respectively. g'_n is the resonant coupling constant between the cavity mode and the $|1\rangle \leftrightarrow |2\rangle$ transition of atom n . \tilde{g}_n, \hat{g}_n , and \hat{g}'_n are the off-resonant coupling constants between the cavity mode and the corresponding two-level transition of atom n . $\Delta = \omega_{20} - \omega_c$, $\Delta' = \omega_{21} - \omega_c$, and $\tilde{\Delta} = \omega_c - \omega_{20}$.

Our above analysis demonstrates that the realization of a five-qubit controlled- U gate is possible using SQUIDs and a cavity within the present technology. In addition, we point out that a quantum controlled Hadamard gate with a larger number of control qubits can in principle be obtained by increasing the length of the cavity though the conditions of $\tau \ll \gamma_2^{-1}$, κ^{-1} becomes increasingly difficult to satisfy.

VII. n -QUBIT CONTROLLED- U GATES WITH ATOMS

In this section, we discuss how to extend the above method to realize an n -qubit controlled- U gate with three-level atoms, by the use of one cavity.

Consider n identical atoms $(1, 2, \dots, n)$ each having a Λ -type level configuration formed by two ground states and an excited state (Fig. 7). In accordance with the previous section, we use $|0\rangle$ and $|1\rangle$ to represent the two ground states and $|2\rangle$ to indicate the excited state. The dipole transition between $|0\rangle$ and $|1\rangle$ is forbidden due to the definite parity of the wave function. In the following, the two logical states of a qubit are represented by the two ground states $|0\rangle$ and $|1\rangle$ of each atom. And, atoms $(1, 2, \dots, n-1)$ act as control qubits while atom n is the target qubit.

We note that an n -qubit quantum controlled- U gate described by Eq. (27) can be realized with three-level atoms using the prescription of Eq. (28), by performing the following operations:

(a) Apply a $\pi/2$ classical pulse with $\phi = -\pi/2$ to atom 1, resonant with the $|1\rangle \leftrightarrow |2\rangle$ transition of atom 1. This pulse leads to the transformation $|1\rangle \rightarrow |2\rangle$ of atom 1, which accomplishes the transformation U_1 of Eq. (28).

(b) Send atom 1 through the cavity. The cavity mode is initially in the vacuum state $|0\rangle_c$ and resonant with the $|0\rangle \rightarrow |2\rangle$ transition of atom 1 [Fig. 7(a)]. Choose the atomic velocity appropriately so that the passage time of atom 1 through the cavity equals to $\pi/(2g_1)$. Thus, after atom 1 exits the cavity, the state $|0\rangle_1|0\rangle_c$ remains unchanged, while the state $|2\rangle_1|0\rangle_c$ changes to $-i|0\rangle_1|1\rangle_c$. This operation implements the transformation U_{1c} in Eq. (28).

(c) Send atoms $(2, 3, \dots, n-1)$ through the cavity one after another in a way that no more than one atom stays in the cavity simultaneously. The cavity mode is resonant with the $|0\rangle \rightarrow |2\rangle$ transition of each atom [Fig. 7(a)]. Choose the atomic velocity appropriately so that the duration of atom l in the cavity equals to $\pi/(2g_l)$ ($l = 2, 3, \dots, n-1$). As a result, there is no change for the state $|0\rangle_l|0\rangle_c$, $|1\rangle_l|0\rangle_c$, and $|1\rangle_l|1\rangle_c$; while the state $|0\rangle_l|1\rangle_c$ becomes $-i|2\rangle_l|0\rangle_c$. This process completes the transformation $\prod_{l=2}^{n-1} U_{lc}$ of Eq. (28).

(d) The transformation U_{nc} in Eq. (30) is performed on atom n and the cavity mode, which is realized as follows:

(d.1) First, apply a $\pi/2$ classical pulse with $\phi = -\pi/2$ to atom n , resonant with the $|1\rangle \leftrightarrow |2\rangle$ transition of atom n and resulting in the transformation $|1\rangle \rightarrow |2\rangle$. Second, adjust the cavity frequency [40] to obtain an off-resonant interaction between the cavity mode and the $|0\rangle \leftrightarrow |2\rangle$ transition of atom n [Fig. 7(b)] and then send atom n through the cavity. After an interaction time $\delta\tilde{\Delta}/g_n^2$, the state $|0\rangle_n|1\rangle_c$ goes to $e^{-i\delta}|0\rangle_n|1\rangle_c$ while the state $|2\rangle_n|1\rangle_c$ changes to $e^{i\delta}|2\rangle_n|1\rangle_c$. Last, apply a $\pi/2$ pulse (with $\phi = \pi/2$) to atom n , resonant with the $|1\rangle \leftrightarrow |2\rangle$ transition of atom 1 and resulting in the transformation $|2\rangle \rightarrow |1\rangle$. After these operations, a rotation $R_z(\delta)$ on the two states $|0\rangle$ and $|1\rangle$ of atom n is obtained while the cavity mode remains in one-photon state.

(d.2) First, adjust the cavity frequency so that the cavity mode is resonant with the $|1\rangle \leftrightarrow |2\rangle$ transition of atom n [Fig. 7(c)]. Then send atom n through the cavity for an interaction time $\pi/(2g'_n)$ so that the state $|0\rangle_n|1\rangle_c$ does not change while the state $|1\rangle_n|1\rangle_c$ goes to $-i|2\rangle_n|0\rangle_c$. Second, adjust the cavity frequency so that the cavity mode is resonant with $|0\rangle \leftrightarrow |2\rangle$ transition of atom n [Fig. 7(a)]. Then send atom n back through the cavity for an interaction time $\gamma/(2g_n)$, leading to the rotation $|0\rangle_n|1\rangle_c \rightarrow \cos(\gamma/2)|0\rangle_n|1\rangle_c - i \sin(\gamma/2)|2\rangle_n|0\rangle_c$ and $|2\rangle_n|0\rangle_c \rightarrow -i \sin(\gamma/2)|0\rangle_n|1\rangle_c + \cos(\gamma/2)|2\rangle_n|0\rangle_c$. Last, have the cavity mode resonant with the $|1\rangle \leftrightarrow |2\rangle$ transition of atom n via the adjustment of the cavity frequency [Fig. 7(c)]; and then send atom n through the cavity for an interaction time $3\pi/(2g_n)$. As a result, the state $|0\rangle_n|1\rangle_c$ remains unchanged while the state $|2\rangle_n|0\rangle_c$ becomes $i|1\rangle_n|1\rangle_c$. It can be seen that after these operations, a rotation gate $R_y(\gamma)$ on the two states $|0\rangle$ and $|1\rangle$ of atom n is realized in the same manner as shown in Eq. (23).

(d.3) To achieve a rotation $R_z(\beta)$ on the two states $|0\rangle$ and $|1\rangle$ of atom n , one just needs to perform the same operations as described in (d.1) by simply setting the time of atom n crossing the cavity as $\beta\tilde{\Delta}/g_n^2$.

(d.4) A common phase $e^{i\alpha}$ for the two states $|0\rangle$ and $|1\rangle$ of atom n can be obtained as follows. First, adjust the cavity frequency to obtain an off-resonant interaction between the cavity mode and the $|0\rangle \leftrightarrow |2\rangle$ transition of atom n [Fig. 7(d)]

and then send atom n through the cavity for an interaction time $\alpha\Delta/\hat{g}_n^2$, resulting in $|0\rangle_n|1\rangle_c \rightarrow e^{i\alpha}|0\rangle_n|1\rangle_c$ while no change for the state $|1\rangle_n|1\rangle_c$. Second, adjust the cavity frequency to obtain an off-resonant interaction between the cavity mode and the $|1\rangle \leftrightarrow |2\rangle$ transition of atom n [Fig. 7(e)]. Then, send atom n back through the cavity for an interaction time $\alpha\Delta'/\hat{g}_n'^2$, which leads to $|1\rangle_n|1\rangle_c \rightarrow e^{i\alpha'}|1\rangle_n|1\rangle_c$.

The operations described above have accomplished a general transformation U on the target qubit (atom n) with the assistance of the cavity photon. However, *it is noted that the original states of the cavity mode and atoms $(1, 2, \dots, n-1)$ have also changed after the above operations.* Therefore, it is necessary to return the cavity mode and atoms $(1, 2, \dots, n-1)$ (control qubits) to their original states, which can be done through the following operations:

(e) Send atoms $(2, 3, \dots, n-1)$ through the cavity one after another, without more than one atom staying in the cavity simultaneously. The cavity frequency is adjusted to have the cavity mode resonant with the $|0\rangle \leftrightarrow |2\rangle$ transition of each atom [Fig. 7(a)]. Choose the atomic velocity appropriately so that the duration of atom l in the cavity equals to $3\pi/(2g_l)$ ($l=2, 3, \dots, n-1$). As a result, nothing happens to the states $|0\rangle_l|0\rangle_c$ and $|1\rangle_l|0\rangle_c$ while the state $|2\rangle_l|0\rangle_c$ becomes $i|0\rangle_l|1\rangle_c$ ($l=2, 3, \dots, n-1$). After this operation, atoms $(2, 3, \dots, n-1)$ return to their original states and the photon, originally absorbed by atoms $(2, 3, \dots, n-1)$, is emitted back to the cavity. This process realizes the transformation $\prod_{l=2}^{n-1} U_{lc}^+$ of Eq. (28).

(f) Send atom 1 back through the cavity. The cavity mode is resonant with the $|0\rangle \leftrightarrow |2\rangle$ transition of atom 1 [Fig. 7(a)]. Choose the atomic velocity appropriately so that the passage time of atom 1 through the cavity equals to $3\pi/(2g_1)$. Thus, after atom 1 exits the cavity, the state $|0\rangle_1|0\rangle_c$ remains unchanged but the state $|0\rangle_1|1\rangle_c$ changes to $i|2\rangle_1|0\rangle_c$. After this operation, the cavity mode returns to the original vacuum state $|0\rangle_c$ and the transformation U_{1c}^+ of Eq. (28) is obtained.

(g) Finally, apply a $\pi/2$ classical pulse (with $\phi = \pi/2$) to atom 1. The pulse is resonant with the $|1\rangle \leftrightarrow |2\rangle$ transition of atom 1. This pulse leads to $|2\rangle \rightarrow |1\rangle$, completing the transformation U_1^+ of Eq. (28).

The present scheme has the following advantages: (i) No adjustment of the level spacings for each atom is required during the operations; (ii) only one cavity is required; (iii) no identical atom-cavity coupling constants are needed; and (iv) the total number of basic operations is $2n+11$, which is much less than that required by the conventional gate-decomposition protocols [20,22] when n is a larger number ($n \geq 5$).

VIII. CONCLUSION

We have presented a method to realize a multiqubit quantum controlled- U gate with SQUIDs coupled to a microwave cavity. The method operates essentially by creating a single photon through one of the control SQUIDs and then exchanging the photon between the control SQUIDs and the cavity mode before and after a unitary transformation U is performed on the target SQUID. The method has these advantages (i) since no tunneling between the qubit levels $|0\rangle$ and $|1\rangle$ is required, decay from the level $|1\rangle$ can be made negligibly small during the operation, via prior adjustment of the barrier of the double-well potential [33]; (ii) as neither the measurement on SQUIDs/photons nor auxiliary SQUID is needed, the operation is simplified and hardware resources saved; (iii) because coupling constants of SQUIDs with the cavity are not required to be identical, inevitable nonuniformity in device parameters is tolerable and nonexact placement of SQUIDs is allowed; (iv) the method can in principle be applied to obtain an n -qubit controlled- U gate with a large number n , and (v) more interestingly, the gate operations are significantly simplified as the number of qubits increases, when compared with the use of the conventional gate-decomposition protocols. As shown above, the present method can be extended to implement a multiqubit controlled- U gate with atoms in cavity QED. Finally, it is noted that the method is also applicable to the realization of a multiqubit controlled- U gate with quantum dots in cavity QED [41].

Before we conclude, it should be mentioned that the idea of realizing multiqubit controlled phase gates with superconducting flux qubits or charge qubits has been proposed previously [42,43]. Our present work, however, deals with the realization of a multiqubit controlled- U gate (a multiqubit controlled “arbitrary transformation”). Therefore, it is much more general than the previous works [42,43]. To the best of our knowledge, no one has yet demonstrated how to realize an n -qubit controlled- U gate within cavity QED. We believe that this work is of great importance since it provides a simple protocol to realize a multiqubit controlled- U gate with SQUID qubits or atomic qubits within cavity QED.

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