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Classification of tripartite entanglement with one qubit

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We present a method to find the decompositions of tripartite entangled pure states with one qubit which are smaller than two successive Schmidt decompositions. We use this method to get a classification of the entangled states according to their decompositions. Furthermore, we also use this method to classify the entangled states that can be interconverted through stochastic local operations and classical communication. More general tripartite systems are briefly discussed.

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I. INTRODUCTION

Increasing interest in quantum information theory (QIT) has motivated the study of general types of quantum entanglement. Although entanglement is well understood only for systems either of small dimensionality or involving few parties [1], there is no doubt of the fact that it has a central importance in QIT. This is mainly due to simple applications, albeit without any classical analogy, of bipartite entanglement to quantum communication like teletransportation [2,3], superdense code [3,4], and Einstein-Podolsky-Rosen (EPR) protocol to quantum secret key distribution [3,5]. Furthermore, strong drive for the development of quantum computers is provided by their apparent intrinsic advantages, as indicated, e.g., by the Shor factoring and Grover search quantum algorithms [3]. In these algorithms, the coherence of an entangled state of many qubits is crucial. In this way, a better understanding of general entanglement is desirable.

To this effect, we notice that the achieved understanding of bipartite entanglement is mainly based on simple decompositions such as the Schmidt decomposition [3,6] and also the relative states decomposition [7–9]. Moreover, these decompositions are at the heart of fundamentals papers on the meaning of entanglement in quantum mechanics since the well-known von Neumann theory of measurement [10] and the Einstein-Podolsky-Rosen incompleteness argument [11]. Nowadays, these decompositions have also been central in the description of the dynamics of the quantum correlations [12] and in studies dealing with the emergence of a "classical" world through the phenomenon of decoherence [13]. Another feature of these decompositions is that they are simple, in the sense that entangled states are written as a superposition of the smallest possible number of product states [23]. This smallest number is usually referred to as the Schmidt rank of the entangled state.

For tripartite systems, these decompositions may be applied recursively. In particular, for three-qubit entangled states they give many ways of representing the entangled state as a superposition of four factorable states. The use of recursive Schmidt decompositions is discussed by Partovi [14]. However, it is already known from the works of Dür, Vidal, and Cirac [15] and of Acín *et al.* [16] that a simpler decomposition with two or three product states exists, depending on the particular tripartite entangled state one is dealing with. States that can be written in terms of two product states are usually said to be of the Greenberger-Horne-Zeilinger (GHZ) type, or of generalized-GHZ type, and states which require at least three product components are said to be of the W type.

Furthermore, many applications of entangled states in QIT are related to the nonlocal aspect of the quantum entanglement. For these applications, in which the qubits are spatially separated, it is important to know in which states $|\psi'\rangle$ an entangled state $|\psi\rangle$ can be transformed through local operations. There are many types of local operations which were extensively discussed by Bennett *et al.* [17]. Here we consider only the class of stochastical local operations with classical communication (SLOCC). In this case, if spatially separated observers share an entangled state $|\psi\rangle$, and are allowed to perform local operations (including measurements and interacting ancillary systems) on their respective subsystem and to communicate with each other classically, then they can convert, with nonvanishing probability of success, the state $|\psi\rangle$ to another state $|\psi'\rangle$. If one restricts oneself to reversible SLOCC, one gets a partition of the set of all states in classes of different types of entangled states [15]. In this way, Dür, Vidal, and Cirac [15] have shown that states of type GHZ and type W correspond to distinct SLOCC classes. The SLOCC classification was extended to the four-qubit case by Verstraete *et al.* [18] and to the case of two qubits and one n-level system by Miyake and Verstraete [19]. More general aspects of SLOCC classification were also discussed by Miyake [20].

In this work, we start from the observation of Dür, Vidal, and Cirac [15] that the number of product states in the smallest decomposition of a state is in general invariant through SLOCC. We then show how to find these decompositions for tripartite systems with one qubit. In general, we show that there are many decompositions which are smaller than that resulting from two successive Schmidt decompositions, which we call "sub-Schmidt decompositions" for short. Particularly, for tripartite systems involving one qubit and local supports with dimensions n, n, and 2 (we call local support the subspaces in which the reduced density matrices of each of the subsystems are nonvanishing), we get a classification of all decompositions which we use to characterize all possible SLOCC classes.

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The paper is organized as follows. In Sec. II we show how to find sub-Schmidt decompositions for entangled states with local supports n, n, and 2 and give some examples. In Sec. III we show how the method for finding such decompositions can be used to define SLOCC classes. As our demonstrations are all constructive, our treatment also provides for a way to find SLOCC protocols to transform entangled states. We close the paper with a discussion on the difficulties of extending the results to more general tripartite entanglement in Sec. IV.

II. ENTANGLED TRIPARTITE STATES WITH ONE QUBIT

In order to get a better understanding of the physical and geometrical meaning of the algebra which follows, we start with some remarks on known results obtained by Dür, Vidal, and Cirac [15] for the entangled three-qubit system and on the results obtained by Sanpera *et al.* [21] concerning planes in $C^2 \otimes C^2$ spaces. Sanpera *et al.* [21] have shown that a plane generated by two entangled states of a two-qubit system contains either one or two pure states. This result is important because, when we trace out one of the qubits of the three-qubit system, the local support of the Hilbert space of the two other qubits is at most bi-dimensional, i.e., a plane. This implies that one does not need a basis for the complete $C^2 \otimes C^2$ space in order to represent an entangled three-qubit state, since a basis for its local support in $C^2 \otimes C^2$ is sufficient. Thus we can always find a basis for its local support with either one or two pure states, and these alternatives correspond, respectively, to the states of class W and of class GHZ [15]. In this way, mathematicaly, we will follow this line of reasoning showing how to find the states which are less entangled in planes belonging to $C^n \otimes C^n$. Physically, the existence of such states will be related to the existence of some particular states of the subsystem that, when measured, project the whole tripartite state onto a pure separable state.

Consider then a pure state $|\psi\rangle$ in a space $C_a^n \otimes C_b^n \otimes C_c^2$, where we have labeled the subsystems as s_a, s_b (two n-dimensional subsystems), and s_c (the qubit). Suppose also that our state $|\psi\rangle$ has local supports with dimensions n, n, and 2 on the subsystem spaces C_a^n, C_b^n, and C_c^2, respectively. Note that we need to specify the dimension of the local supports of the three subsystems, in contrast with bipartite entanglement, where the local supports always have the same dimension. Of course, the dimension of any local support cannot be greater than the product of the other two. We will express this situation by saying that $|\psi\rangle$ is an entangled state of dimensionality n by n by 2 or that the entanglement of $|\psi\rangle$ has dimensionality n by n by 2 if their local supports on the subsystems have dimensions n, n, and 2, respectively. We will also denote the dimensionality of $|\psi\rangle$ by $(n,n,2)$ as a shorthand. Then, if the tripartite system as a whole is in an entangled state of dimensionality $(n,n,2)$, the local support in $s_{ab}=s_a+s_b$ is a bidimensional plane $\mathcal{P} \subset C_a^n \otimes C_b^n$. This plane can be easily found from any relative states decomposition [7–9] of $|\psi\rangle$. Explicitly, let $\{|k\rangle\}_{k=0,1}$ be an orthonormal basis in C_c^2. We can write

$$|\psi\rangle = \sum_{k=0,1} c_k |r_k\rangle |k\rangle, \tag{1}$$

where $|r_k\rangle \in \mathcal{P} \subset C_a^n \otimes C_b^n$ is the relative state of $|k\rangle \in C_c^2$ and $|c_k|^2$ is the probability of finding s_c in state $|k\rangle$ or s_{ab} in state $|r_k\rangle$. In this way the two states $|r_k\rangle$ span the plane $\mathcal{P}$.

Let us now look for the entangled states in $\mathcal{P}$ having Schmidt rank smaller than n and use them to span $\mathcal{P}$ and write $|\psi\rangle$. Any state $|\phi\rangle$ in $\mathcal{P}$ can be written as a linear combination of the two states $|r_k\rangle$,

$$|\phi\rangle = \alpha_0 |r_0\rangle + \alpha_1 |r_1\rangle, \tag{2}$$

where α_0 and α_1 are complex coefficients. In order not to carry unimportant normalization factors, we ignore the normalization constraint on the coefficients α_0 and α_1. Of course the state can easily be normalized at the end. Each bipartite entangled state $|\phi\rangle$ can be seen as a linear mapping of C_a^{n*} on C_b^n (where C_a^{n*} is the dual of C_a^n) defined by the partial scalar product of any $\langle u_a| \in C_a^{n*}$ with $|\phi\rangle$, $\langle u_a|\phi\rangle \in C_b^n$. The rank of this linear mapping is the Schmidt rank of the state $|\phi\rangle$. We are then looking for α_0 and α_1 such that $|\phi\rangle$ has Schmidt rank less than n, i.e., we are looking for α_0 and α_1 such that the equation

$$\langle u_a|(\alpha_0 |r_0\rangle + \alpha_1 |r_1\rangle) = 0 \tag{3}$$

has at least one nontrivial solution $\langle u_a| \in C_a^{n*}$.

We must emphasize, at least for the moment, that the nature of the state $|u_a\rangle$ is in fact irrelevant, the relevant question being: which are the α_0 and α_1 such that some non-null $|u_a\rangle$ satisfying Eq. (3) exists? The state $|u_a\rangle$ has an interesting physical meaning, however. Suppose there is some nonvanishing $|u_a\rangle$ for some also nonvanishing values of α_0 and α_1 (since we have not yet proved that they exist), and suppose further that we make a measurement on subsystem s_a and find it in state $|u_a\rangle$. Then, although the state of s_{ab} is mixed (in particular it can be seen as a mixture involving $|r_0\rangle$ and $|r_1\rangle$), we get also a pure state for s_b and, consequently, also for s_c so that the whole system is reduced to a product state. This happens because the validity of Eq. (3) implies that $\alpha_0\langle u_a|r_0\rangle = -\alpha_1\langle u_a|r_1\rangle$, i.e., the vectors $\langle u_a|r_0\rangle$ and $\langle u_a|r_1\rangle$ in C_b^n are linearly dependent. In physical terms, the relative state for $|u_a\rangle$ is the same whether the state of s_{ab} is $|r_0\rangle$, $|r_1\rangle$ or in fact any state in $\mathcal{P}$ (other than $\alpha_0|r_0\rangle + \alpha_1|r_1\rangle$). If we could find two linearly independent $|u_a\rangle$'s for the same α_0 and α_1, we would have a subspace of $|u_a\rangle$'s with the same relative state for all states in $\mathcal{P}$. This subspace would actually be the null space of the linear mapping of C_a^{n*} on C_b^n defined by those particular superpositions of $|r_0\rangle$ and $|r_1\rangle$ such that $\alpha_0\langle u_a|r_0\rangle = -\alpha_1\langle u_a|r_1\rangle$. In this way, these particular superpositions would have Schmidt rank n less the dimension of this subspace of $|u_a\rangle$'s. The existence of states $|u_a\rangle$ with this property has already been observed for three qubits entanglement of dimensionality (2,2,2) by Acín *et al.* [16].

In order to find a solution of Eq. (3) we choose a basis $\{|i\rangle\}$ in C_a^n and a basis $\{|j\rangle\}$ in C_b^n, so that we can rewrite it in matrix form as

$$(\alpha_0 R_0 + \alpha_1 R_1) u_a^* = 0, \tag{4}$$

where the matrix R_k has components $[R_k]_{ij} = \langle ji | r_k \rangle$ and the vector u_a^* has components $u_{a_i}^* = \langle u_a | i \rangle$. The Schmidt rank of the state $|\phi\rangle = \alpha_0 |r_0\rangle + \alpha_1 |r_1\rangle$ is the rank of the matrix $(\alpha_0 R_0 + \alpha_1 R_1)$. Now we assume, without loss of generality, that the state $|r_1\rangle$ has Schmidt rank n. Otherwise we would either have a problem of dimensionality lower than $(n,n,2)$ or $|r_1\rangle$ would already be a solution of our problem. We will show, however, that the set of solutions of Eq. (4) is of null measure in $\mathcal{P}$. We can rewrite Eq. (4) so that it looks like an eigenvalue equation

$$(R_1^{-1} R_0 - \lambda) u_a^* = 0, \tag{5}$$

where $\lambda = -\alpha_1 / \alpha_0$. However, we must keep in mind that we are not solving an eigenvalue problem since λ in Eq. (5) depends on the ratio of coefficients α_1 and α_0, which in turn depends on the basis we have chosen in Eq. (1) for $\mathcal{P}$. But the number of distinct states with Schmidt rank smaller than n in $\mathcal{P}$ obviously cannot depend on the basis chosen for $\mathcal{P}$, neither can their respective Schmidt rank.

In this way, we can ask what would change in Eq. (5) if we would have chosen another basis for $\mathcal{P}$ in Eq. (2). Let us call this basis $\{|\phi_k\rangle\}$, with $|\phi_0\rangle = a|r_0\rangle + b|r_1\rangle$ and $|\phi_1\rangle = c|r_0\rangle + d|r_1\rangle$ where a, b, c, and d are complex and $(ad - bc) = 1$. Thus, analogously with Eq. (2), any state $|\phi\rangle$ in $\mathcal{P}$ can be written as

$$|\phi\rangle = \beta_0 |\phi_0\rangle + \beta_1 |\phi_1\rangle.$$

With every state $|\phi_k\rangle$, we can associate the matrix $[\Phi_k]_{ij} = \langle ji | \phi_k \rangle$. We can also suppose without loss of generality that Φ_1 is invertible. Instead of Eq. (5), we would thus get

$$(\Phi_1^{-1} \Phi_0 - \mu) u_a^* = 0,$$

where $\mu = -\beta_1 / \beta_0$. What aspects are common to matrices $R_1^{-1} R_0$ and $\Phi_1^{-1} \Phi_0$ and how are their respective eigenvalues λ_l and μ_l related? To answer this question we define the concept of the *Jordan family*. We will say that two matrices, A and B, are at the same Jordan family if for each eigenvalue λ_l of A there is an eigenvalue μ_l of B such that the rank of the matrices $(A - \lambda_l)^k$ and $(B - \mu_l)^k$ are equal for every positive integer k. This is equivalent to saying that the Jordan blocks of the matrices A and B in their Jordan canonical form have the same structure ([22] Sec. 3.2), although they can differ in the numerical values of the eigenvalues. The following theorem asserts that the matrices $R_1^{-1} R_0$ and $\Phi_1^{-1} \Phi_0$ are in the same Jordan family.

Theorem 1. Let R_0 and R_1 be two n by n matrices, R_1 invertible, and $\Phi_0 = aR_0 + bR_1$, $\Phi_1 = cR_0 + dR_1$ two linear combinations of R_0 and R_1 such that Φ_1 is also invertible and $(ad - bc) = 1$. Then the matrices $R_1^{-1} R_0$ and $\Phi_1^{-1} \Phi_0$ belong to the same Jordan family. Moreover, the relation between the eigenvalues λ_l of $R_1^{-1} R_0$ and μ_l of $\Phi_1^{-1} \Phi_0$, such that, for all positive integer k, $\mathrm{rank}(R_1^{-1} R_0 - \lambda_l)^k = \mathrm{rank}(\Phi_1^{-1} \Phi_0 - \mu_l)^k$ is given by

$$\mu_l = \frac{a\lambda_l + b}{c\lambda_l + d}. \tag{6}$$

Proof. Let us consider first the case $c = 0$, so that $a \neq 0$, $d \neq 0$, and straightforward evaluation gives

$$(\Phi_1^{-1} \Phi_0 - \mu_l)^k = \left[(dR_1)^{-1} (aR_0 + bR_1) - \frac{a\lambda_l + b}{d} \right]^k = \frac{a^k}{d^k} (R_1^{-1} R_0 - \lambda_l)^k,$$

from which the desired result follows. In case $c \neq 0$, we must have $c\lambda_l + d \neq 0$, since $c\lambda_l + d$ is an eigenvalue of Φ_1^{-1} which is invertible. In this case we first notice that

$$\begin{aligned} \Phi_1^{-1} \Phi_0 &= (cR_0 + dR_1)^{-1} (aR_0 + bR_1) \\ &= [R_1 (cR_1^{-1} R_0 + d)]^{-1} R_1 (aR_1^{-1} R_0 + b) \\ &= (cR_1^{-1} R_0 + d)^{-1} (aR_1^{-1} R_0 + b). \end{aligned}$$

Dividing the polynomial $az + b$ by $cz + d$ we get complex γ and δ such that

$$az + b = \gamma(cz + d) + \delta,$$

for any complex z. Thus it is clear that $aR_1^{-1} R_0 + b = \gamma(cR_1^{-1} R_0 + d) + \delta$. Thus we have

$$\begin{aligned} \Phi_1^{-1} \Phi_0 &= (cR_1^{-1} R_0 + d)^{-1} [\gamma(cR_1^{-1} R_0 + d) + \delta] \\ &= \gamma + \delta(cR_1^{-1} R_0 + d)^{-1} \end{aligned}$$

and

$$\begin{aligned} (\Phi_1^{-1} \Phi_0 - \mu_l)^k &= \left(\gamma + \delta(cR_1^{-1} R_0 + d)^{-1} - \gamma - \frac{\delta}{c\lambda_l + d} \right)^k \\ &= \delta^k c^k (c\lambda_l + d)^{-k} (cR_1^{-1} R_0 + d)^{-k} (\lambda_l - R_1^{-1} R_0)^k. \end{aligned}$$

As Φ_0 and Φ_1 are linearly independent, $\delta \neq 0$. Moreover, as Φ_1 is invertible, $c\lambda_l + d \neq 0$, since its an eigenvalue of Φ_1, and $cR_1^{-1} R_0 + d$ is also invertible. Then we have that $\mathrm{rank}(\Phi_1^{-1} \Phi_0 - \mu_l)^k = \mathrm{rank}(\lambda_l - R_1^{-1} R_0)^k$ as desired. ∎

Therefore for each eigenvalue λ_l found using a basis $\{|r_0\rangle, |r_1\rangle\}$, the use of a different basis $\{|\phi_0\rangle, |\phi_1\rangle\}$, would also give a corresponding eigenvalue μ_l. Moreover, the rank of the matrix $(R_1^{-1} R_0 - \lambda_l)^k$ is equal to the rank of $(\Phi_1^{-1} \Phi_0 - \mu_l)^k$. This rank for $k = 1$ (first rank for short) is simply the Schmidt rank of the state $|\phi_l\rangle = \alpha_{0l} |r_0\rangle + \alpha_{1l} |r_l\rangle = \beta_{0l} |\phi_0\rangle + \beta_{1l} |\phi_1\rangle$, where α_{0l} and α_{1l} are such that $\lambda_l = -\alpha_{1l} / \alpha_{0l}$ and β_{0l} and β_{1l} are such that $\mu_l = -\beta_{1l} / \beta_{0l}$. Thus the first rank can be understood as a property of the state $|\phi_l\rangle$ alone, since it will not change if the same state $|\phi_l\rangle$ is also found in another plane. The same is not true for the higher ranks $(k \geqslant 2)$, which can be different for the same $|\phi_l\rangle$ in different planes. In this way, we must understand these higher ranks as invariant properties of the state $|\phi_l\rangle$ inside the plane $\mathcal{P}$ and also as invariant properties of the whole tripartite entanglement state $|\psi\rangle$ from which $\mathcal{P}$ is obtained. The distinction of these higher ranks is important for $n \geqslant 4$. We write these states explicitly for entangled states of dimensionality (4, 4, 2) in example 3.

If subsystems s_a and s_b are interchanged the result is equivalent. In this case the states $|r_0\rangle$ and $|r_1\rangle$ in Eq. (3) must be understood as linear mappings from C_b^{n*} in C_a^n and, instead of the matrix R_0 and R_1 in Eq. (4), we will get their respective transposes R_0^T and R_1^T. Thus in place of the matrix $R_1^{-1} R_0$

in Eq. (5), we will get the matrix $(R_0 R_1^{-1})^T$ which is similar to it.

We can now use the solutions of Eq. (5) to find states in $\mathcal{P}$ with Schmidt rank smaller than n and then rewrite the state $|\psi\rangle$ in terms of a smaller number of product states. More explicitly, suppose, we have found two solutions and then write its respective normalized bipartite states $|\phi_1\rangle$ and $|\phi_2\rangle$ in $\mathcal{P}$ with Schmidt rank smaller than n. Using the basis $\{|\phi_1\rangle, |\phi_2\rangle\}$ to span $\mathcal{P}$, we have

$$|\psi\rangle = |\phi_1\rangle |c_1\rangle + |\phi_2\rangle |c_2\rangle, \tag{7}$$

where $|c_1\rangle$ and $|c_2\rangle$ are appropriate non-normalized states in C_c^2, given by

$$|c_1\rangle = \sum_k c_k |k\rangle (g_{11}\langle\phi_1|r_k\rangle + g_{12}\langle\phi_2|r_k\rangle),$$

$$|c_2\rangle = \sum_k c_k |k\rangle (g_{21}\langle\phi_1|r_k\rangle + g_{22}\langle\phi_2|r_k\rangle).$$

The metric coefficients g_{ij} appear from the fact that $|\phi_1\rangle$ and $|\phi_2\rangle$ are in general nonorthogonal states. We must observe that $|c_1\rangle$ and $|c_2\rangle$ are in general also nonorthogonal. From Eq. (7), it is easy to see that $|\psi\rangle$ may be written in terms of a number of products given by the sum of the Schmidt ranks of $|\phi_1\rangle$ and $|\phi_2\rangle$. Equation (5) may have from one to n distinct eigenvalues. In the case of a single eigenvalue, we have just one state with Schmidt rank smaller than n and will have to choose another state in $\mathcal{P}$ with Schmidt rank n. When we find $m \geq 2$ solutions of Eq. (5), we have

$$\begin{pmatrix} m \\ 2 \end{pmatrix}$$

distinct combinations of $|\phi_1\rangle$ and $|\phi_2\rangle$ to write $|\psi\rangle$ in Eq. (7). Of course, it is always possible to choose a state with Schmidt rank smaller than n and another with Schmidt rank n and each $|\phi_k\rangle$ has also infinitely many bipartite decompositions. Then there will be always infinitely many sub-Schmidt decompositions of $|\psi\rangle$ in Eq. (7).

Therefore for each Jordan family to which the matrix $R_1^{-1}R_0$ may belong we can associate a family of entangled states $|\psi\rangle$. These entangled states will be all of dimensionality $(n,n,2)$, except for the family of matrices proportional to the identity matrix which implies that matrices R_0 and R_1 are proportional and therefore that the qubit is not entangled with the other two n-dimensional subsystems, i.e., in this case, we have an ordinary bipartite entanglement with Schmidt rank n of the two n-dimensional subsystems.

We will see in the following section that states which belong to distinct families in fact belong to distinct SLOCC classes. Before discussing the relation between the Jordan families and SLOCC classification, it is convenient to discuss some examples of Jordan families and their respective sub-Schmidt decomposition of their corresponding entangled states.

A. Example 1: Three qubits

The properties of three qubit entangled states are well known [1,15,16]. Here we reproduce known results for this case in terms of the procedure described above as an example of its use. A state $|\psi\rangle$ with entanglement of dimensionality (2, 2, 2) can be identified with one of the following two Jordan families:

$$\text{(a)}:\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}, \quad \text{(b)}:\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where $\lambda_1 \neq \lambda_2$. In case (a), there is only one state,

$$|\phi_{\lambda_1}\rangle = \alpha_{\lambda_1 0}|r_0\rangle + \alpha_{\lambda_1 1}|r_1\rangle,$$

where $\lambda_1 = -\alpha_{\lambda_1 1}/\alpha_{\lambda_1 0}$, with Schmidt rank 1 in $\mathcal{P}$, i.e., $|\phi_{\lambda_1}\rangle$ is the only unentangled state in $\mathcal{P}$. Then, if we want to span $\mathcal{P}$, we have to choose another state $|\phi_2\rangle \in \mathcal{P}$ with Schmidt rank 2 in $\mathcal{P}$. From Eq. (7), it follows that $|\psi\rangle$ can be written as a superposition of three product states. This means that $|\psi\rangle$ is in class W [15], since it can be converted through some SLOCC to the state

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle).$$

In case (b), on the other hand, we have two unentangled states in $\mathcal{P}$, one for each λ_l, given by

$$|\phi_{\lambda_l}\rangle = \alpha_{\lambda_l 0}|r_0\rangle + \alpha_{\lambda_l 1}|r_1\rangle,$$

where $\lambda_l = -\alpha_{\lambda_l 1}/\alpha_{\lambda_l 0}$. Then we can write $|\psi\rangle$ as a superposition of two product states, meaning that $|\psi\rangle$ belongs to class GHZ, since it can be converted through some SLOCC to the state [15]

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

We observe that our method provides a way to decide whether a given state is in class W or in class GHZ which is alternate to that proposed in Ref. [15]. The dimensionality of entanglement can easily be obtained from the determinant of the reduced density matrices of subsystems. Once we have verified that a state $|\psi\rangle$ involves an entanglement of dimensionality (2, 2, 2), we just have to verify whether Eq. (5) has one or two solutions. With little further calculation we can also get the sub-Schmidt decompositions. We will also see in Sec. III that we can constructively determine the SLOCC which transforms the considered state into the state $|\text{W}\rangle$ or $|\text{GHZ}\rangle$. All other states with entanglement dimensionality smaller than (2, 2, 2) (smaller meaning that at least one of local supports has smaller dimensionality and none has higher) show ordinary bipartite entanglement or are completely unentangled states.

B. Example 2: One qubit and two three-level systems

In this example, we show new families of entangled states which are simple to write down and which provide insight for more general systems with higher entanglement dimensionality. Let $|\psi\rangle$ be an entangled state of dimensionality (3, 3, 2). Then $|\psi\rangle$ must be in one of the following five Jordan families:

$$(a):\begin{pmatrix}\lambda_1 & 1 & 0\\ 0 & \lambda_1 & 1\\ 0 & 0 & \lambda_1\end{pmatrix},\quad (b):\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_1 & 1\\ 0 & 0 & \lambda_1\end{pmatrix},$$

$$(c):\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 1\\ 0 & 0 & \lambda_2\end{pmatrix},\quad (d):\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_2\end{pmatrix},$$

$$(e):\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix},$$

where $\lambda_l \neq \lambda_{l'}$ for $l \neq l'$. For each one of this families:

(a): There is only one $|\phi_{\lambda_1}\rangle$ with Schmidt rank 2 in $\mathcal{P}$. Then, to span $\mathcal{P}$, we need to choose another $|\phi_2\rangle \in \mathcal{P}$ with Schmidt rank 3. From Eq. (7), we get that $|\psi\rangle$ can be written as a superposition of five product states. We choose as the characteristic example of this family the state

$$|\psi_a\rangle = \frac{1}{\sqrt{5}}[(|10\rangle + |21\rangle)|0\rangle + (|00\rangle + |11\rangle + |22\rangle)|1\rangle].$$

(b): There is only one state $|\phi_{\lambda_1}\rangle$ with Schmidt rank 1 in $\mathcal{P}$. Then we need to choose another $|\phi_2\rangle \in \mathcal{P}$ with Schmidt rank 3 to write $|\psi\rangle$ as a superposition of four product states. We choose as the characteristic example of this family the state

$$|\psi_b\rangle = \frac{1}{2}[|21\rangle|0\rangle + (|00\rangle + |11\rangle + |22\rangle)|1\rangle].$$

(c): There are two states, $|\phi_{\lambda_1}\rangle$ and $|\phi_{\lambda_2}\rangle$ with Schmidt rank 2 in $\mathcal{P}$. Then we can use them to span $\mathcal{P}$ and, using Eq. (7), write $|\psi\rangle$ as a superposition of four product states. We choose as the characteristic example of this family the state

$$|\psi_c\rangle = \frac{1}{2}[(|00\rangle + |21\rangle)|0\rangle + (|11\rangle + |22\rangle)|1\rangle].$$

(d): There is one state $|\phi_{\lambda_1}\rangle$ with Schmidt rank 1 and also one state $|\phi_{\lambda_1}\rangle$ with Schmidt rank 2 in $\mathcal{P}$. Then we can use them to write $|\psi\rangle$ as a superposition of three product states. We choose as the characteristic example of this family the state

$$|\psi_d\rangle = \frac{1}{\sqrt{3}}[|00\rangle|0\rangle + (|11\rangle + |22\rangle)|1\rangle].$$

(e): There are three states $|\phi_{\lambda_l}\rangle$ with Schmidt rank 2 in $\mathcal{P}$. As we need only two to span $\mathcal{P}$, we have three ways in Eq. (7) to write $|\psi\rangle$ as a superposition of four product states. We choose as the characteristic example of this family the state

$$|\psi_e\rangle = \frac{1}{2}[(|00\rangle + |11\rangle)|0\rangle + (|11\rangle + |22\rangle)|1\rangle].$$

Therefore an entangled state $|\psi\rangle$ of dimensionality (3, 3, 2) can be classified in five distinct Jordan families which correspond to five distinct ways of sub-Schmidt decomposing it in terms of 3, 4, or 5 product states. Moreover, we see that there are three families with a sub-Schmidt decomposition of four product states. We see therefore that is not just the number of product states that distinguishes entangled states, but also the nature of the decomposition and the number of distinct decompositions [compare, e.g., cases (b), (c), and (d), which involve four product states]. As will be shown in Sec. III, each one of these families corresponds to a distinct SLOCC class.

C. Example 3: One qubit and two four level subsystems

We will not list explicitly all families for the entangled states of dimensionality (4, 4, 2). There are in all 13 families, and we will limit ourselves to discuss those which emphasize some aspects that did not arise in connection with example 2.

We start with the following situation. Suppose that Alice, Bob, and Carol share three qubits in a state $|\mathrm{GHZ}\rangle$ or $|\mathrm{W}\rangle$ and that Alice and Bob also share two qubits in the Bell state $|\phi^+\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle)$. Then we can consider Alice's and Bob's two qubits as our four-level subsystems and the state of all the five qubits is an entangled state of dimensionality (4, 4, 2). The sub-Schmidt decompositions of these two states are most easily obtained directly from the evaluation of the tensor products $|\mathrm{GHZ}\rangle \otimes |\phi^+\rangle$ and $|\mathrm{W}\rangle \otimes |\phi^+\rangle$, i.e.,

$$|\mathrm{GHZ}\rangle \otimes |\phi^+\rangle = \frac{1}{2}[(|00,00\rangle + |01,01\rangle)|0\rangle + (|10,10\rangle + |11,11\rangle)|1\rangle]$$

and

$$|\mathrm{W}\rangle \otimes |\phi^+\rangle = \frac{1}{\sqrt{6}}[(|00,00\rangle + |01,01\rangle)|1\rangle + (|00,10\rangle + |01,11\rangle + |10,00\rangle + |11,01\rangle)|0\rangle].$$

Our procedure further reveals that these sub-Schmidt decompositions are the smallest ones and that these states belong respectively to the following Jordan families:

$$(a):A = \begin{pmatrix}\lambda_1 & 0 & 0 & 0\\ 0 & \lambda_1 & 0 & 0\\ 0 & 0 & \lambda_2 & 0\\ 0 & 0 & 0 & \lambda_2\end{pmatrix} \text{ and } (b):B = \begin{pmatrix}\lambda_1 & 1 & 0 & 0\\ 0 & \lambda_1 & 0 & 0\\ 0 & 0 & \lambda_1 & 1\\ 0 & 0 & 0 & \lambda_1\end{pmatrix}.$$

Note that the Jordan family corresponding to B differs from that corresponding to

$$(c):C = \begin{pmatrix}\lambda_1 & 0 & 0 & 0\\ 0 & \lambda_1 & 1 & 0\\ 0 & 0 & \lambda_1 & 1\\ 0 & 0 & 0 & \lambda_1\end{pmatrix}$$

only in that the ranks of $(B-\lambda_1)^k$ and $(C-\lambda_1)^k$ differ for $k=2$. In this way, the local support planes in $C_b^4 \otimes C_c^4$, $\mathcal{P}_b$ and $\mathcal{P}_c$, of any state belonging to one of the families (b) or (c) will have only one state $|\phi_{\lambda_1}\rangle$ with Schmidt rank 2 and all other states in $\mathcal{P}_b$ and $\mathcal{P}_c$ will have Schmidt rank 4. Thus any state in families (b) or (c) will have a smallest decomposition with six products states. An example of a state in this family (c) is

$$|\psi_c\rangle = \frac{1}{\sqrt{6}}[(|10,01\rangle + |11,10\rangle)|1\rangle + (|00,00\rangle + |01,01\rangle + |10,10\rangle + |11,11\rangle)|0\rangle].$$

Other Jordan families that differ in the higher-k ranks are

$$(d):D = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} \text{ and}$$

$$(e):E = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}.$$

The ranks of $(D-\lambda_1)^k$ and of $(E-\lambda_1)^k$ differ for $k \geqslant 2$, while the ranks of $(D-\lambda_2)^k$ and of $(E-\lambda_2)^k$ differ for $k \geqslant 3$. Their respective sub-Schmidt decompositions will also involve six product states. Examples of states of these classes are

$$|\psi_d\rangle = \frac{1}{\sqrt{6}}[(|11\rangle + |22\rangle + |33\rangle)|0\rangle + (|00\rangle + |21\rangle + |32\rangle)|1\rangle]$$

and

$$|\psi_e\rangle = \frac{1}{\sqrt{6}}[(|10\rangle + |22\rangle + |33\rangle)|0\rangle + (|00\rangle + |11\rangle + |32\rangle)|1\rangle]$$

belonging to Jordan families (d) and (e), respectively.

Two other families which are distinguished in higher-k ranks, $k \geqslant 2$, for either of the two eigenvalues are

$$(f):\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} \text{ and } (g):\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}.$$

Examples of states in these families are

$$|\psi_f\rangle = \frac{1}{\sqrt{5}}[(|11\rangle + |22\rangle + |33\rangle)|0\rangle + (|00\rangle + |23\rangle)|1\rangle]$$

and

$$|\psi_g\rangle = \frac{1}{\sqrt{5}}[(|22\rangle + |33\rangle + |10\rangle)|0\rangle + (|00\rangle + |11\rangle)|1\rangle].$$

Another interesting family is

$$(h)\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix},$$

which is the only one at this entanglement dimensionality that needs to be subdivided into an infinity of SLOCC classes as will be seen in Sec. III. The existence of infinitely many SLOCC classes for entanglements of higher dimensionality was already noted by Dür *et al.* [15] using a counting parameter argument. We can write as an example of a state in this family the state

$$|\psi_h\rangle = \frac{1}{\sqrt{4+2|a|^2}}[(|11\rangle + a|22\rangle + |33\rangle)|0\rangle + (|00\rangle + a|11\rangle + |22\rangle)|1\rangle],$$

where $a \neq 0$, so that the associated component does not vanish, and $a \neq 1$, which reflects the fact that all components cannot be made simultaneously equal. This state has also five more sub-Schmidt decompositions.

III. SUB-SCHMIDT DECOMPOSITIONS AND SLOCC

In this section we discuss the relation between the results of Sec. II and transformation of entangled states through SLOCC protocols. We start by establishing Lemma 1 relating the Jordan canonical forms of the matrices $R_1^{-1}R_0$ in Eq. (5) resulting from two states that are interconvertible through some SLOCC. After this, with further analysis of the relation between the eigenvalues given by Eq. (6), we establish our criterion to verify whether two given entangled states of dimensionality $(n,n,2)$ are related by SLOCC. The result is stated as Theorem 2.

In order to determine whether a pure state $|\psi\rangle$ can be transformed into a state $|\psi'\rangle$ through SLOCC we can use the following theorem given in Ref. [15]: a pure state $|\psi\rangle$ can be transformed into a pure state $|\psi'\rangle$ through a SLOCC if a local linear operator $A \otimes B \otimes C$ exists such that

$$|\psi'\rangle = A \otimes B \otimes C|\psi\rangle, \tag{8}$$

where A, B, and C are linear operators in C_a^n, C_b^n, and C_c^2, respectively [24]. If we consider only invertible local linear operators in Eq. (8) we get an equivalence relation between $|\psi\rangle$ and $|\psi'\rangle$ which corresponds to the same equivalence relation defined by invertible SLOCC.

Let us consider what happens when using this result on the decompositions developed in the preceding section. Suppose that relation (8) is satisfied for some entangled states $|\psi\rangle$ and $|\psi'\rangle$ of dimensionality $(n,n,2)$ and some invertible linear operators A, B, and C. Then, writing $|\psi\rangle$ as in Eq. (1) and inserting in Eq. (8), we have

$$|\psi'\rangle = \sum_{k=0,1} A \otimes B|r_k\rangle C(c_k|k\rangle) = \sum_{k=0,1} |\phi_k'\rangle|c_k'\rangle, \tag{9}$$

where $|c_k'\rangle = C(c_k|k\rangle)$ and $|\phi_k'\rangle = A \otimes B|r_k\rangle$. Note that the states $|c_k'\rangle$ are in general nonorthogonal, and that therefore the states $|\phi_k'\rangle$ are not in general the corresponding relative states of $|c_k'\rangle$ for $|\psi'\rangle$. Since we did not normalize the operators A, B, and C, the states $|c_k'\rangle$, $|\phi_k'\rangle$, and consequently $|\psi'\rangle$ are not normalized. We observe that the invertible linear operator C can transform the states $c_k|k\rangle$ into any two distinct states $|c_k'\rangle$. The operator $A \otimes B$ is obviously not so general in $C_a^n \otimes C_b^n$. A well-known fact is that it preserves the Schmidt rank of any state [15]. In order to understand this, we will study the relation between the local supports $\mathcal{P}$ and $\mathcal{P}'$ of $|\psi\rangle$ and $|\psi'\rangle$, respectively, in $C_a^n \otimes C_b^n$. It is clear that, for any $|\phi\rangle \in \mathcal{P}$, there is a unique $|\phi'\rangle \in \mathcal{P}'$ such that $|\phi'\rangle = A\otimes B|\phi\rangle$. Particularly, for each one of the states $|r_k\rangle$, we have

$$|\phi_k'\rangle = A\otimes B|r_k\rangle.$$

Writing the operators A and B in the local bases of their respective subsystems, i.e., $\{|i\rangle\}$ for A in C_a^n and $\{|j\rangle\}$ for B in C_b^n, and the matrices $[R_k]_{ij}=\langle ji|r_k\rangle$ and $[\Phi_k']_{ij}=\langle ji|\phi_k'\rangle$ as we did in Eq. (4), we get

$$\Phi_k' = BR_kA^T,$$

where the matrices R_k and Φ_k' have the components $[R_k]_{ij}=\langle ji|r_k\rangle$ and $[\Phi_k']_{ij}=\langle ji|\phi_k'\rangle$, respectively, as in Sec. II. Thus if we use the basis $\{|\phi_k'\rangle\}$ to evaluate Eq. (5) for the state $|\psi'\rangle$, we will get

$$\Phi_1'^{-1}\Phi_0' = A^{T^{-1}}R_1^{-1}R_0A^T. \tag{10}$$

Then, we see that the matrices $R_1^{-1}R_0$ and $\Phi_1'^{-1}\Phi_0'$ are similar, i.e., they have the same Jordan canonical form. Note that Φ_1' is invertible if R_1 is invertible.

The fact that the operator B no longer appears in Eq. (10) does not imply that it is unimportant, since the plane $\mathcal{P}'$ generated by the basis $\{A\otimes B|r_0\rangle, A\otimes B|r_1\rangle\}$ is obviously distinct from the plane $\mathcal{P}''$ generated by the basis $\{A\otimes I_b|r_0\rangle, A\otimes I_b|r_1\rangle\}$, where I_b is the identity operator in C_b^n, although they result in the same matrix $A^{T^{-1}}R_1^{-1}R_0A^T$. Moreover, if we interchange the roles of subsystem s_a and s_b, we find that A is the operator which that no longer appears between matrices $R_1^{T^{-1}}$ and R_0^T, that is,

$$(\Phi_0'\Phi_1'^{-1})^T = B^{T^{-1}}(R_0R_1^{-1})^TB^T. \tag{11}$$

We notice also that matrices $(R_0R_1^{-1})^T$ and $(\Phi_0'\Phi_1'^{-1})^T$ are similar if the matrices $R_1^{-1}R_0$ and $\Phi_1'^{-1}\Phi_0'$ are also similar. Thus the existence of a matrix B such that the matrices $(R_0R_1^{-1})^T$ and $(\Phi_0'\Phi_1'^{-1})^T$ are similar is equivalent to the existence of a matrix A such that the matrices $R_1^{-1}R_0$ and $\Phi_1'^{-1}\Phi_0'$ are similar. The existence of either of the matrices A or B in Eqs. (10) and (11) is thus equivalent to the initial supposition that the states $|\psi\rangle$ and $|\psi'\rangle$ are in the same SLOCC class. We have therefore proved the following lemma:

Lemma 1. Two entangled states of dimensionality $(n,n,2)$, $|\psi\rangle$ and $|\psi'\rangle$, are interconvertible through SLOCC if, and only if, for any basis $\{|r_k\rangle\}$ for the local support of $|\psi\rangle$ in $C_a^n\otimes C_b^n$, a basis $\{|\phi_k'\rangle\}$ for the local support of $|\psi'\rangle$ in $C_a^n\otimes C_b^n$ exists such that the respective matrices $R_1^{-1}R_0$ and $\Phi_1'^{-1}\Phi_0'$ are similar.

From this Lemma 1 and Theorem 1, it is clear that two given states, $|\psi\rangle$ and $|\psi'\rangle$, are interconvertible through some SLOCC only if they are in the same Jordan family. However, if $|\psi\rangle$ and $|\psi'\rangle$ belong to the same Jordan family the situation is not to simple and we need further work on Eq. (6) to verify whether $|\psi\rangle$ and $|\psi'\rangle$ are in the same SLOCC class. This can be stated as follows. Let $|\psi\rangle$ and $|\psi'\rangle$ be two entangled states of dimensionality $(n,n,2)$ and suppose we get the bases $\{|r_k\rangle\}$ and $\{|r_k'\rangle\}$ for the local support planes $\mathcal{P}$ and $\mathcal{P}'$ in $C_a^n\otimes C_b^n$ from the states $|\psi\rangle$ and $|\psi'\rangle$ as in Eq. (2). Using the method of Sec. II, we find that the matrices $R_1^{-1}R_0$ and $R_1'^{-1}R_0'$ are in the same Jordan family and have the eigenvalues $\{\lambda_{l,r}\}$ and $\{\lambda_{l,r'}\}$, respectively, where the index $l=1,2,\ldots L$ (L being the number of distinct eigenvalues) is such that rank $(R_1^{-1}R_0-\lambda_{l,r})^k$=rank $(R_1'^{-1}R_0'-\lambda_{l,r'})^k$, that is, the Jordan blocks corresponding to the eigenvalues $\lambda_{l,r}$ and $\lambda_{l,r'}$ have the same structure. Thus we need to verify whether we can find a basis $\{|\phi_k'\rangle\}$ for $\mathcal{P}'$ such that the eigenvalues of the respective matrix $\Phi_1'^{-1}\Phi_0'$, $\{\mu_{l,\phi'}\}$, are all equal to $\lambda_{l,r}$ for each l. Using Eq. (6) to get $\mu_{l,\phi'}$ as a function of $\lambda_{l,r'}$, we must require that the equation

$$\lambda_{l,r} = \mu_{l,\phi'} = \frac{a\lambda_{l,r'}+b}{c\lambda_{l,r'}+d}$$

or

$$\lambda_{l,r}\lambda_{l,r'}c + \lambda_{l,r}d - \lambda_{l,r'}a - b = 0 \tag{12}$$

have at least one solution for all l's with the additional condition that $(ad-bc)=1$, that is, we have a linear system with L equations with an additional constraint for the variables a, b, c, and d which correspond to the coefficients of the linear transformation $\Phi_0'=aR_0'+bR_1'$ and $\Phi_1'=cR_0'+dR_1'$. Notice that $c\lambda_{l,r'}+d\neq 0$, since it is an eigenvalue of $\Phi_1'^{-1}$, which is invertible. We observe that any nontrivial solution of the linear system (12) intersects the surface defined by the additional constraint $(ad-bc)=1$ at two opposite points [25], that is, if the linear system (12) has a nontrivial solution, then there are always at least two solutions satisfying also the additional constraint $(ad-bc)=1$ and differing by a sign. In this way, we have reduced the problem of deciding if $|\psi\rangle$ and $|\psi'\rangle$ are in the same SLOCC class to the existence of a nontrivial solution of the homogeneous linear system (12) with L equations.

In case the determinant of the linear system (12) has some nonvanishing minor of dimension larger than 3, its unique solution is the trivial one which is incompatible with the condition $(ad-bc)=1$. Thus the states $|\psi\rangle$ and $|\psi'\rangle$ will not be in the same SLOCC class. Hence a Jordan family of states with more than three distinct eigenvalues can be subdivided into an infinity of SLOCC classes defined by the constraints that all minors of Eq. (12) of dimension greater than 3 must vanish. In case that the greatest nonvanishing minor of Eq. (12) has dimension 3, then there are always two solutions which differ only by a sign. Thus, in this case, we can find a, b, c, and d such that the transformation from the basis $\{|r_0'\rangle, |r_1'\rangle\}$ to $\{|\phi_0'\rangle, |\phi_1'\rangle\}$ will result in a matrix $\Phi_1'^{-1}\Phi_0'$ similar to $R_1^{-1}R_0$ and we can find also invertible local operators A, B, and C such that the expression (8) holds, and so the states $|\psi\rangle$ and $|\psi'\rangle$ are interconvertible through SLOCC. The operator A can be obtained from Eq. (10). Similarly, the operator B can be obtained from Eq. (11). The operator C can now be obtained from Eq. (9). Hence if we have $L\leqslant 3$ it is always possible to find a nontrivial solution of Eq. (12), and every Jordan family with less than three eigenvalues is equivalent to a SLOCC class. We have therefore proved the following theorem:

Theorem 2. Let $|\psi\rangle$ and $|\psi'\rangle$ two entangled states of dimensionality $(n,n,2)$ and let $\{|r_k\rangle\}$ and $\{|r_k'\rangle\}$ being basis for the local support of $|\psi\rangle$ and $|\psi'\rangle$ in $C_a^n \otimes C_b^n$, respectively. Then the states $|\psi\rangle$ and $|\psi'\rangle$ are interconvertible through SLOCC if, and only if, the respective matrices $R_1^{-1}R_0$ and $R_1'^{-1}R_0'$ are in the same Jordan family and a nontrivial solution of the linear system (12) exist. Particularly, if $R_1^{-1}R_0$ and $R_1'^{-1}R_0'$ belong to a same Jordan family with three or less distinct eigenvalues, then $|\psi\rangle$ and $|\psi'\rangle$ are interconvertible through SLOCC.

When the greatest nonvanishing minor of Eq. (12) has dimension smaller than 3, there will be one or two free parameters in Eq. (12). These free parameters in principle may allow for the existence of an infinity of SLOCC protocols depending on SLOCC class. This may easily be seen to be actually the case in specific examples, e.g., two states in class W. However, examples also can be found in which the matrix $\Phi_1'^{-1}\Phi_0'$ turns out to be independent of the remaining free parameter, e.g., two states in class GHZ.

When more than one distinct eigenvalue is associated with the same Jordan block structure there will be more than one way to label the two set of eigenvalues, $\{\lambda_{l,r}\}$ and $\{\lambda_{l,r'}\}$. The considered states, $|\psi\rangle$ and $|\psi'\rangle$, will be in the same SLOCC class provided at least one labeling can be found for which we can get a solution to Eq. (12). In many cases, it will be possible to find a solution to Eq. (12) for many labelings. Particularly, for two states in class GHZ, there will be always a solution for each one of the two possible labelings.

IV. DISCUSSION: MORE GENERAL TRIPARTITE ENTANGLED STATES

It is easy to obtain an equation similar to Eq. (4) for general tripartite systems. However, it appears to be very difficult to classify the possible solutions. In the case involving one qubit we have the bonus that we could transform Eq. (4) into an eigenvalue problem. In the general case it appears to be very difficult to avoid being led to a system of polynomial equations in many variables.

As an example, suppose we have an entangled state $|\psi\rangle$ of dimensionality (n,n,n). Using the reasoning of Sec. II, we get that the sum in Eq. (1) now has n terms and the local support for the subsystem s_{ab} is some n-dimensional hyperplane $\mathcal{P} \in C_a^n \otimes C_b^n$. We could easily get a basis for this hyperplane using some bipartite decomposition as done in Eq. (1) in Sec. II, i.e., we can write any state $|\phi\rangle \in \mathcal{P}$ as

$$|\phi\rangle = \sum_k \alpha_k |r_k\rangle,$$

where α_k are complex coefficients and $|r_k\rangle \in C_a^n \otimes C_b^n$ is the relative state of $|k\rangle \in C_c^n$, $\{|k\rangle\}$ being an orthonormal basis in C_c^n. Using a basis $\{|i\rangle\}$ in C_a^n, a basis $\{|j\rangle\}$ in C_b^n, and a state $|u_a\rangle \in C_a^n$, we get an equation similar to Eq. (4), but with n matrices

$$\left(\sum_k \alpha_k R_k\right) u_a^* = 0, \qquad (13)$$

where, like in Eq. (4), R_k has the components $[R_k]_{ij} = \langle ji | r_k\rangle$ and the vector u_a^* has the components $u_{a_i}^* = \langle u_a | i\rangle$. Keeping in mind that we are looking for the superpositions of matrices with rank less than n, we can easily get from

$$\det\left(\sum_k \alpha_k R_k\right) = 0 \qquad (14)$$

that there is at least one $(n-1)$-dimensional surface $S \subset \mathcal{P}$ in which the states have Schmidt rank $(n-1)$ or less. As the constraint defined by Eq. (14) is obviously nonlinear, there are n states in S that span $\mathcal{P}$. This means that there are infinitely many sub-Schmidt decompositions of $|\psi\rangle$ with $n(n-1)$ products. In order to identify the smaller sub-Schmidt decompositions, we need to verify the existence of some set of α_k's which makes null all $(n-1)$ minors of the matrix $(\Sigma_k \alpha_k R_k)$. For the simplest case with $n=3$, this gives us a system of nine polynomials in two variables. Therefore, for general systems, we cannot avoid very impractical conditions. However, we know that states with sub-Schmidt decompositions exist, since we can explicitly write entangled states of dimensionality (n,n,n) with less than $n(n-1)$ products. As an example, we take some kind of general GHZ state in n dimensions,

$$|\psi_{\mathrm{GHZ}}\rangle = \sum_k d_k |a_k\rangle |b_k\rangle |c_k\rangle, \qquad (15)$$

where the d_k are some complex coefficient and $\{|a_k\rangle\}$, $\{|b_k\rangle\}$, and $\{|c_k\rangle\}$ are linearly independent states in C_a^n, C_b^n, and C_c^n, respectively. Particularly, it is easy to see that the family of all states that can be written in the form (15) are in a single SLOCC class, since there are always invertible local operators A in n dimensions taking an arbitrary set of n linearly independent vectors $d_k|a_k\rangle$ into any other n linearly independent vectors $d_k'|a_k'\rangle$. It is similar for B taking $|b_k\rangle$ into $|b_k'\rangle$ and C taking $|c_k\rangle$ into $|c_k'\rangle$. Thus there are always invertible local operators A, B, and C to satisfy Eq. (8).

We must also observe that we cannot claim to have obtained a full classification of entangled states in a space $C_a^n \otimes C_b^n \otimes C_c^2$. In the space $C_a^3 \otimes C_b^3 \otimes C_c^2$ of example 2, for example, we can have factorable states, bipartite states of the three types and the two classes of example 1, which were all previously known, and furthermore, the five additional classes of example 2. However, in this space, we also have entangled states of dimensionality (3, 2, 2). In this case, when we use the method of Sec. II we will get a local support in $C_a^3 \otimes C_b^3$ with no state with Schmidt rank 3 and almost all with Schmidt rank 2. Thus all the matrices in Eq. (4) are noninvertible and we cannot reduce the problem to an eigenvalue form like Eq. (5). This case was in fact solved by Miyake and Verstraete [20], however, there are many other cases in which the local supports of Alice and Bob are distinct and we do not know a solution, e.g., an entangled state of dimensionality (4, 3, 2).

V. CONCLUSION

We have described a constructive method to find decompositions of tripartite entangled pure states with one qubit which involve a number of terms smaller than one obtains using two successive Schmidt decompositions. These decompositions have been called sub-Schmidt decompositions for short. Particularly for entangled states of dimensionality $(n,n,2)$, we found a one-to-one correspondence between the concept of Jordan families and the sub-Schmidt decompositions and use this correspondence to classify all sub-Schmidt decompositions of entangled states in this dimensionality. Moreover, from this classification of sub-Schmidt decompositions, we got a classification of these states according with their interconvertibility under SLOCC. We also briefly discussed the difficulties in generalizing our methods to more general systems. We expect that these results will contribute to the understanding of higher dimensional and multipartite entanglement.

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[23] In the case of the relative states decomposition, not all decompositions have this property, but it is always possible to fulfill it in infinitely many ways.

[24] In fact, we must have $A^{\dagger}A$, $B^{\dagger}B$, and $C^{\dagger}C \leqslant 1$, since A must come from the POVM defined by operators $\sqrt{w_A}A$ and $\sqrt{1_A - w_A A^{\dagger}A}$ performed by Alice, where w_A is some positive weight and 1_A is the identity in C_a^n, and similarly for B and C. Note that $w_A \leqslant 1/\lambda_{max}$ where λ_{max} is the greatest eigenvalue of $A^{\dagger}A$. However, we do not normalize the operators to simplify the calculation. The final state can always be easily normalized [15].

[25] In fact, this is true for any nonvanishing value chosen for the determinant $(ad - bc)$, and ultimately allows for making it equal to 1. In particular, the resulting matrix $\Phi_1'^{-1}\Phi_0'$ does not depend on the value of this determinant.