Potential barrier classification by short-time measurement

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We investigate the short-time dynamics of a delta-function potential barrier on an initially confined wave packet. There are mainly two conclusions: (A) At short times the probability density of the first particles that passed through the barrier is unaffected by it. (B) When the barrier is absorptive (i.e., its potential is imaginary) it affects the transmitted wave function at shorter times than a real potential barrier. Therefore, it is possible to distinguish between an imaginary and a real potential barrier by measuring its effect at short times only on the transmitting wave function.

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I. INTRODUCTION

One of the methods in imaging a turbid or a diffusive medium with optical radiation is the time-gating technique [1–5]. In this technique a temporally narrow pulse is injected into the medium. Owing to the diffusivity of the medium, when the pulse exits the medium it becomes considerably wider. However, if the first arriving photons are separated from the rest of the pulse, then it is possible to use these, so called, ballistic (or quasiballistic) photons to reconstruct the ballistic image of the medium.

Therefore, by employing a short time-gating technique the multiscattering effect can be eliminated. Indeed, such methods were employed in recognizing hidden objects and informative signals in diffusive media [6]. Naively, one might expect that this technique can be implemented for electron imaging to "see" absorptive objects inside the scattering medium. That is, one can send a short pulse of electrons to one end of the medium, while at the other end only the first arriving electrons will be measured. By doing so, all the noise caused by the multiscattering should be eliminated. On the other hand, the presence of absorptive regions (like imaginary potentials [7]) will be felt in the amount of the early arriving electrons. However, electrons are governed by the Schrödinger equation, and unlike the Maxwell wave equation, has a parabolic dispersion relation. As a consequence, any localized wave packet suffers from strong dispersion, since each spectral component propagates at a different velocity. The fastest particles are the most energetic ones, which pass through the medium unaffected, since the barrier's potential energy is negligible compared to their kinetic energy. In particular, when the medium is a certain barrier (or well), we conclude that when only the first-arriving particles are measured, there should be no trace of the barrier's presence. It does not matter what shape or height the barrier has—the first particles that pass though the barrier should be indifferent to it. Thus, one may argue that the time-gating technique cannot be implemented to electron imaging, at least not in its naive form. However, we show that the short-time measurement reveals information about the nature of the barrier—whether it is imaginary or real.

There is a peculiar distinction between an absorbing medium (e.g., an imaginary potential) and a nonabsorbing one (e.g., a real potential). While they both have no effect on the wave packet (both transmitted and reflected) at $t\to 0$, the imaginary barrier influences the wave packet sooner. In other words, in the temporal Taylor expansion of the probability density the imaginary potential appears at a smaller order than a real potential. It is then clear that we can classify the barrier as an absorptive one simply by measuring the wave packet at short times. Note that, in general, it is required to measure both reflection and transmission coefficients to figure out if the barrier is absorptive or not.

Recently [8], it has been demonstrated even experimentally that it is feasible to investigate the one-dimensional (1D) scattering of a Bose-Einstein condensate by a narrow defect. Therefore, it seems that there is a good chance of witnessing these effects in the laboratory in the near future.

In this paper we demonstrate this effect rigorously (both analytically and by a numerical simulation) for the delta function potential. That is, we show that it is possible to identify an absorptive potential by measuring the short time dynamics of only the transmitted wave function.

The initial state we consider is a wave packet, which is confined to one side of the barrier. It is then demonstrated that the wave function at the other side is independent of the barrier for short times, while the temporal dependence depends on the exact nature of the barrier (absorptive or not).

II. SYSTEM DESCRIPTION AND DYNAMICS

Evidently, in order to confine the initial wave packet to one side of the barrier, there has to be a certain singularity in the wave packet. In this paper we focus on a step function to simplify the problem; however, it has been demonstrated elsewhere that most of the conclusions are valid, even in the

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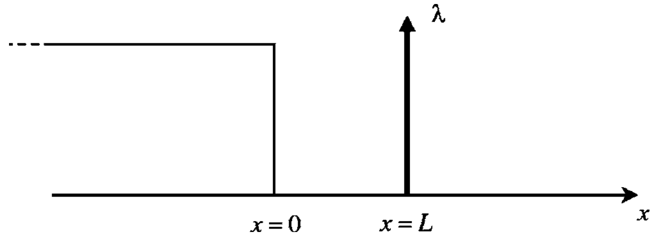

FIG. 1. System schematic: a semi-infinite wave packet hitting a delta function potential.

continuous case, provided the measurements are taken at specific ranges (see Ref. [9]). It is also demonstrated at the end of the paper that the main conclusions are valid even when the initial wavepacket is a Gaussian. For simplicity we take a 1D delta function as the potential barrier.

The system illustration is depicted in Fig. 1. Initially, the wave packet has the form [7,9,10]

$$\psi(x,t=0)=\theta(-x)\exp(ik_0x), \tag{1}$$

and at a distance L from its front we place a delta-function barrier $V(x)=\lambda\delta(x-L)$ (see Fig. 1). That is, the Schrödinger equation reads as

$$-\frac{\partial^2}{\partial x^2}\psi+\lambda\,\delta(x-L)\psi=i\frac{\partial\psi}{\partial t}. \tag{2}$$

Since the Fourier transform of $\psi(x,t=0)$ is

$$\varphi(k)=\frac{i}{k-k_0+i0}, \tag{3}$$

then in the free case where $L\to\infty$, i.e., when the barrier is absent, the wave packet satisfies

$$\psi_{free}(x,t\geqslant 0)=\frac{1}{2\pi}\int dk\ \varphi(k)\exp(ikx-ik^2t), \tag{4}$$

while for a finite L the free-space eigenfunctions $\exp(ikx)$ should be replaced with the barrier's eigenfunctions,

$$\chi(k,x)\equiv\exp(ikx)+\frac{i\lambda/2}{k-i\lambda/2}\exp(ik|x-L|+ikL), \tag{5}$$

and therefore

$$\psi(x,t\geqslant 0)=\frac{1}{2\pi}\int dk\ \varphi(k)\chi(k,x)\exp(-ik^2t). \tag{6}$$

Carrying out the integral, one finds (for $x>0$)

$$\psi(x,t)=\frac{e^{ix^2/4t}}{2}w\left[\sqrt{it}\left(\frac{x}{2t}-k_0\right)\right]+\frac{e^{iy^2/4t}}{2}\frac{i\lambda/2}{k_0-i\lambda/2}\times\left\{w\left[\sqrt{it}\left(\frac{y}{2t}-k_0\right)\right]-w\left[\sqrt{it}\left(\frac{y}{2t}-i\frac{\lambda}{2}\right)\right]\right\}, \tag{7}$$

where $w(z)\equiv\exp(-z^2)\mathrm{erfc}(-iz)$ [11] and $y\equiv L+|x-L|$. In short times one can expand this expression in powers of t. Up to $t^{5/2}$,

$$\psi(x,t)\simeq\sqrt{\frac{it}{\pi}}\frac{e^{ix^2/4t}}{x}\left(1+\frac{2(k_0x-i)}{x^2}t+\frac{4(k^2x^2-3ik_0x-3)}{x^4}t^2\right)+\frac{(it)^{3/2}}{\sqrt{\pi}}\frac{e^{iy^2/4t}}{y^2}\lambda\left(1+\frac{i(\lambda y-6)+2k_0y}{y^2}t\right), \tag{8}$$

for $x>L$,

$$\psi(x,t)\simeq\sqrt{\frac{it}{\pi}}\frac{e^{ix^2/4t}}{x}\left(1+\frac{2(k_0x-i)}{x^2}t+\frac{4(k^2x^2-3ik_0x-3)}{x^4}t^2+\frac{i\lambda}{x}t-\frac{\lambda(\lambda x-6-2ik_0x)}{x^3}t^2\right),$$

and to the third order of t,

$$|\psi(x>L,t)|^2\simeq\frac{t}{\pi x^2}\left[1+4\frac{k_0t}{x}+4\frac{3(k_0x)^2-5}{x^4}t^2+\frac{\lambda}{x^2}t^2\left(\frac{8}{x}-\lambda\right)\right]. \tag{9}$$

We can see that the barrier's presence is felt only at the third order of t.

Even when $k_0\ll\lambda$ and $x\to\infty$, the barrier's presence has a significant influence when the measurement is taken in the range $4k_0x/\lambda^2\ll t\ll x/\lambda$.

On the other hand, for $0<x<L$, owing to the reflection from the barrier, the dependence of the probability density on the barrier appears at order $t^{3/2}$, i.e.,

$$\psi(x,t)\simeq\sqrt{\frac{i}{\pi}}\frac{e^{ix^2/4t}}{x}\left[t^{1/2}+\frac{2t^{3/2}}{x}\left(k_0+\frac{i}{x}\right)\right]+i\lambda\sqrt{\frac{i}{\pi}}\frac{e^{i(2L-x)^2/4t}t^{3/2}}{(2L-x)^2}, \tag{10}$$

and

$$|\psi(x,t)|^2\simeq\frac{t}{\pi}\left[\frac{1}{x^2}\left(1+\frac{4tk_0}{x}\right)-\frac{2\lambda t}{x(2L-x)^2}\sin\left(\frac{L(L-x)}{t}\right)\right]. \tag{11}$$

That is, the dependence (i.e., λ) appears in the probability density in the coefficient of t^2. Obviously, this approximation applies only when the argument $L(L-x)/t$ is not too small.

In Fig. 2 we plot a comparison between the propagation of the wave packet in case the barrier is absent (upper panel) and when it is present (lower panel). In Fig. 3 the difference between the two (with and without the barrier, i.e., $\Delta|\psi|^2=|\psi|^2_{with}-|\psi|^2_{free}$) is plotted. Despite the fact that the packet passes *through* the potential, its effect beyond the barrier $x>L$ is miniscule, and for $|x|^2\gg 1$ the two solutions are essentially identical. Moreover, the difference between the $x>L$ and $x<L$ regimes is clear from the figure. In the latter regime the influence of the potential is felt for longer distances, but still when $|x|^2/t\to\infty$ its influence decays to zero.

When the potential is absorptive the Schrödinger equation may be rewritten as

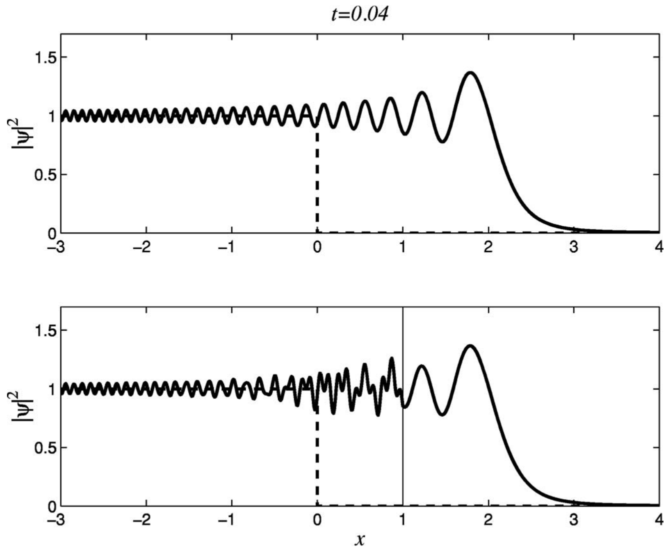

FIG. 2. A comparison between the solution with (lower panel) and without (upper panel) the barrier, which is represented by the horizontal line at $x=1$. The initial state is represented by the dashed line. The parameters in this case are $L=1$, $k_0=30$, $t=0.04$, and $\lambda=3$.

$$-\frac{\partial^2}{\partial x^2}\psi + i\lambda\,\delta(x-L)\psi = i\frac{\partial \psi}{\partial t}, \tag{12}$$

with the solution

$$\psi(x,t) = \sqrt{\frac{it}{\pi}}\frac{e^{ix^2/4t}}{x}\left(1+\frac{2(k_0x-i)-\lambda x}{x^2}t+O(t^2)\right), \tag{13}$$

and the probability density satisfies

$$|\psi(x>L,t)|^2 = \frac{t}{\pi x^2}\left(1+2\frac{2k_0-\lambda}{x}t+O(t^2)\right). \tag{14}$$

In this case the potential presence appears in the probability density at the second order of t [and not the third as in Eq. (9)].

In case the wave function vanishes initially at $x=0$ ([9,10,12]), i.e., when

$$\psi(x,t=0) = \theta(-x)\sin(k_0x), \tag{15}$$

the general solution is

$$\begin{aligned}\psi(x,t) = {} & \frac{e^{ix^2/4t}}{4i}w\left[\sqrt{it}\left(\frac{x}{2t}-k_0\right)\right] + \frac{e^{iy^2/4t}}{4}\frac{\lambda/2}{k_0-i\lambda/2}\\ & \times\left\{w\left[\sqrt{it}\left(\frac{y}{2t}-k_0\right)\right]-w\left[\sqrt{it}\left(\frac{y}{2t}-i\frac{\lambda}{2}\right)\right]\right\}\\ & -\left(\frac{e^{ix^2/4t}}{4i}w\left[\sqrt{it}\left(\frac{x}{2t}+k_0\right)\right]-\frac{e^{iy^2/4t}}{4}\frac{\lambda/2}{k_0+i\lambda/2}\right.\\ & \left.\times\left\{w\left[\sqrt{it}\left(\frac{y}{2t}+k_0\right)\right]-w\left[\sqrt{it}\left(\frac{y}{2t}-i\frac{\lambda}{2}\right)\right]\right\}\right).\end{aligned} \tag{16}$$

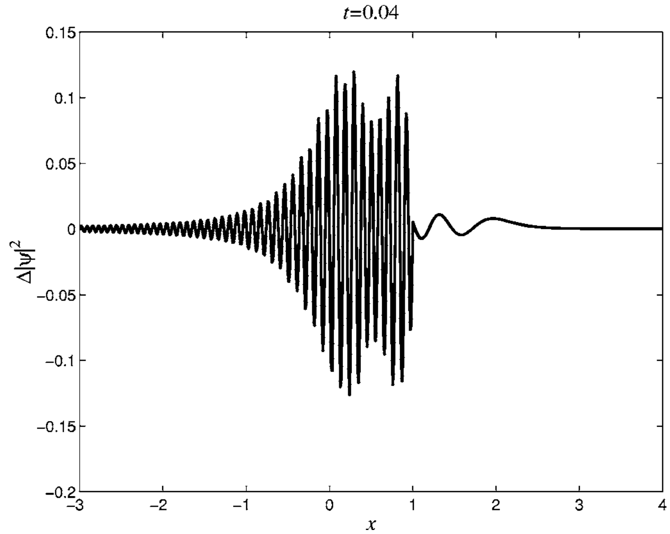

FIG. 3. The difference between the solution when the barrier is present and when it is absent. Clearly, as $|x|^2/t \gg 1$ the two solutions are identical.

In short times one can expand this expressions in powers of t. Up to $t^{5/2}$,

$$\psi(x,t) \simeq -\sqrt{\frac{i}{\pi}}\frac{2k_0t^{3/2}e^{ix^2/4t}}{x^2}\left(i+\frac{6}{x^2}t\right)+\sqrt{\frac{i}{\pi}}\frac{2k_0t^{5/2}e^{iy^2/4t}}{y^3}\lambda, \tag{17}$$

for $x>L$,

$$\psi(x,t) \simeq \frac{t^{3/2}}{\sqrt{i\pi}}\frac{e^{ix^2/4t}}{x^2}2k_0\left[1+\frac{it}{x}\left(\lambda-\frac{6}{x}\right)\right] \tag{18}$$

and

$$|\psi|^2 \cong \frac{4k_0^2t^3}{\pi x^4}+O(t^4). \tag{19}$$

We can see that the barrier is not felt even at the third order of t (the coefficient of t^3 is independent of λ). Although the phase of ψ is sensitive to the barrier, even in the first order of t, the leading term is proportional to t^{-1} for large x, i.e.,

$$\arg(\psi) \cong -\frac{\pi}{4}+\frac{x^2}{4t}+(x\lambda-6)\frac{t}{x^2}. \tag{20}$$

III. THE CONTINUOUS CASE: GAUSSIAN DYNAMICS

The fact that we used a singular initial wave function may raise skepticism about the physical validity of the conclusions. However, the main conclusions of the semi-infinite plane wave can be deduced even when the initial wave packet is a continuous wave function, such as a Gaussian. However, owing to the finite spectral width of the Gaussian, high energy particles are very rare in the packet, and therefore at short times there is not enough energy to make any substantial difference. The variations in the wave packet at

$t \to 0$ is negligible. Therefore, we should expect to find the same conclusions only in a certain intermediate period (as in Ref. [9]).

If the initial wavepacket is a Gaussian, i.e.,

$$\psi(x,t=0) = \sqrt{\frac{2}{\pi}\frac{1}{\sigma}} \exp\left[-\left(\frac{x}{\sigma}\right)^2 + ik_0 x\right], \qquad (21)$$

then

$$\psi(x>L,t\geq 0) = \frac{1}{\sqrt{2\pi}}\int dk \frac{\exp[-(k-k_0)^2\sigma^2/4 + ikx - ik^2 t]}{1 - i\lambda/2k}, \qquad (22)$$

and carrying out the integration, one finds

$$\psi(x>L,t\geq 0) = \left\{\frac{1}{\sqrt{2\pi s}} - \frac{\lambda}{\sqrt{8}} w\left(i\frac{\lambda s}{2} - \frac{(\sigma^2 k_0/2 + ix)}{2s}\right)\right\} \times \exp\left\{\frac{1}{4}\frac{(\sigma^2 k_0/2 + ix)^2}{s^2} - \frac{\sigma^2 k_0^2}{4}\right\}, \qquad (23)$$

where $s \equiv \sqrt{\sigma^2/4 + it}$. For small s (i.e., short times), by factoring out $(\sqrt{2\pi s})^{-1}$ and taking the limit $s \to 0$,

$$\psi(x>L,t\geq 0) \simeq \frac{1}{\sqrt{2\pi s}} \frac{1}{1 - i\dfrac{\lambda s^2}{\sigma^2 k_0/2 + ix}} \times \exp\left(\frac{1}{4}\frac{(\sigma^2 k_0/2 + ix)^2}{s^2} - \frac{\sigma^2 k_0^2}{4}\right), \qquad (24)$$

where for large distances $x + L \gg \sigma^2 k_0$ can be approximated in the two extreme cases: $t \ll \sigma^2/4$ and $t \gg \sigma^2/4$. In the former case,

$$\psi(x>L,t\geq 0) \cong \frac{\sqrt{2}}{\sqrt{\pi}\sigma}\frac{1}{1 - \dfrac{\lambda\sigma^2}{4x}} \exp\left(-\frac{x^2}{\sigma^2}\right), \qquad (25)$$

the barrier influence is independent of time, and, in fact, very far from the barrier $x \gg \lambda\sigma^2$ the barrier's influence is negligible. However, we see here that in the limit $t \to 0$, due to the finite spectral width $\sim\sigma$, the presence of the barrier is always felt. (Note that the singular case $4x = \lambda\sigma^2$ is not consistent with the above approximation.) This is to be expected, since in a Gaussian distribution the number of particles with extremely large enegies is exponentially small.

When $t \gg \sigma^2/4$,

$$\psi(x>L,t\geq 0) = \frac{1}{\sqrt{2\pi it}}\frac{1}{1 - i\dfrac{\lambda t}{x}} \exp\left(\frac{i}{4}\frac{x^2}{t} - \frac{\sigma^2 k_0^2}{4}\right),$$

we recognize a penetration velocity. When $x/t \ll \lambda$, the barrier has a large effect, however, if the particles' velocity is very large $x/t \gg \lambda$ the barrier's influence is negligible.

And similarly, in the temporal period $\sigma^2/4 \ll t \ll x/\lambda$, the difference between real and imaginary barrier is apparent.

For a real barrier,

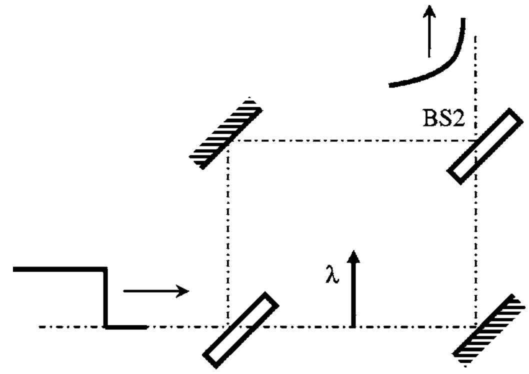

FIG. 4. Schematic illustration of a Mach-Zehnder interferometer for barrier classification.

$$|\psi(x>L,t>0)|^2 \cong \frac{1 - (\lambda t/x)^2}{2\pi t} \exp\left(-\frac{\sigma^2 k_0^2}{4}\right), \qquad (26)$$

while for an imaginary one,

$$|\psi(x>L,t>0)|^2 \cong \frac{1 - 2\lambda t/x}{2\pi t} \exp\left(-\frac{\sigma^2 k_0^2}{4}\right). \qquad (27)$$

Again, the the influence of the absorptive potential appears in a lower order term.

IV. SCHEMATIC EXPERIMENTAL REALIZATION

One of the methods of emphasizing the impact of the potential is by placing the potential barrier (delta function in our case) in one arm of a Mach-Zehnder interferometer (see Fig. 4).

Let us denote by c_1 and ic_2 the transmission and reflection coefficients of each of the two interferometers' beam splitters (BS1 and BS2 in Fig. 4). With this notation we assume (without loss of generality) that both c_1 and c_2 are real, and the conservation of energy implies $c_1^2 + c_2^2 = 1$. Then, at the interferometer's exit,

$$\psi(x,t) = \sqrt{\frac{it}{\pi}}\frac{e^{ix^2/4t}}{x}\left[\left(1 + \frac{2(k_0 x - i)}{x^2}t + \frac{4(k_0^2 x^2 - 3ik_0 x - 3)}{x^4}t^2\right) \times (c_1^2 - c_2^2) + \left(\frac{i\lambda}{x}t - \frac{\lambda(\lambda x - 6 - 2ik_0 x)}{x^3}t^2\right)c_1^2\right].$$

Let us further assume that both BS' are almost 50:50, i.e., $c_1^2 = 0.5(1-\varepsilon), c_2^2 = 0.5(1+\varepsilon)$, and $\varepsilon \ll 1$. In this case, for a short time,

$$\psi(x,t) \simeq \sqrt{\frac{it}{\pi}}\frac{e^{ix^2/4t}}{x}\left(\varepsilon + \frac{i\lambda}{2x}t\right), \qquad (28)$$

and the probability density can be approximated by

$$|\psi(x,t)|^2 \simeq \frac{t}{\pi x^2}\left[\left(\varepsilon - \frac{\Im\lambda}{2x}t\right)^2 + \left(\frac{\Re\lambda}{2x}t\right)^2\right]. \qquad (29)$$

Thus, when the potential is real, the potential-dependent term has a cubic dependence on time,

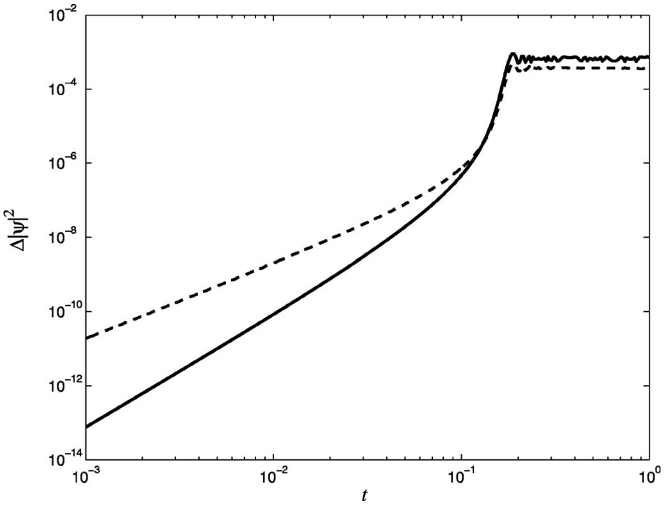

FIG. 5. The temporal evolution of $\Delta|\psi|^2$ outside the interferometer (a distance $x=10$ from the barrier) for a real potential (solid line) and an imaginary one (dashed line). $c_1=\sqrt{0.49}$ and the other system's parameters are the same as in Fig. 2.

$$|\psi(x,t)|^2 \simeq \frac{t}{\pi x^2}\left[\varepsilon^2 + \left(\frac{\lambda}{2x}t\right)^2\right], \tag{30}$$

while if the potential is imaginary, the temporal dependence of the potential-dependent term is parabolic,

$$|\psi(x,t)|^2 \simeq \frac{t\varepsilon}{\pi x^2}\left(\varepsilon - \frac{\Im\lambda}{x}t\right). \tag{31}$$

To emphasize the difference, we define $\Delta|\psi|^2 \equiv |\psi|^2 - |\psi|^2_{free}$ as the difference between the probability density at the interferometer exit when the barrier is present ($|\psi|^2$) and when it is absent ($|\psi|^2_{free}$).

In Fig. 5 we plot the temporal evolution of $\Delta|\psi|^2$, which is measured at the exit of the interferometer (at $x=10$ from the barrier) for the two cases (real and imaginary potentials). The only difference between the two plots is the potential ($i\lambda$ instead of λ). While the two plots are similar after long times, their temporal differences are considerably different for short times, as Eqs. (30) and (31) imply (like t^3 and t^2, respectively).

V. SUMMARY

The short-time influence of a delta-function potential barrier on an initially confined wave packet was investigated. The analysis was taken on either compact support and Gaussian wave function. It was shown that at short times the barrier presence has a negligible influence, if any, on the compact support wave packet dynamics. This result applies also for the probability density of the particles that passed through the barrier. It was also demonstrated that at short times an absorptive barrier (i.e., imaginary) has a different impact on the dynamics than a nonabsorptive (i.e., real) one. Namely, at short times the effect of an absorptive barrier appears in the coefficient of the t^2 term, while the effect of a nonabsorptive barrier appears only in the coefficient of the t^3 term. Therefore, it is possible to distinguish between an imaginary and a real potential barrier by measuring its effect at short times only on the transmitted wave function. There is no need to measure the transmission and reflection coefficient simultaneously. It is also demonstrated that a similar distinction is possible even when the initial wave packet has a Gaussian shape.

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