Coherent-state analysis of the quantum bouncing ball

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Gaussian-Klauder coherent states are applied to the bound "quantum bouncer," a gravitating particle above an infinite potential boundary. These Gaussian-Klauder states, originally created for Rydberg atoms, provide an overcomplete set of wave functions that mimic classical trajectories for extended times through the utilization of energy localization. For the quantum bouncer, analytic methods are applied presently to compute first and second moments of position and momentum operators, and from these results, at least two scalings of Gaussian-Klauder parameters are highlighted, one of which tends to remains localized for markedly more bounces than comparable states that are Gaussian in position (by an order of magnitude in some cases). We close with a connection that compares Gaussian-Klauder states and positional Gaussian states directly for the quantum bouncer, relating the two through a known energy-position duality of Airy functions. Our results, taken together, ultimately reemphasize the primacy of energy localization as a key ingredient for long-lived classical correspondence in systems with smooth spectra.

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I. INTRODUCTION

The generalization of the quantum harmonic oscillator coherent states—i.e., those of the Heisenberg-Weyl (H_4) group—has been a point of research for over three decades (consult [1] for a review of earlier work). Coherent states, as compared to energy eigenfunctions, provide an overcomplete decomposition of quantum phase space into localized wave packets whose expectation values follow their classical trajectories for long times. In this vein, a complete knowledge of coherent states makes precise the concept of classical correspondence for a Hamiltonian system—e.g., in the analysis of quantum chaos and Husimi-Wigner distributions [2–4].

The generalized definition of "coherent" is dependent on which properties are retained from the H_4 coherent states (reference the Introduction in [5]), and at least two approaches, based on different motivations, have developed in the literature. The first and most developed approach demands that the dynamical group structure determine coherent states, much as H_4 coherent states are determined through the displacement operator $e^{\alpha a^\dagger - \alpha^* a}$ [1,6]. The group-based approach has had several successes, including the automatic construction of H_4 and SO(3) coherent states, and additionally, this approach puts coherent states in direct correspondence with a dynamical quotient group. Unfortunately, purely group-theoretic attempts to describe Rydberg atoms (e.g., [7]) fail to localize states along the classical elliptical orbit, precluding arguments of true classical correspondence. Even apart from the difficulties with Rydberg atoms, the inability to address Hamiltonians without strong group structure leaves the group approach wanting.

An alternative approach for coherent-state construction takes a more direct route by "giving up the group" in favor of the Klauder construction [8], a general method Klauder introduced for creating a set of (so-called) temporally coherent and complete states. However, initial skepticism of this method arose due to the first attempts failing to give long-lived Rydberg atom coherent states at moderate energies [9]. Fox rectified this problem through the construction of Gaussian-Klauder states [10], based on the principle of energy localization in long-lived packet coherence (similar observations in a different context have been made elsewhere in the study of revivals [11,12]). In short, the driving connection with the H_4 coherent states is that localization in energy space approximates the relevant spectrum for a wave packet by a nearly equally spaced (harmonic) spectrum, with corrections reliant on the degree of energy localization. Gaussian-Klauder states have since had successes for Rydberg coherent states [10] and a myriad of other systems [13–15].

Our current paper extends the literature of Gaussian-Klauder coherent states to include the case of a quantum particle confined to a semi-infinite gravity well, a system labeled the quantum bouncer [16–21]. The quantum bouncer satisfies at least two generic properties of one-dimensional quantum systems: an energy spectrum containing internal times of all orders (see [11]) and a Hamiltonian possessing no clear nontrivial continuous symmetry. That a set of coherent states exist for such a system significantly relaxes the constraint that coherent states must be motivated by group structure and, thus, provides a welcome invitation to a wide host of Hamiltonians.

Several results are presented in the course of this work, organized in the paper as follows. Section II introduces Gaussian-Klauder states and defines the quantum bouncer. Sections III and IV treat expectation values for Gaussian-Klauder states at initial times, using nontrivial Airy function identities and summation techniques. Sections V and VI similarly discuss expectation values at later times (including packet breakup and classical correspondence). Finally, Sec. VII compares Gaussian-Klauder states to the often chosen positional Gaussian states (Gaussian in position and momentum), with the final claim that energy localization establishes Gaussian-Klauder states as the more "coherent" of the two.

II. GAUSSIAN-KLAUDER COHERENT STATES FOR A GRAVITATIONAL WELL

We begin with the basic construction of a Gaussian-Klauder (GK) coherent state for a nondegenerate spectrum, as parametrized by n_0, σ, and ϕ_0 [8,10]. Labeling $|k\rangle$ the kth energy eigenket and $E_k = e_k \hbar\omega$ its energy (with some energy scaling $\hbar\omega$), the GK coefficients are

$$c_k = \frac{1}{\sqrt{N}} e^{-(k-n_0)^2/4\sigma^2} e^{ie_k\phi_0}, \quad \sum_k |c_k|^2 = 1, \tag{1}$$

where $\phi \equiv \phi_0 - \omega t$ acts as an angle variable for the evolution of the GK state. Localization in energy space is clear from Eq. (1). If we label states described by Eq. (1) with $|n_0, \phi_0\rangle$, we have explicitly

$$|n_0, \phi_0\rangle = \sum_k \frac{1}{\sqrt{N}} e^{-(k-n_0)^2/4\sigma^2} e^{ie_k\phi_0} |k\rangle. \tag{2}$$

Resolution of the identity is given by

$$\int_{-\infty}^{\infty} dn_0 \lim_{\Phi\to\infty} \frac{1}{2\Phi} \int_{-\Phi}^{\Phi} d\phi_0 K(n_0) |n_0, \phi_0\rangle\langle n_0, \phi_0| = 1, \tag{3}$$

where we have defined

$$K(n_0) \equiv \frac{N}{\sqrt{2\pi\sigma^2}}. \tag{4}$$

We illustrate three initial-time GK coherent states for the quantum bouncer in Fig. 1. For the purposes of coherent states in this paper, we set $\phi_0 = 0$, which happens to define initial states at the classical upper turning point (away from the infinite potential boundary of the quantum bouncer).

We close this section with an outline of the physical system itself. The Hamiltonian for this paper is (defined for $z > 0$)

$$H = \frac{p_z^2}{2m} + mgz, \tag{5}$$

with z the position, p_z the momentum, g the local gravitational acceleration, and m the mass. There exists an infinite potential barrier for $z < 0$, which can be treated as a boundary condition at $z = 0$ for many purposes [29]. We rescale into natural units (length, energy, and time, respectively):

$$\delta_0 = \left(\frac{\hbar^2}{2m^2 g}\right)^{1/3}, \quad \epsilon_0 = \left(\frac{1}{2}\hbar^2 m g^2\right)^{1/3}, \quad \tau_0 = \hbar/\epsilon_0. \tag{6}$$

We will always work in natural units in this paper, unless otherwise stated. The weakness of the gravitational force leads to a large quantum length for atomic systems, much greater than atomic widths. This is relevant, as we will show in the next section, because GK states are approximately bounded to have at least this positional width for all choices of parameters. Thus, GK states tend to be "large" in this sense.

In natural units, we have the Hamiltonian H and Schrödinger equation

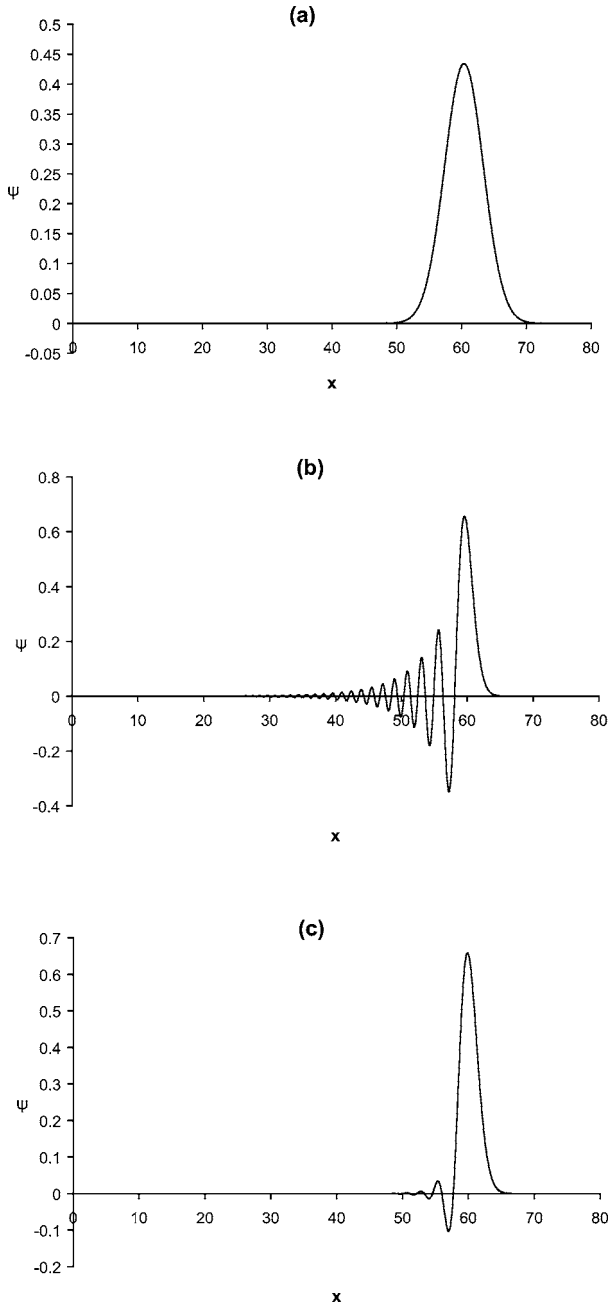

FIG. 1. Three GK initial states, constructed as in the text (each is real valued). Each is evaluated at $n_0 = 100$ for (a) $\sigma \approx 5.24$, (b) $\sigma \approx 1.02$, and (c) $\sigma \approx 1.96$. (a) and (b) represent the same $\langle \Delta x^2 \rangle$; however, they clearly have different properties (localization in phase space, coherence in time, etc.). (c) is a GK state with minimum $\langle \Delta x^2 \rangle$ for given n_0. Details are elaborated in Sec. III.

$$H = -\frac{\partial^2}{\partial x^2} + x,$$

$$\left(i\frac{\partial}{\partial t} - H\right)\psi = 0. \tag{7}$$

Equations (7) have eigenfunctions in terms of the bounded Airy function $\mathrm{Ai}(x - x_m)$, where quantization only requires a

set of x_m such that $\mathrm{Ai}(-x_m)=0$. We use integers $m \geqslant 1$ to label the zeros of the Airy function. It is a simple matter to show that the energy eigenvalue of such functions is $E_m = x_m$.

Asymptotic expansions exist for $x_m = x(m)$ as a function of the now continuous variable m, as listed in Appendix A. A knowledge of this expansion is important for the evaluation of energies and matrix elements, also stated in Appendix A.

We define

$$\omega_0 \equiv \left.\frac{\partial x_n}{\partial n}\right|_{n_0} \tag{8}$$

in accordance with the group frequency that determines the classical frequency of oscillation [11]. Evaluation of this quantity also follows from the asymptotic expansion of x_m.

III. GK INITIAL CONDITION ANALYSIS

Computation of expectation values for GK coherent states (remembering that we choose $\phi_0=0$) becomes straightforward at initial time $t=0$. We will here carry through several algebraic manipulations particular to Airy functions, though the results pertaining to the expectation values will be postponed until the following section. This mentioned, we proceed with our analysis.

For an operator, V, symmetric in the Airy energy eigenbasis, the expectation value for our GK states at any time is (remembering that the c_k have been chosen to be real through $\phi_0=0$)

$$\langle V\rangle = V_1 + V_2(t),$$

$$V_1 \equiv \sum_m c_m^2 \langle m|V|m\rangle,$$

$$V_2(t) \equiv \sum_{m_1>m_2} 2c_{m_1}c_{m_2}\cos[(x_{m_1}-x_{m_2})t]\langle m_1|V|m_2\rangle, \tag{9}$$

where summations are done over all states with the given restriction. The separation of this series into V_1 and $V_2(t)$ respects the stationary and dynamic (here, diagonal and off-diagonal) portions, respectively. When we wish to take zero time $t=0$, then we simply label $V_2(0)=V_2$, and in particular, we label the corresponding series for x and x^2 to be $X_j^{(1)}$ and $X_j^{(2)}$, respectively, with $j=1,2$.

Using this terminology and the matrix elements listed in Appendix A, we have the expectation values at $t=0$:

$$\langle x\rangle = X_1^{(1)} + X_2^{(1)},$$

$$\langle \Delta x^2\rangle = X_1^{(2)} + X_2^{(2)} - (X_1^{(1)} + X_2^{(1)})^2,$$

$$\langle p\rangle = 0,$$

$$\langle \Delta p^2\rangle = \frac{1}{2}X_1^{(1)} - X_2^{(1)}. \tag{10}$$

The calculation of terms in $X_1^{(1)}$ and $X_1^{(2)}$ follows effectively from Gaussian integration (refer to Appendix B), and little more needs to be written. In contrast, $X_2^{(1)}$ and $X_2^{(2)}$ may be simplified slightly, which we do now.

We define the polynomials $f_r(p,k)$ from the Taylor expansion in p and k of

$$f_r(p,k) = \left[\frac{x_{n_0+p}-x_{n_0+k}}{\omega_0(p-k)}\right]^r, \tag{11}$$

which should (at least in the asymptotic sense) converge for p and k relevant to our problem. Then, through a straightforward use of Eq. (9) with GK coefficients and the matrix elements from Appendix A, we have an expression for $X_2^{(1)}$ and $X_2^{(2)}$:

$$X_2^{(1)} = \frac{4}{\omega_0^2}\sum_{p>k}(-1)^{p+k+1}\frac{e^{-(p^2+k^2)/4\sigma^2}}{N}\frac{f_{-2}(p,k)}{(p-k)^2},$$

$$X_2^{(2)} = \frac{48}{\omega_0^4}\sum_{p>k}(-1)^{p+k+1}\frac{e^{-(p^2+k^2)/4\sigma^2}}{N}\frac{f_{-4}(p,k)}{(p-k)^4}, \tag{12}$$

where we have assumed n_0 is an integer in Eq. (1) and have shifted the summations in Eq. (9) through $m_1=n_0+p$ and $m_2=n_0+k$.

Such series are simplified by using a different basis. We define

$$\xi_1 = p+k, \quad \xi_2 = p-k. \tag{13}$$

This transformation leaves the Gaussian term diagonal:

$$e^{-(p^2+k^2)/4\sigma^2} = e^{-(\xi_1^2+\xi_2^2)/8\sigma^2}. \tag{14}$$

Equation (11) is then written

$$(x_{n_0+p}-x_{n_0+k})^r = \omega_0^r \xi_2^r f_r(\xi_1,\xi_2). \tag{15}$$

From this form, it is clear that $f_r(\xi_1,\xi_2)$ must either be even or odd in ξ_2, due to oddness of $(x_{n_0+p}-x_{n_0+k})$ in ξ_2. This observation is important for the computation of expectation values, since we can then often restrict ourselves (for certain measurements) only to the even-parity summations in Appendix B.

This new basis splits the summation:

$$\sum_{p>k} = \sum_{\xi_2>0}{}' + \sum_{\xi_2>0}{}'', \tag{16}$$

with $\Sigma'_{\xi_1\xi_2}$ over odd ξ_1 and ξ_2 and with $\Sigma''_{\xi_1\xi_2}$ over even ξ_1 and ξ_2 (evaluated using $\xi_2>0$ while ξ_1 varies over all such possible indices.) These summations are otherwise unrestricted.

Since $(-1)^{p+k+1}=(-1)^{\xi_1+1}$

$$\left[\sum_{\xi_2>0}{}' + \sum_{\xi_2>0}{}''\right](-1)^{p+k+1} \to \left[\sum_{\xi_2>0}{}' - \sum_{\xi_2>0}{}''\right]. \tag{17}$$

Equation (12) can then be rewritten

$$X_2^{(1)} = \frac{4}{\omega_0^2}\left[\sum_{\xi_2>0}{}' - \sum_{\xi_2>0}{}''\right]\frac{e^{-(\xi_1^2+\xi_2^2)/8\sigma^2}}{N}\frac{f_{-2}(\xi_1,\xi_2)}{\xi_2^2},$$

$$X_2^{(2)} = \frac{48}{\omega_0^4}\left[\sum_{\xi_2>0}{}' - \sum_{\xi_2>0}{}''\right]\frac{e^{-(\xi_1^2+\xi_2^2)/8\sigma^2}}{N}\frac{f_{-4}(\xi_1,\xi_2)}{\xi_2^4}. \tag{18}$$

Equations (18) are suggestive: $X_2^{(1)}$ and $X_2^{(2)}$ are the differences of two similar quantities that differ only by parity of the summation variables. The precise statements about summations such as these—e.g., (approximate) alternating series in ξ_2—are given in Appendix B.

We do not carry out these calculations explicitly, due to the routine nature of the expansions. Use of an algebraic manipulation program can quickly compute the asymptotic series needed.

To simplify results, we use a shifted form of n_0,

$$n \equiv n_0 - 1/4, \tag{19}$$

due to ubiquitous factors of $\frac{1}{4}$ in the asymptotic series for x_m. We additionally scale σ with the parameter β,

$$\sigma^2 = \beta n, \tag{20}$$

since results in terms of β often appear simpler.

IV. GK INITIAL CONDITION RESULTS

Using the symbols from the previous section, the lowest-order (in n) initial expectation values are

$$\langle x \rangle = x_n + O(1/n),$$

$$\langle p \rangle = 0,$$

$$\langle \Delta x^2 \rangle = \left[\frac{1}{96} \left(\frac{12n}{\pi^8} \right)^{1/3} \right] \left[32\pi^4 \beta + \frac{1}{n} \left(\frac{9}{\beta^2} - 24\pi^2 + 16\pi^4 \beta^2 \right) + O(1/n^2) \right],$$

$$\langle \Delta p^2 \rangle = \frac{1}{8} \left(\frac{18}{\pi^4 n} \right)^{1/3} \left[\frac{1}{\beta} + O(1/n^2) \right]. \tag{21}$$

Notice that the positional variance has a minimum for intermediate β values [given shortly in Eq. (25)]. The failure to produce sharp position eigenfunctions should not be alarming, since states with low uncertainty in position have energy profiles far removed from Gaussians (refer to Sec. VII).

The above equations give, to the same order,

$$\langle \Delta x^2 \rangle \langle \Delta p^2 \rangle = \frac{1}{4} + \frac{1}{n} \left(\frac{9}{128} \frac{1}{\pi^4 \beta^3} - \frac{3}{16} \frac{1}{\pi^2 \beta} + \frac{1}{8} \beta \right) + O(1/n^2). \tag{22}$$

In our natural units, the value $1/4$ is the minimum total uncertainty $\hbar^2/4$. Notice as well that whenever we scale with $\beta = O(1)$, then *all* such packets approach minimum uncertainty (as Gaussians in position). A different scaling $\beta \sim n^{-1/3}$ leads to different constant values of total uncertainty, indicative of non-Gaussians in position.

Two special values for β can be found at this order: namely, β_x, leading to minimum positional variance, and β_{xp}, leading to minimum total uncertainty. We define these to be lowest order by definition. They are

$$\beta_x = \frac{1}{4} \left(\frac{36}{\pi^4 n} \right)^{1/3},$$

$$\beta_{xp} = \frac{\sqrt{3}}{2\pi}. \tag{23}$$

The scaling of β_x is an indication that considerations of our coherent states should include the scaling

$$\sigma = (\gamma n)^{1/3}. \tag{24}$$

Though such states are slightly less "packet like" than minimum total uncertainty states, they have several interesting properties. For instance, $\langle \Delta x^2 \rangle$ and $\langle \Delta p^2 \rangle$ both approach constant values (equally well for their product), and additionally, the low-energy uncertainty implies that these states have long lifetimes (discussed further in Sec. VI).

Also worthy of further mention are the states evaluated at $\beta = \beta_{xp}$. These states make zero the first-order correction in Eq. (22), leading to $O(1/n^2)$ convergence to minimum total uncertainty. When attempting to draw close parallels between minimum-uncertainty Gaussians (in position space) and GK states, $\beta = \beta_{xp}$ can be considered the point of closest approach.

We evaluate positional variance and total uncertainty at both of these β values (the subscript of the following quantities identify the β with which they are evaluated):

$$\langle \Delta x^2 \rangle \langle \Delta p^2 \rangle|_x = \frac{3}{8} + O(1/n^{2/3}) > \frac{1}{4},$$

$$\langle \Delta x^2 \rangle \langle \Delta p^2 \rangle|_{xp} = \frac{1}{4} + O(1/n^2),$$

$$\langle \Delta x^2 \rangle|_x = \frac{432^{1/3}}{8} + O(1/n^{2/3}) \sim 0.945 + O(1/n^{2/3}),$$

$$\langle \Delta x^2 \rangle|_{xp} = \frac{(4\pi 3^{5/2} n)^{1/3}}{6} [1 + O(1/n^2)] \sim 0.968 n^{1/3} [1 + O(1/n^2)], \tag{25}$$

where the positional variances can be compared to the approximate squared bouncing amplitude, $x_n^2 \sim n^{1+1/3}$. For instance, the minimum-total-uncertainty states scale as

$$\sqrt{\langle \Delta x^2 \rangle|_{xp}} / x_n \sim 1/\sqrt{n}. \tag{26}$$

Typical widths for minimum-uncertainty packets are thus relatively small, despite their increasing absolute value (see Fig. 2).

The series computed in this and other sections are almost certainly asymptotic in n, but the limits of convergence for β are also a matter of questioning. Numerical comparison of Eq. (21) with the full series evaluation hints at validity approximately when $\sigma > 1$ and $\sigma \ll n$ (as could be expected). This paper will consider convergence properties only in passing, and in this manner, we move on.

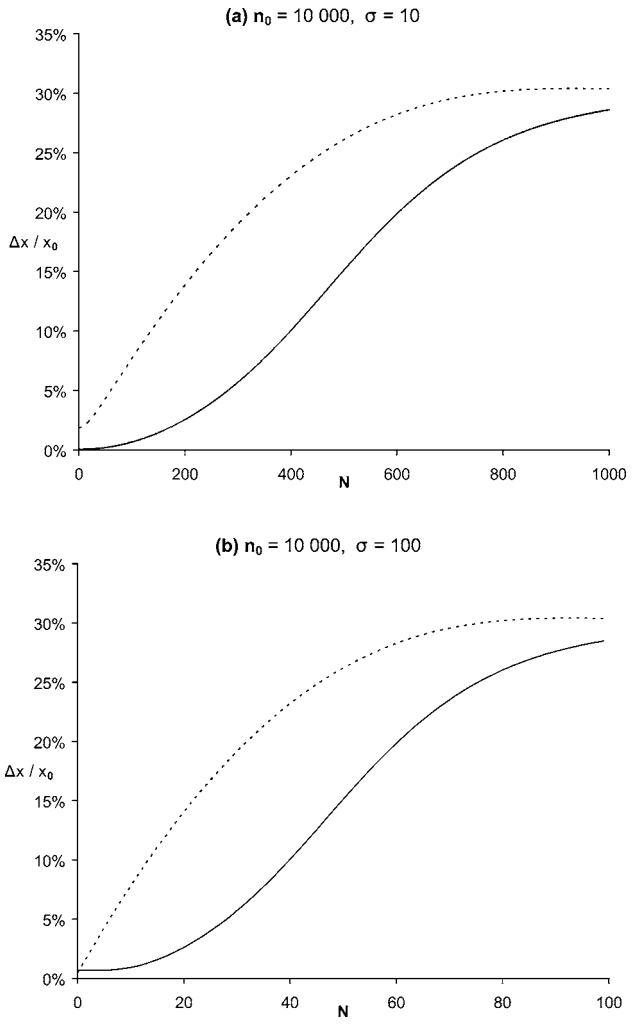

FIG. 2. Displays the rms deviation of a GK state (with parameters n_0 and σ) vs the number of "bounces" of the state, as measured through the quantity $\tau_0=2\pi/\omega_0$. The solid line is measured at integral periods from the starting time (upper turning point), while the dashed line is measured at half-integral periods. The overall shape for each graph is similar, due fairly general scaling relations for the time of packet breakup [11], though the behavior for a small number of bounces differs in a manner similar to the difference between a "minimum-position" and "minimum-uncertainty" GK state (refer to the text for further details).

V. CLASSICAL LIMIT

Before a more detailed treatment of the dynamics, it is simple to show that, with the assumption that variances remain small, expectation values for position of the quantum and classical states are equal to lowest order. Similar results can be found elsewhere for the particle in a box [13]. The application of Eq. (9) to the position operator, along with the lowest-order approximation

$$x_{n_0+k} - x_{n_0+p} \to \omega_0(k-p), \tag{27}$$

produces a Fourier series with coefficients (to lowest order)

$$d_0 = (2/3)x_{n_0},$$

$$d_k = (-1)^{k+1}4/k^2\omega_0^2. \tag{28}$$

Dividing Eqs. (28) by the amplitude of oscillation (using the lowest-order expansion x_{n_0}):

$$d_0/x_{n_0} = (2/3),$$

$$d_k/x_{n_0} = (-1)^{k+1}4/k^2\pi^2, \tag{29}$$

which are the Fourier coefficients for a classical bouncing ball (of unit bouncing amplitude). These classical coefficients can be easliy derived through use of the integral $\frac{\partial^2}{\partial\omega^2}\int_{-\pi}^{\pi}e^{i\omega x}dx=-\int_{-\pi}^{\pi}x^2e^{i\omega x}dx$, where the necessary steps should be clear.

VI. DYNAMICAL EVALUATIONS

The investigation of GK states in time is the true test of (temporal) coherence, though direct evaluation of expectation values becomes increasingly difficult. As such, we provide in this section three different measures of GK-state breakup (approximate time for the positional width of the packet to become comparable to the classical bouncing amplitude) that all agree in the scaling of GK packet lifetimes. As we will show, the scaling for number of bounces before breakup is

$$N_{breakup} \propto \sqrt{n/\beta}, \tag{30}$$

consistent with earlier expectations in a different context [12].

We first examine polynomial growth of the positional variance at integer periods of the bouncing time—i.e., at the upper turning point. By Taylor expanding small arguments of cosines in equations of the form Eq. (9) (after removing first-order terms that are integer multiples of 2π), a straightforward computation reveals the following increase in variance, relative to initial variance:

$$\Delta\langle\Delta x^2\rangle_N/\langle\Delta x^2\rangle_{N=0} = \frac{N^2}{n}\left(\frac{1}{\pi^2\beta}+\frac{28}{9}\beta\right)+\frac{N^4}{n}\frac{8}{9}\beta+O(1/n^2), \tag{31}$$

with N the number of bounces. We can show, by examination of the stroboscopically evaluated series for positional variance (noticing how powers of N^2 carry terms of ξ_2^2 that lead to vanishing series in Appendix B), that the above equation only has powers N^2 and N^4 to all orders of n, ignoring exponentially small terms from the Poisson summation formula. Indeed, often only the N^4 term is significant for most of the growth, leading to relatively stable variance at the upper turning point compared to other parts of the trajectory.

Numerical tests show a departure from this polynomial growth once the positional variance becomes comparable to system size, which is a result to be expected for a bounded system. We can speculate on the source of this departure, such as due to the secular aspect of the series, but we will not pursue definite answers here.

Equation (31) predicts its own failure by the time when the GK state undergoes breakup. Determining when $\langle\Delta x^2\rangle$ is

some fraction θ^2 of the total squared amplitude of oscillation provides an equation for the number of bounces before breakup:

$$N_\theta^2 \approx \frac{9\sqrt{2}}{8}\frac{n}{\beta}\theta. \tag{32}$$

For minimum total uncertainty packets, $N_\theta^2 \sim n$, and thus, the number of coherent bounces for these states scales as $\sim\sqrt{n}$. Of course, we can always allow β to become very small to increase N_θ, though such an action typically reduces overall localization of the coherent state.

As a numerical example, allow $\theta=0.1$ (of order 10% relative width). Using β values from Eq. (23), Eq. (32) predicts, for $N_\theta=20$ (20 bounces),

$$n \sim 98 \quad \text{(minimum positional uncertainty packet)},$$

$$n \sim 693 \quad \text{(minimum total uncertainty packet)},$$

or, instead, with $\theta=0.01$ (of order 1% relative width),

$$n \sim 550 \quad \text{(minimum positional uncertainty packet)},$$

$$n \sim 6931 \quad \text{(minimum total uncertainty packet)}.$$

These results make tangible the importance of energy uncertainty in packet stability (related to the time before packet breakup).

Analysis of the turning point is special, and we desire a more general measurement. A second measure for our system describes how the average positional variance increases with time, a quantity that we label $\Delta x_{F,0}^2$ (to remind us of the similarity to the zeroth Fourier coefficient). If we label $\{*\}_\tau$ as the time average of some time-dependent quantity, over a normalized Gaussian centered at zero time and with variance τ, then we define

$$\Delta x_{F,0}^2(\tau) \equiv \{\langle \Delta x^2 \rangle\}_\tau. \tag{33}$$

Such a definition is inspired by the transformation $\{a+bt^2\}_\tau = a+b\tau^2$, with constants a,b. We expect that GK states have a remnant of free particle growth, where variance in position acts as $\sigma^2(t)=\sigma_0^2(1+t^2/t_0^2)$, and so we compute $\Delta x_{F,0}^2$ to lowest order to verify this expectation.

When $\langle \Delta x^2 \rangle$ is expanded, the time-dependent quantities are all cosines or products of cosines, with the energy difference in their argument $\omega_{kp} \equiv x_{n_0+k} - x_{n_0+p}$. Their time average is

$$\{\cos(\omega_{kp}t)\}_\tau = e^{-\tau^2\omega_{kp}^2/2},$$

$$\{\cos(\omega_{kp}t)\cos(\omega_{rs}t)\}_\tau = \frac{1}{2}e^{-\tau^2(\omega_{kp}+\omega_{rs})^2/2} + \frac{1}{2}e^{-\tau^2(\omega_{kp}-\omega_{rs})^2/2}. \tag{34}$$

We assume here, for simplicity, that ω_{kp} and ω_{rs} are strictly positive—i.e., $k>p$ and $r>s$. For τ larger than the natural period of oscillation of the system, $2\pi/\omega_0$, then all terms in Eq. (34), excepting possibly $e^{-\tau^2(\omega_{kp}-\omega_{rs})^2/2}$, will be exponentially small. We restrict also $\tau\omega_0 \ll n$, so that we can Taylor expand $e^{-\tau^2(\omega_{kp}-\omega_{rs})^2/2}$ for small arguments, though this last condition is generally not sufficient when computing the series expansions to follow.

To lowest order, similar calculations as those leading to Eq. (31) give (being careful that we no longer have an alternating series in ξ_2^2)

$$\Delta x_{F,0}^2(N)/\langle x \rangle_{N=0}^2 = \frac{1}{n}\left(\frac{32}{135}\beta + \frac{1}{3\pi^2\beta} + \frac{16}{27}\beta N^2\right) + O(1/n^{3/2}), \tag{35}$$

where, as before, N labels the number of bounces, but we are allowed here to vary N *continuously* (simply a reparametrization of time τ). One can show that predictions from Eq. (35) for packet breakup time of the GK state agree with Eq. (30).

The asymptotic convergence of the full series in Eq. (35), as a function of N and β, will not be pursued here, though we notice the relative error in Eq. (35), predicted to be at least of order $O(1/\sqrt{n})$, decreases slowly with n.

The final measure of wave packet breakup explored here is the autocorrelation function of a wave packet, $|\Psi\rangle$ (used previously—e.g., [9,10,30]):

$$C(t) = |\langle \Psi | e^{-i\hat{H}t/\hbar} | \Psi \rangle|^2. \tag{36}$$

Assuming that $|\Psi\rangle = \Sigma c_k |k\rangle$, for an energy eigenbasis $|k\rangle$ (assumed nondegenerate in this paper) and denoting $\hbar\omega_k = E_k$ the eigenenergies, the coefficients in Eq. (1) ($\phi_0=0$) give

$$C(t) = |S(t)|^2,$$

$$S(t) = \sum_k \frac{1}{N} e^{-(k-n_0)^2/2\sigma^2} e^{i\omega_k t}, \tag{37}$$

where we have replaced $\omega_k \to -\omega_k$ as a matter of convenience [we can take the complex conjugate of $S(t)$ inside the norm]. Wave packets, such as $S(t)$ above, have been specifically studied by Leichtle *et al.* [11]. If we Taylor expand the continuous approximation for the eigenenergies—i.e., the Airy roots—around n_0,

$$x(n_0+k) = x(n_0) + \eta_1 \frac{2\pi}{T_1}k + \eta_2 \frac{2\pi}{T_2}k^2 + \cdots,$$

$$\omega_0 T_1 = 2\pi, \quad \eta_1 = 1,$$

$$\omega_0 T_2 = 12\pi(n_0 - 1/4) + O(1/n), \quad \eta_2 = -1, \tag{38}$$

defined with T_1 the bouncing period and T_2 the revival time. Leichtle *et al.* then derives

$$S(t) = \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{1 - i\eta_2 4\pi\sigma^2 t/T_2}} \times \exp\left[-\frac{2\pi^2\sigma^2}{1 - i\eta_2 4\pi\sigma^2 t/T_2}\left(\frac{t}{T_1} - k\right)^2\right], \tag{39}$$

valid approximately until $k_{max} \sim T_2/(4\sqrt{2}\sigma T_1)$ bounces [k_{max} is consistent with Eq. (30)]. Equation (39) is typical of GK coherent states in general and bears similarity to previ-

ously calculated GK correlation functions [10]. If we assume that separate Gaussian terms interfere little with each other in Eq. (39)—say, up to k_{max} bounces—then $C(t)$ evaluated at the turning point reduces to the simple form ($t_m = mT_1$, with m an integer)

$$C(t_m) = \frac{1}{\sqrt{1 + (4\pi\sigma^2)^2 (t_m/T_2)^2}}, \tag{40}$$

which can be made to decay quite slowly for small $\sigma^2 \ll T_2/t_m$. Computing the correlation function for a different initial time, by replacing $|\Psi\rangle \to e^{-i\hat{H}t_0/\hbar}|\Psi\rangle$, will not change these results.

VII. GP STATES

When comparing GK packets to wave packets that are Gaussian in position (label them GP) and assuming the packet is localized away from the boundary, the following integral is useful [17,20]:

$$\int_{-\infty}^{\infty} \mathrm{Ai}(x - x_n) e^{-(x - x_0)^2/4\sigma^2} dx = \sqrt{4\pi\sigma^2}\, e^{\sigma^2(x_0 - x_n + 2/3\sigma^4)} \mathrm{Ai}(x_0 - x_n + \sigma^4), \tag{41}$$

where we label σ *for this section* as the uncertainty in position, not the principle quantum number. This expression provides, with proper normalization, the projection of GP states onto the energy eigenbasis. Upon inspection, there are two clear directions to take this expression, $\sigma \to 0$ and $\sigma \to \infty$.

$0 < \sigma \ll 1$ is the limit of a Dirac δ function, which we know must diverge in energy—i.e., contains a high-energy uncertainty. In this limit, the simple exponential factor (compare to a Gaussian factor) on the right-hand side (RHS) of Eq. (41) becomes modulated by the oscillations in the Airy function at higher energies, though smaller energies have exponential damping due to this same Airy function. Such GP states experience rapid breakup due to this large, skewed uncertainty in energy.

$\sigma \gg 1$ presents a quite different scenario. Using the asymptotic expansion of the Airy function for large argument and likewise taking asymptotic expressions for large σ, Eq. (41) becomes (we abbreviate $x_0 - x_n$ by Δx_n)

$$\int_{-\infty}^{\infty} \mathrm{Ai}(x - x_n) e^{-(x - x_0)^2/4\sigma^2} dx \underset{\sigma \gg 1}{(\to)} \exp\left[\sigma^2\left(\Delta x_n + \frac{2}{3}\sigma^4\right) - \frac{2}{3}(\Delta x_n + \sigma^4)^{3/2}\right] \underset{\sigma \gg 1}{(\to)} e^{-(x_n - x_0)^2/4\sigma^2}. \tag{42}$$

Hence, the projection onto the eigenbasis is a Gaussian in the energy. If the energy uncertainty is not so large as to disallow the linear approximation $x_n - x_0 \approx \omega_0 (n - n_0)$, then the result of Eq. (42) becomes

$$e^{-(x_n - x_0)^2/4\sigma^2} \approx e^{-\omega_0^2 (n - n_0)^2/4\sigma^2}. \tag{43}$$

In other words, the GP state is well approximated by a GK state with uncertainty $\sigma_{GK}^2 = \sigma^2/\omega_0^2$. Thus, we expect both types of states to behave similarly in this regime.

From the discussions in this section, we have good reason to expect that, given equal $\langle \Delta x^2 \rangle$ and $\langle x \rangle$, GK states are better localized in quantum number than GP states—i.e., $\Delta n_{GK} < \Delta n_{GP}$. Numerically and sensibly (as tested for the GK minimal positional uncertainty states versus their GP equivalents), GK states remain coherent for longer times than GP states (though this difference becomes small for high n and $\beta = \beta_{xp}$).

VIII. CONCLUSION

The hope that all properties of H_4 coherent states will be reproduced for a general bound Hamiltonian has largely diminished, with even the purely group-theoretic (nondispersive) coherent states of Rydberg atoms failing to be localized along their azimuthal angle. In light of these difficulties, Gaussian-Klauder coherent states are a viable avenue of pursuit—e.g., as with Rydberg atoms [10]. Localization in energy leads to long-lived coherence in the Gaussian-Klauder construction, taking advantage of an approximate uniformity of energy level spacings for large principle quantum number, and the Gaussian character of the energy profile additionally supplies an analytic handle for evaluation of expectation values, meaning there exists computability as well.

As a particular example in this paper, we have examined Gaussian-Klauder coherent states for the quantum bouncer, at both initial and later times, and established details of quantum-classical correspondence for these states. Notable results include identification of two relevant scalings that are characteristic of either minimum positional or total variance, where the former tends to retain localization for longer times than comparable Gaussian wave functions in position space, on the principle of lower energy uncertainty. Of note on more general grounds is the great satisfaction that a set of generalized coherent states even exist for this system, despite repeated exposure to infinite potential collisions near a nonconstant potential.

Intriguingly, experiments in roughly the last decade have formed gravitational quantum bouncers using neutrons [22–24] and atomic clouds [25–28]. The energies involved with these physical systems can be many times the ground-state energy but are always uncomfortably far from a classical limit—we can expect an initially localized state can grow to system size in observational times. In this regime, Gaussian-Klauder states are nonequivalent to positional Gaussians when considering long-lived localization in quantum phase space, and studies based on these two different approaches (e.g., wavelet analysis) will differ both quantitatively and also in the interpretation of the "classical-ness" of the underlying basis states. Ambitious future work, alongside a use of quantum-state preparation theory, may even directly compare these theoretical implications through direct state construction in an experimental environment.

APPENDIX A: AIRY FUNCTION ROOTS AND MATRIX ELEMENTS

Eigenfunctions and their matrix elements for the gravitational problem have been deduced elsewhere in terms of integrals over Airy functions [17–21].

The explicit form for the eigenfunctions, in terms of the normalized and bounded Airy function $\mathrm{Ai}(x)$ and the derivative of $\mathrm{Ai}(x)$, $\mathrm{Ai}'(x)$:

$$\psi_m(x) = \mathrm{Ai}(x - x_m)/|\mathrm{Ai}'(-x_m)|, \tag{A1}$$

where x_m is the positive value of the mth Airy root, where $\mathrm{Ai}(-x_m)=0$.

The asymptotic form of the Airy roots has been worked out as well [31], with the expansion

$$x_n \sim t^{2/3}\sum_{j=0}^{\infty} Y_j/t^{2j},$$

$$t \equiv \frac{3\pi}{2}\left(n - \frac{1}{4}\right),$$

$$Y_0 = 1, \quad Y_1 = 5/48, \quad Y_2 = -5/36, \quad \ldots . \tag{A2}$$

We can define Taylor expansions around a given n as well, owing to the analytic form of this expression.

Once we know x_m, the basic matrix elements used in this paper are simple functions of these roots [19]:

$$\langle n|x|n\rangle = \frac{2}{3}x_n, \tag{A3}$$

$$\langle n|x|m\rangle = (-1)^{n+m+1}\frac{2}{(x_n - x_m)^2}, \tag{A4}$$

$$\langle n|x^2|n\rangle = \frac{8}{15}x_n^2, \tag{A5}$$

$$\langle n|x^2|m\rangle = (-1)^{n+m+1}\frac{24}{(x_n - x_m)^4}, \tag{A6}$$

$$\langle n|p|m\rangle = \frac{i}{2}(x_n - x_m)\langle n|x|m\rangle, \tag{A7}$$

$$\langle n|p^2|m\rangle = x_n\delta_{mn} - \langle n|x|m\rangle, \tag{A8}$$

etc., with $n \neq m$ in Eqs. (A4) and (A6). All of these matrix elements can be derived through Airy-integral identities.

We caution that the matrix elements of operators that are commutators of the Hamiltonian can result in contradictions if done improperly. For instance, the matrix elements of $[H,[H,x]]$, done carelessly, lead to such contradictions. This issue can be treated through modification of the Hamiltonian to include a term that accounts for the infinite potential wall at the boundary [29].

APPENDIX B: POISSON SUMMATION

A number of approximations are needed for the series presented in this paper, and these are essentially always summations over a normalized Gaussian distribution multiplied by a power (or inverse power) of summation variable:

$$W_n(\alpha) = \sum_{k=-\infty}^{\infty} k^n e^{-\alpha k^2},$$

$$Q_n(\alpha) = \sum_{k=1}^{\infty} k^n e^{-\alpha k^2},$$

$$T_n(\alpha) = \sum_{k=1}^{\infty} (-1)^{k+1} k^n e^{-\alpha k^2}. \tag{B1}$$

These summations are not independent and can be related to one another through differentiating:

$$\frac{\partial}{\partial\alpha}W_n(\alpha) = -W_{n+2}(\alpha), \quad \text{etc.} \tag{B2}$$

Thus, even when we allow n to become negative for, say, T_n, the summation is related to one with moments of positive powers (by appropriate successive constants of integration in α).

In many cases for our purposes, good estimates can be generated with use of the Poisson summation formula [11,32]:

$$\sum_{m=-\infty}^{\infty} f_m = \sum_{l=-\infty}^{\infty}\int_{m=-\infty}^{\infty} dm f(m)e^{-2\pi i l m}, \tag{B3}$$

where $f(m)$ is a continuous continuation of the discrete variable f_m. We will also assume, for this section, that $f(-m) = f(m)$.

We can directly relate this to ranges over the half real line by extending summations to be even over the whole interval, being careful to handle any contribution from f_0 that may occur that does not appear in the one-sided series.

The Poisson summation formula reduces Eq. (B1) to Gaussian (possibly one sided) integrals. Some observations are following.

(i) The series in Eq. (B1) with even n have expressions (neglecting an exponentially small error term) with a finite number of terms. In particular for T_n with n even and for $n \geqslant 2$, these summations are zero to the same approximation. Zeta functions are useful for determining integration constants between summations when $n \leqslant -2$.

(ii) W_n, with odd n, is zero. However, Q_n and T_n, with odd n, may best be expressed in an infinite asymptotic series, with large quantity $1/\alpha$.

For even n, we ignore the exponentially small terms when energy uncertainties are comparable to a few energy levels. We claim that for the particle in the gravitational well problem, the exponentially small factors are of the form (neglecting the subexponential part):

$$\text{error} \sim \exp(-\kappa/\alpha), \tag{B4}$$

with κ positive and of order 1.

Finally, we list the "generating" summations, where taking derivatives provides the remaining needed summations

for this paper. We have, for $\alpha \ll 1$ (exponentially small terms ignored),

$$W_0(\alpha) = \sqrt{\frac{\pi}{\alpha}}, \tag{B5}$$

$$Q_{-4}(\alpha) = \frac{\pi^4}{90} - \frac{\pi^2}{6}\alpha + \frac{2}{3}\sqrt{\frac{\pi}{\alpha}}\alpha^2 - \frac{1}{4}\alpha^2, \tag{B6}$$

$$T_{-4}(\alpha) = \frac{7}{720}\pi^4 - \frac{\pi^2}{12}\alpha + \frac{1}{4}\alpha^2. \tag{B7}$$

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