# Reply to "Comment on 'Zero-range potentials for Dirac particles: Scattering and related continuum problems' "

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We disprove the critique of our recent work [R. Szmytkowski, Phys. Rev. A **71**, 052708 (2005)] contained in the Comment by Coutinho and Nogami [Phys. Rev. A **73**, 026701 (2006)]. We show that the model of zero-range potentials for scattering processes involving Dirac particles, proposed by us in the aforementioned work, admits a definition of a scalar product under which the plane-wave excited scattering wave function is normalizable to the Dirac delta function in the momentum space. In addition, we prove that the eigenchannel wave functions, introduced in the aforementioned paper, are orthogonal with respect to that scalar product.

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In our recent work [1], we have proposed the model of zero-range potentials (ZRPs) for continuum processes involving Dirac particles. The basic assumptions of this model are as follows.

(a) The time-independent wave function $\Psi^{(+)}(E,\mathbf{r})$ describing a particle of energy $E$ ($|E|>mc^2$), scattered from a system of $N$ spherically symmetric ZRPs located at the points $\{\mathbf{r}_n\}$ $(n=1,\ldots,N)$, satisfies the free-particle time-independent Dirac equation

$$[-ic\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+mc^2\beta-E\mathcal{I}]\Psi^{(+)}(E,\mathbf{r})=0$$
$$(\mathbf{r}\neq\mathbf{r}_n;n=1,\ldots,N) \qquad (1)$$

everywhere in $\mathbb{R}^3$ except at the locations of the ZRPs.

(b) If the exciting wave (satisfying the free-particle Dirac equation everywhere in $\mathbb{R}^3$) is $\Phi(E,\mathbf{r})$, then the scattering wave function $\Psi^{(+)}(E,\mathbf{r})$ is explicitly given by

$$\Psi^{(+)}(E,\mathbf{r})=\Phi(E,\mathbf{r})+\sum_{n=1}^{N}\begin{pmatrix}h_0^{(+)}(k|\mathbf{r}-\mathbf{r}_n|)\chi_n^{(+)}(E)\\ \varepsilon h_1^{(+)}(k|\mathbf{r}-\mathbf{r}_n|)\boldsymbol{\mu}_n(\mathbf{r})\cdot\boldsymbol{\sigma}\chi_n^{(+)}(E)\end{pmatrix}, \qquad (2)$$

with $k$, $\varepsilon$, $\boldsymbol{\mu}_n(\mathbf{r})$, and the functions $h_0^{(+)}(z)$ and $h_1^{(+)}(z)$ defined in Eqs. (2.8)–(2.11) of Ref. [1].

(c) Interaction between the particle and the ZRPs is modeled by imposing the following limiting conditions on $\Psi^{(+)}(E,\mathbf{r})$ at the scatterers locations

$$\lim_{\mathbf{r}\to\mathbf{r}_n}[i(\mathbf{r}-\mathbf{r}_n)\cdot\boldsymbol{\alpha}_+ +\varepsilon(|\mathbf{r}-\mathbf{r}_n|\mathcal{K}_n+k^{-1}\beta_+)]\Psi^{(+)}(E,\mathbf{r})=0$$
$$(n=1,\ldots,N); \qquad (3)$$

the $4\times4$ matrices $\boldsymbol{\alpha}_+$, $\beta_+$, and $\mathcal{K}_n$ have been defined in Eqs. (2.15) and (2.20) of Ref. [1]. Substitution of Eq. (2) into the conditions (3) leads to the inhomogeneous system of algebraic equations for the spinor coefficients $\chi_n^{(+)}(E)$, given in Eq. (2.23) from Ref. [1].

In their Comment [2], Coutinho and Nogami point out that the integral

$$\int_{\mathbb{R}^3} d^3\mathbf{r}\;\Psi'^{(+)\dagger}(E',\mathbf{r})\Psi^{(+)}(E,\mathbf{r}) \qquad (4)$$

cannot be considered as the scalar product of two scattering wave functions $\Psi'^{(+)}(E',\mathbf{r})$ and $\Psi^{(+)}(E,\mathbf{r})$ from the class defined by Eqs. (1)–(3) [the function $\Psi'^{(+)}(E',\mathbf{r})$ is induced by the free wave $\Phi'(E',\mathbf{r})$], since, due to the singularity

$$h_1^{(+)}(k|\mathbf{r}-\mathbf{r}_n|)\overset{\mathbf{r}\to\mathbf{r}_n}{\sim}\frac{i}{k^2|\mathbf{r}-\mathbf{r}_n|^2} \qquad (5)$$

exhibited by the (rescaled) spherical Hankel function $h_1^{(+)}(k|\mathbf{r}-\mathbf{r}_n|)$, this integral diverges at the locations of the ZRPs. The authors of the Comment [2] see in this fact (as we shall show below—unjustly) a deficiency of the model proposed in Ref. [1].

In response, we would like to point out two facts. First, the idea to use in quantum mechanics singular (hence, not normalizable in the standard sense) wave functions is not new [3]. For instance, such singular functions emerge in the natural way when one attempts to generalize the nonrelativistic model of zero-range potentials to point obstacles diffracting also higher partial waves [4–10]. Second, and still more importantly, below we shall show that it is possible to define a scalar product of two scattering functions of the form (2) in such a manner that it appears to be free of the deficiency afflicting the integral (4).

We begin our reasoning with defining a spherical region $\mathbb{V}_R\subset\mathbb{R}^3$ of radius $R$, centered at the coordinate origin, and $N$ spherically shaped domains $\mathbb{V}_n\subset\mathbb{R}^3$, $(n=1,\ldots,N)$, of identical radii $\rho$, with the $n$th domain $\mathbb{V}_n$ centered at the point $\mathbf{r}_n$ (i.e., at the location of the $n$th ZRP):

$$\mathbb{V}_R=\{\mathbf{r}\in\mathbb{R}^3:|\mathbf{r}|<R\},\quad \mathbb{V}_n=\{\mathbf{r}\in\mathbb{R}^3:|\mathbf{r}-\mathbf{r}_n|<\rho\}. \quad (6)$$

Henceforth, we shall be assuming that the radii $R$ and $\rho$ are such that it holds that

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$$\mathbb{V}_n \subset \mathbb{V}_R \quad (n=1,\ldots,N). \tag{7}$$

Next, consider the scattering functions $\Psi^{(+)}(E,\mathbf{r})$ and $\Psi'^{(+)}(E',\mathbf{r})$ in the domain

$$\mathbb{V} = \mathbb{V}_R \backslash \bigcup_{n=1}^{N} \bar{\mathbb{V}}_n. \tag{8}$$

Evidently, both these functions are free of singularities in $\mathbb{V}$. Premultiplying the Dirac equation (1) for $\Psi^{(+)}(E,\mathbf{r})$ by $\Psi'^{(+)\dagger}(E',\mathbf{r})$, and vice versa, and then subtracting from the first resulting expression the complex conjugate of the other, after some further obvious manipulations one obtains

$$\int_{\mathbb{V}} d^3\mathbf{r}\, \Psi'^{(+)\dagger}(E',\mathbf{r})\Psi^{(+)}(E,\mathbf{r}) = \frac{c\hbar}{E'-E}\int_{\mathbb{V}} d^3\mathbf{r}\, \boldsymbol{\nabla}\cdot[\Psi'^{(+)\dagger}(E',\mathbf{r})i\boldsymbol{\alpha}\Psi^{(+)}(E,\mathbf{r})]. \tag{9}$$

Since the vector field $\Psi'^{(+)\dagger}(E',\mathbf{r})i\boldsymbol{\alpha}\Psi^{(+)}(E,\mathbf{r})$ is regular in the domain $\mathbb{V}$, one may apply the Gauss divergence theorem to the integral on the right-hand side of Eq. (9). This yields

$$\int_{\mathbb{V}} d^3\mathbf{r}\, \boldsymbol{\nabla}\cdot[\Psi'^{(+)\dagger}(E',\mathbf{r})i\boldsymbol{\alpha}\Psi^{(+)}(E,\mathbf{r})] = I_R^{(+)}(E';E) - \sum_{n=1}^{N} I_n^{(+)}(E';E), \tag{10}$$

where

$$I_R^{(+)}(E';E) = R^2\oint_{4\pi} d^2\mathbf{n}_R \Psi'^{(+)\dagger}(E',\mathbf{R})i\mathbf{n}_R\cdot\boldsymbol{\alpha}\Psi^{(+)}(E,\mathbf{R}), \tag{11}$$

with

$$\mathbf{n}_R = \mathbf{R}/R \tag{12}$$

and

$$I_n^{(+)}(E';E) = \rho^2\oint_{4\pi} d^2\mathbf{n}_n \Psi'^{(+)\dagger}(E',\mathbf{r}_n+\rho\mathbf{n}_n) \times i\mathbf{n}_n\cdot\boldsymbol{\alpha}\Psi^{(+)}(E,\mathbf{r}_n+\rho\mathbf{n}_n), \tag{13}$$

with $\mathbf{n}_n$ denoting the (variable) *outward* unit vector normal to the spherical surface $\partial\mathbb{V}_n$ surrounding $\mathbb{V}_n$. On combining Eqs. (9) and (10), after obvious rearrangements we obtain

$$\int_{\mathbb{V}} d^3\mathbf{r}\, \Psi'^{(+)\dagger}(E',\mathbf{r})\Psi^{(+)}(E,\mathbf{r}) + \sum_{n=1}^{N}\frac{c\hbar}{E'-E}I_n^{(+)}(E';E) = \frac{c\hbar}{E'-E}I_R^{(+)}(E';E). \tag{14}$$

As $\rho\to 0$, the volume integral on the left-hand side of the above equation exhibits a singularity of the order $\rho^{-1}$ [observe that this is just this singularity which makes it impossible to define the scalar product of $\Psi'^{(+)}(E',\mathbf{r})$ and $\Psi^{(+)}(E,\mathbf{r})$ in the form (4)]; the similar statement is true for each of the surface integrals $I_n^{(+)}(E';E)$, hence, also for their sum. On the other hand, the right-hand side of Eq. (14) is completely independent of $\rho$. This means that the singularities in the volume and surface integrals of the left-hand side of Eq. (14) cancel out. This, in turn, suggests one may define the "regularized" (or "renormalized") scalar product of the two singular wave functions $\Psi'^{(+)}(E',\mathbf{r})$ and $\Psi^{(+)}(E,\mathbf{r})$ as

$$\langle\Psi'^{(+)}(E')|\Psi^{(+)}(E)\rangle \overset{\text{def}}{=} \lim_{\rho\to 0}\Bigg[\int_{\mathbb{R}^3\backslash\cup_{n=1}^{N}\bar{\mathbb{V}}_n} d^3\mathbf{r}\, \Psi'^{(+)\dagger}(E',\mathbf{r})\Psi^{(+)}(E,\mathbf{r}) + \sum_{n=1}^{N}\frac{c\hbar\rho^2}{E'-E}\oint_{4\pi} d^2\mathbf{n}_n\Psi'^{(+)\dagger}(E',\mathbf{r}_n+\rho\mathbf{n}_n) \times i\mathbf{n}_n\cdot\boldsymbol{\alpha}\Psi^{(+)}(E,\mathbf{r}_n+\rho\mathbf{n}_n)\Bigg]. \tag{15}$$

For practical purposes, the use of the relationship

$$\langle\Psi'^{(+)}(E')|\Psi^{(+)}(E)\rangle = \lim_{R\to\infty}\frac{c\hbar R^2}{E'-E}\oint_{4\pi} d^2\mathbf{n}_R\Psi'^{(+)\dagger}(E',\mathbf{R})i\mathbf{n}_R\cdot\boldsymbol{\alpha}\Psi^{(+)}(E,\mathbf{R}), \tag{16}$$

deducible from Eqs. (14) and (15), may appear to be advantageous over the direct use of the definition (15).

The scalar product (15) possesses several interesting, and desirable, properties. Below we shall list, and prove, some of them.

First, the product (15) is applicable to any two nonsingular solutions $\Phi'(E',\mathbf{r})$ and $\Phi(E,\mathbf{r})$ of the free-particle Dirac equation as well. Defining

$$\langle\Phi'(E')|\Phi(E)\rangle \overset{\text{def}}{=} \lim_{\rho\to 0}\Bigg[\int_{\mathbb{R}^3\backslash\cup_{n=1}^{N}\bar{\mathbb{V}}_n} d^3\mathbf{r}\, \Phi'^{\dagger}(E',\mathbf{r})\Phi(E,\mathbf{r}) + \sum_{n=1}^{N}\frac{c\hbar\rho^2}{E'-E}\oint_{4\pi} d^2\mathbf{n}_n\Phi'^{\dagger}(E',\mathbf{r}_n+\rho\mathbf{n}_n) \times i\mathbf{n}_n\cdot\boldsymbol{\alpha}\Phi(E,\mathbf{r}_n+\rho\mathbf{n}_n)\Bigg], \tag{17}$$

in view of the regularity of $\Phi'(E',\mathbf{r})$ and $\Phi(E,\mathbf{r})$ in $\mathbb{R}^3$, *including the points* $\{\mathbf{r}_n\}$, we see that in the limit $\rho\to 0$ the second term in the square bracket on the right-hand side of Eq. (17) vanishes; at the same time, the first term (i.e., the volume integral) does not exhibit any singularity, reducing in the limit $\rho\to 0$ to the integral over $\mathbb{R}^3$, so that one has

$$\langle\Phi'(E')|\Phi(E)\rangle = \int_{\mathbb{R}^3} d^3\mathbf{r}\, \Phi'^{\dagger}(E',\mathbf{r})\Phi(E,\mathbf{r}). \tag{18}$$

Second, it appears that the singular scattering wave function

$$\Psi^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) = \Phi(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) + \sum_{n=1}^{N}\begin{pmatrix} h_0^{(+)}(k|\mathbf{r}-\mathbf{r}_n|)\chi_n^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0) \\ \varepsilon h_1^{(+)}(k|\mathbf{r}-\mathbf{r}_n|)\boldsymbol{\mu}_n(\mathbf{r})\cdot\boldsymbol{\sigma}\chi_n^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0)\end{pmatrix}, \tag{19}$$

excited by the Dirac plane wave

$$\Phi(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) = e^{ik\mathbf{n}_0\cdot\mathbf{r}}\begin{pmatrix}\eta(\boldsymbol{\nu}_0) \\ \varepsilon\mathbf{n}_0\cdot\boldsymbol{\sigma}\eta(\boldsymbol{\nu}_0)\end{pmatrix} \tag{20}$$

(see Sec. III A in Ref. [1]), is normalizable under the scalar product (15) to the Dirac delta function in the momentum space, since it holds that

$$\langle\Psi^{(+)}(E',\boldsymbol{\nu}_0',\mathbf{n}_0')|\Psi^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0)\rangle = (2\pi)^3(1+\varepsilon^2)\delta^{(3)}(k'\mathbf{n}_0'-k\mathbf{n}_0)\eta^{\dagger}(\boldsymbol{\nu}_0')\eta(\boldsymbol{\nu}_0). \tag{21}$$

To prove this, consider the integral

$$I_R^{(+)}(E',\boldsymbol{\nu}_0',\mathbf{n}_0';E,\boldsymbol{\nu}_0,\mathbf{n}_0) = R^2\oint_{4\pi} d^2\mathbf{n}_R\Psi^{(+)\dagger}(E',\boldsymbol{\nu}_0',\mathbf{n}_0',\mathbf{R})i\mathbf{n}_R\cdot\boldsymbol{\alpha}\Psi^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{R}). \tag{22}$$

On one hand, in virtue of Eq. (16) it holds that

$$\lim_{R\to\infty}\frac{c\hbar}{E'-E}I_R^{(+)}(E',\boldsymbol{\nu}_0',\mathbf{n}_0';E,\boldsymbol{\nu}_0,\mathbf{n}_0) = \langle\Psi^{(+)}(E',\boldsymbol{\nu}_0',\mathbf{n}_0')|\Psi^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0)\rangle. \tag{23}$$

On the other hand, assuming that $R$ is large, and observing that from Eqs. (3.16) and (3.17) of Ref. [1] it follows that

$$\Psi^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{R}) \overset{R\to\infty}{\sim} \frac{2\pi i}{k}\left[\frac{e^{-ikR}}{R}\delta^{(2)}(\mathbf{n}_R+\mathbf{n}_0)\begin{pmatrix}\eta(\boldsymbol{\nu}_0) \\ \varepsilon\mathbf{n}_0\cdot\boldsymbol{\sigma}\eta(\boldsymbol{\nu}_0)\end{pmatrix} - \frac{e^{ikR}}{R}\begin{pmatrix}S^{(+)}(E,\mathbf{n}_R,\mathbf{n}_0)\eta(\boldsymbol{\nu}_0) \\ \varepsilon\mathbf{n}_R\cdot\boldsymbol{\sigma}S^{(+)}(E,\mathbf{n}_R,\mathbf{n}_0)\eta(\boldsymbol{\nu}_0)\end{pmatrix}\right], \tag{24}$$

where $\delta^{(2)}(\mathbf{n}-\mathbf{n}')$ is the Dirac delta function on the unit sphere and $S^{(+)}(E,\mathbf{n}',\mathbf{n})$ is the $2\times 2$ scattering kernel (see Eq. (3.12) in Ref. [1]), we have

$$\begin{aligned} I_R^{(+)}(E',\boldsymbol{\nu}_0',\mathbf{n}_0';E,\boldsymbol{\nu}_0,\mathbf{n}_0) \overset{R\to\infty}{\sim} & \frac{4\pi^2}{k'k}(\varepsilon'+\varepsilon)\sin[(k'-k)R]\eta^{\dagger}(\boldsymbol{\nu}_0')\left[\Delta^{(+)}(E',\mathbf{n}_0';E,\mathbf{n}_0)+\delta^{(2)}(\mathbf{n}_0'-\mathbf{n}_0)I\right]\eta(\boldsymbol{\nu}_0) \\ & + i\frac{4\pi^2}{k'k}(\varepsilon'+\varepsilon)\cos[(k'-k)R]\eta^{\dagger}(\boldsymbol{\nu}_0')\left[\Delta^{(+)}(E',\mathbf{n}_0';E,\mathbf{n}_0)-\delta^{(2)}(\mathbf{n}_0'-\mathbf{n}_0)I\right]\eta(\boldsymbol{\nu}_0) \\ & + i\frac{4\pi^2}{k'k}(\varepsilon'-\varepsilon)\exp[i(k'+k)R]\eta^{\dagger}(\boldsymbol{\nu}_0')S^{(+)}(E,-\mathbf{n}_0',\mathbf{n}_0)\eta(\boldsymbol{\nu}_0) \\ & - i\frac{4\pi^2}{k'k}(\varepsilon'-\varepsilon)\exp[-i(k'+k)R]\eta^{\dagger}(\boldsymbol{\nu}_0')S^{(+)\dagger}(E',-\mathbf{n}_0,\mathbf{n}_0')\eta(\boldsymbol{\nu}_0), \end{aligned} \tag{25}$$

with

$$\Delta^{(+)}(E',\mathbf{n}_0';E,\mathbf{n}_0) = \oint_{4\pi} d^2\mathbf{n}_R S^{(+)\dagger}(E',\mathbf{n}_R,\mathbf{n}_0')S^{(+)}(E,\mathbf{n}_R,\mathbf{n}_0). \tag{26}$$

After dividing both sides of Eq. (25) by $E'-E$ and after subsequent passing to the limit $R\to\infty$, the third and fourth terms on the right-hand side of the resulting equation, considered as functions of $E$, oscillate infinitely rapidly with the finite amplitude, being thus effectively zero in the distributional sense. The same happens with the second term, because

$$\Delta^{(+)}(E,\mathbf{n}_0';E,\mathbf{n}_0) = \delta^{(2)}(\mathbf{n}_0'-\mathbf{n}_0), \tag{27}$$

as may be easily deduced from Eq. (26) and from Eqs. (5.30) and (4.17) in Ref. [1]. The situation with the first term is different, since it holds that

$$\lim_{R\to\infty}\frac{\sin[(k'-k)R]}{E'-E} = \pi\frac{dk}{dE}\delta(k'-k) = \frac{\pi E}{c^2\hbar^2 k}\delta(k'-k) \tag{28}$$

(the second equality follows after applying Eq. (2.9) of Ref. [1]). On combining Eqs. (23), (25), (27), and (28), one obtains

$$\langle\Psi^{(+)}(E',\boldsymbol{\nu}_0',\mathbf{n}_0')|\Psi^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0)\rangle = \frac{16\pi^3\varepsilon E}{c\hbar k^3}\delta(k'-k)\delta^{(2)}(\mathbf{n}_0'-\mathbf{n}_0)\eta^{\dagger}(\boldsymbol{\nu}_0')\eta(\boldsymbol{\nu}_0). \tag{29}$$

From this, Eq. (21) follows immediately. Proceeding in the same way as above, one may show that the "final-state" wave functions, defined in Sec. VI of Ref. [1] for the future use in the theory of photodetachment, obey the orthogonality relation analogous to that in Eq. (21):

$$\langle\Psi^{(-)}(E',\boldsymbol{\nu}_0',\mathbf{n}_0')|\Psi^{(-)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0)\rangle = (2\pi)^3(1+\varepsilon^2)\delta^{(3)}(k'\mathbf{n}_0'-k\mathbf{n}_0)\eta^\dagger(\boldsymbol{\nu}_0')\eta(\boldsymbol{\nu}_0). \quad (30)$$

It is instructive to compare the result (21) with the scalar product (17) of the Dirac plane waves $\Phi(E',\boldsymbol{\nu}_0',\mathbf{n}_0',\mathbf{r})$ and $\Phi(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r})$, both being of the functional form (20). Exploiting Eq. (18) and proceeding in the standard way, this product is readily shown to be

$$\langle\Phi(E',\boldsymbol{\nu}_0',\mathbf{n}_0')|\Phi(E,\boldsymbol{\nu}_0,\mathbf{n}_0)\rangle = (2\pi)^3(1+\varepsilon^2)\delta^{(3)}(k'\mathbf{n}_0'-k\mathbf{n}_0)\eta^\dagger(\boldsymbol{\nu}_0')\eta(\boldsymbol{\nu}_0), \quad (31)$$

and is seen to be identical with the right-hand side of Eq. (21), i.e., it holds that

$$\langle\Psi^{(+)}(E',\boldsymbol{\nu}_0',\mathbf{n}_0')|\Psi^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0)\rangle = \langle\Phi(E',\boldsymbol{\nu}_0',\mathbf{n}_0')|\Phi(E,\boldsymbol{\nu}_0,\mathbf{n}_0)\rangle. \quad (32)$$

Equation (32) is a particular case of the more general relationship

$$\langle\Psi'^{(+)}(E')|\Psi^{(+)}(E)\rangle = \langle\Phi'(E')|\Phi(E)\rangle, \quad (33)$$

which is the third property of the product (15) we wish to highlight. To prove the relationship (33), we observe that any regular solution $\Phi(E,\mathbf{r})$ of the free-particle Dirac equation may be expressed as the superposition of the plane waves of the form (20):

$$\Phi(E,\mathbf{r}) = \sum_{s=\pm1}\oint_{4\pi} d^2\mathbf{n}_0\, c(E,s\boldsymbol{\nu}_0,\mathbf{n}_0)\Phi(E,s\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}), \quad (34)$$

where

$$c(E,\boldsymbol{\nu}_0,\mathbf{n}_0)\delta(E-E') = \frac{Ek}{(2\pi)^3(1+\varepsilon^2)c^2\hbar^2}\int_{\mathbb{R}^3} d^3\mathbf{r}\,\Phi^\dagger(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r})\Phi(E',\mathbf{r}). \quad (35)$$

After inserting the expansion (34) into either of Eqs. (17) or (18), it follows that

$$\langle\Phi'(E')|\Phi(E)\rangle = \sum_{s,s'=\pm1}\oint_{4\pi} d^2\mathbf{n}_0'\oint_{4\pi} d^2\mathbf{n}_0\, c'^*(E',s'\boldsymbol{\nu}_0,\mathbf{n}_0')c(E,s\boldsymbol{\nu}_0,\mathbf{n}_0) \times\langle\Phi(E',s'\boldsymbol{\nu}_0,\mathbf{n}_0')|\Phi(E,s\boldsymbol{\nu}_0,\mathbf{n}_0)\rangle. \quad (36)$$

On the other hand, from the linearity and homogeneity of Eqs. (1) and (3) [see also Eq. (2.23) in Ref. [1] and the remark following Eq. (3.7) therein], and also from Eqs. (2) and (34), we find that

$$\Psi^{(+)}(E,\mathbf{r}) = \sum_{s=\pm1}\oint_{4\pi} d^2\mathbf{n}_0\, c(E,s\boldsymbol{\nu}_0,\mathbf{n}_0)\Psi^{(+)}(E,s\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}). \quad (37)$$

Combining this with either of Eqs. (15) or (16), we get

$$\langle\Psi'^{(+)}(E')|\Psi^{(+)}(E)\rangle = \sum_{s,s'=\pm1}\oint_{4\pi} d^2\mathbf{n}_0'\oint_{4\pi} d^2\mathbf{n}_0\, c'^*(E',s'\boldsymbol{\nu}_0,\mathbf{n}_0')c(E,s\boldsymbol{\nu}_0,\mathbf{n}_0) \times\langle\Psi^{(+)}(E',s'\boldsymbol{\nu}_0,\mathbf{n}_0')\Psi^{(+)}(E,s\boldsymbol{\nu}_0,\mathbf{n}_0)\rangle. \quad (38)$$

Comparing the right-hand sides of Eqs. (36) and (38), after making use of Eq. (32) we see that they are identical, which proves the validity of the relationship (33).

Before concluding, we cannot resist the temptation to show that the scalar product (15) offers even more than expected: it appears that the eigenchannel wave functions $\mathcal{X}_\gamma(E,\mathbf{r})$, introduced in Sec. IV of Ref. [1], are orthogonal under this scalar product in the sense of

$$\langle\mathcal{X}_{\gamma'}(E')|\mathcal{X}_\gamma(E)\rangle = \frac{\pi}{\sin^2\delta_\gamma(E)}\delta(E'-E)\delta_{\gamma'\gamma}. \quad (39)$$

To prove the relation (39), we exploit the fact that the eigenchannels are singular solutions of the Dirac equation (1). Thus, instead of using directly the definition (15), we apply Eq. (16) and write

$$\langle\mathcal{X}_{\gamma'}(E')|\mathcal{X}_\gamma(E)\rangle = \lim_{R\to\infty}\frac{c\hbar R^2}{E'-E}\oint_{4\pi} d^2\mathbf{n}_R\,\mathcal{X}^\dagger_{\gamma'}(E',\mathbf{R})i\mathbf{n}_R\cdot\boldsymbol{\alpha}\mathcal{X}_\gamma(E,\mathbf{R}), \quad (40)$$

and then transform the right hand of Eq. (40) with the aid of the asymptotic representation

$$\mathcal{X}_\gamma(E,\mathbf{R}) \overset{R\to\infty}{\sim} \frac{\mathrm{sgn}(E)}{\sqrt{1+\varepsilon^2}}\sqrt{\frac{E}{2c^2\hbar^2k}}\frac{1}{i\sin\delta_\gamma(E)} \times\left[\frac{e^{-ikR-i\delta_\gamma(E)}}{R}\begin{pmatrix}\Upsilon_\gamma(E,-\mathbf{n}_R)\\ -\varepsilon\mathbf{n}_R\cdot\boldsymbol{\sigma}\Upsilon_\gamma(E,-\mathbf{n}_R)\end{pmatrix} - \frac{e^{ikR+i\delta_\gamma(E)}}{R}\begin{pmatrix}\Upsilon_\gamma(E,\mathbf{n}_R)\\ \varepsilon\mathbf{n}_R\cdot\boldsymbol{\sigma}\Upsilon_\gamma(E,\mathbf{n}_R)\end{pmatrix}\right] \quad (41)$$

[see Eqs. (4.13) and (4.16) from Ref. [1]]. After some manipulations, exploiting, among others, the identity

$$\oint_{4\pi} d^2\mathbf{n}_R\Upsilon^\dagger_{\gamma'}(E',-\mathbf{n}_R)\Upsilon_\gamma(E,\pm\mathbf{n}_R) = \oint_{4\pi} d^2\mathbf{n}_R\Upsilon^\dagger_{\gamma'}(E',\mathbf{n}_R)\Upsilon_\gamma(E,\mp\mathbf{n}_R), \quad (42)$$

one finds

$$\frac{c\hbar R^2}{E'-E}\oint_{4\pi} d^2\mathbf{n}_R \mathcal{X}^\dagger_{\gamma'}(E',\mathbf{R}) i\mathbf{n}_R\cdot\boldsymbol{\alpha}\mathcal{X}_\gamma(E,\mathbf{R}) \overset{R\to\infty}{\sim} \frac{\mathrm{sgn}(E'E)}{c\hbar\sin\delta_{\gamma'}(E')\sin\delta_\gamma(E)}\sqrt{\frac{E'E}{k'k(1+\varepsilon'^2)(1+\varepsilon^2)}} \times\Bigg[(\varepsilon'+\varepsilon)\frac{\sin\{(k'-k)R+[\delta_{\gamma'}(E')-\delta_\gamma(E)]\}}{E'-E}\oint_{4\pi} d^2\mathbf{n}_R \Upsilon^\dagger_{\gamma'}(E',\mathbf{n}_R)\Upsilon_\gamma(E,\mathbf{n}_R) - \frac{\varepsilon'-\varepsilon}{E'-E}\sin\{(k'+k)R+[\delta_{\gamma'}(E')+\delta_\gamma(E)]\}\oint_{4\pi} d^2\mathbf{n}_R \Upsilon^\dagger_{\gamma'}(E',\mathbf{n}_R)\Upsilon_\gamma(E,-\mathbf{n}_R)\Bigg]. \tag{43}$$

In the limit $R\to\infty$ the second term in the square bracket on the right-hand side of the above equation is zero in the distributional sense. Furthermore, since

$$\frac{\sin\{(k'-k)R+[\delta_{\gamma'}(E')-\delta_\gamma(E)]\}}{E'-E} = \frac{\sin[(k'-k)R]}{E'-E}\cos[\delta_{\gamma'}(E')-\delta_\gamma(E)] + \cos[(k'-k)R]\frac{\sin[\delta_{\gamma'}(E')-\delta_\gamma(E)]}{E'-E} \tag{44}$$

and

$$\oint_{4\pi} d^2\mathbf{n}_R \Upsilon^\dagger_{\gamma'}(E',\mathbf{n}_R)\Upsilon_\gamma(E,\mathbf{n}_R) = \delta_{\gamma'\gamma} + \sum_{n=1}^{\infty}\frac{(E'-E)^n}{n!}\oint_{4\pi} d^2\mathbf{n}_R \frac{\partial^n \Upsilon^\dagger_{\gamma'}(E,\mathbf{n}_R)}{\partial E^n}\Upsilon_\gamma(E,\mathbf{n}_R), \tag{45}$$

in the distributional sense one has

$$\lim_{R\to\infty}\frac{\sin\{(k'-k)R+[\delta_{\gamma'}(E')-\delta_\gamma(E)]\}}{E'-E}\oint_{4\pi} d^2\mathbf{n}_R \Upsilon^\dagger_{\gamma'}(E',\mathbf{n}_R)\Upsilon_\gamma(E,\mathbf{n}_R) = \pi\delta(E'-E)\delta_{\gamma'\gamma}. \tag{46}$$

On combining Eqs. (40), (43), and (46), one arrives at the orthogonality relation (39).

In a later publication, we shall present an extension of the model of zero-range potentials to bound states of Dirac particles.


[1] R. Szmytkowski, Phys. Rev. A **71**, 052708 (2005).

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