Elementary proof of the bound on the speed of quantum evolution

Piotr Kosiński and Magdalena Zych
Department of Theoretical Physics II, Institute of Physics, University of Łódź, ulica Pomorska 149/153, 90-236 Łódź, Poland
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An elementary proof is given of the bound on "orthogonalization time" $t_0 \geqslant \pi\hbar/2\Delta E$.

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In many problems of quantum theory (like, for example, quantum computing [1–4] or fidelity between two quantum states [5,6]) it appears important to estimate the speed of quantum evolution.

An interesting measure of evolution speed is provided by the minimum time t_0 required for the state to be transformed into an orthogonal (i.e., distinguishable) state. The basic estimate concerning t_0 is given by the inequality

$$t_0 \geqslant \frac{\pi\hbar}{2\Delta E} \tag{1}$$

which has been derived and studied by many authors [7–13]. This bound, in terms of the energy dispersion ΔE of the initial state, is very simple and natural (in particular, $\Delta E=0$ implies $t_0=\infty$, as it should since the initial state is then an energy eigenstate). It has been generalized in various directions [14,15,5]; also, a beautiful geometric interpretation in terms of the Fubini-Study metric was given [16] (see also [17]) and the intelligent states saturating Eq. (1) were found [18].

Quite unexpectedly, a few years ago Margolus and Levitin [1] derived a new bound of the form

$$t_0 \geqslant \frac{\pi\hbar}{2(E-E_0)} \tag{2}$$

valid for Hamiltonians bounded from below; here E_0 is the lowest energy while E is the expectation value of the Hamiltonian. They were able to show that, for a large class of states, Eq. (2) provides a more optimal bound than Eq. (1) [on the other hand, for energy eigenstates, except the lowest one, Eq. (2) is useless]. The intelligent states for the inequality (2) were found in Refs. [19,20].

While the standard proof of the bound (1) is based on Heisenberg equations of motion and the uncertainty principle (see, however, [12]), the Margolus-Levitin derivation of the new bound (2) is surprisingly elementary; moreover, the corresponding intelligent states can be easily found [20].

The question arises whether the bound (1) can be derived along the same lines. The aim of the present Brief Report is to provide a positive answer to this question. We shall show that (1) holds provided the Hamiltonian H is self-adjoint and the initial state belongs to its domain. No further restrictions on the properties of H are necessary; in particular, the spectrum may include both discrete and continuous parts and may extend to infinity in both directions.

Let us first sketch a generalization of the elegant approach of Ref. [1]. We assume for simplicity that the spectrum of H is purely discrete; the general case is briefly discussed in the final part of the paper.

Let $\{|n\rangle\}$ be the basis consisting of eigenstates of the Hamiltonian H,

$$H|n\rangle = E_n|n\rangle, \tag{3}$$

and let

$$|\Psi(0)\rangle = \sum_n c_n|n\rangle \tag{4}$$

be some initial state. Then

$$\langle\Psi(0)|\Psi(t)\rangle = \sum_n |c_n|^2 e^{-(iE_n/\hbar)t} = \left\langle \cos\left(\frac{Ht}{\hbar}\right)\right\rangle_0 - i\left\langle \sin\left(\frac{Ht}{\hbar}\right)\right\rangle_0. \tag{5}$$

Here $\langle f(H)\rangle_0 \equiv \Sigma_n f(E_n)|c_n|^2$ denotes the average with respect to the initial state.

Now, since $\langle\Psi(0)|\Psi(t_0)\rangle=0$ one obtains

$$\left\langle \cos\left(\frac{Ht_0}{\hbar}\right)\right\rangle_0 = 0, \quad \left\langle \sin\left(\frac{Ht_0}{\hbar}\right)\right\rangle_0 = 0, \tag{6}$$

or

$$\left\langle A\cos\left(\frac{Ht_0}{\hbar}+\alpha\right)\right\rangle_0 = 0 \tag{7}$$

for arbitrary constants A, α.

Consider now an inequality of the form

$$f(x) \geqslant A\cos(x+\alpha) \tag{8}$$

which is assumed to hold for $-\infty < x < \infty$ or $0 \leqslant x \leqslant \infty$ if the spectrum of H extends in both directions or is nonnegative, respectively. Then

$$\left\langle f\left(\frac{Ht}{\hbar}\right)\right\rangle_0 \geqslant \left\langle A\cos\left(\frac{Ht}{\hbar}+\alpha\right)\right\rangle_0 \tag{9}$$

provided the left-hand side is well defined (i.e., the average exists). Now, due to Eq. (7),

$$\left\langle f\left(\frac{Ht_0}{\hbar}\right)\right\rangle_0 \geqslant 0. \tag{10}$$

The above inequality imposes certain restrictions on t_0. By a judicious choice of $f(x)$ one can learn something interesting about t_0. For example, the bound (2) is obtained by taking the optimal inequality (8) in the class of linear functions $f(x)$ (in this case we have to restrict the range of x to the positive semiaxis). Let us now consider (8) in the class of quadratic functions $f(x)$ and $-\infty < x < \infty$. It is an elementary task to check that the optimal inequality reads now

$$(x+\alpha)^2 - \frac{\pi}{4} \geqslant -\pi\cos(x+\alpha). \tag{11}$$

By assumption, $|\Psi(0)\rangle$ belongs to the domain of H and both $\langle H\rangle_0$ and $\langle H^2\rangle_0$ are well defined [21]. Equation (10) takes now the form

$$\frac{\langle H^2\rangle_0}{\hbar^2}t_0^2 + \frac{2\alpha\langle H\rangle_0}{\hbar}t_0 + \left(\alpha^2 - \frac{\pi^2}{4}\right) \geqslant 0 \tag{12}$$

which implies that t_0 lies outside the open interval

$$\Delta_\alpha \equiv \left(\frac{-2\alpha\langle H\rangle_0 - \sqrt{\pi^2\langle H^2\rangle_0 - 4\alpha^2\Delta E_0^2}}{2\langle H^2\rangle_0/\hbar}, \frac{-2\alpha\langle H\rangle_0 + \sqrt{\pi^2\langle H^2\rangle_0 - 4\alpha^2\Delta E_0^2}}{2\langle H^2\rangle_0/\hbar}\right) \tag{13}$$

where $\Delta E_0^2 \equiv \langle H^2\rangle_0 - \langle H\rangle_0^2$. It follows from Eq. (13) that Δ_α is nonempty provided α belongs to the open interval

$$\Omega \equiv \left(\frac{-\pi\sqrt{\langle H^2\rangle_0}}{2\Delta E_0}, \frac{\pi\sqrt{\langle H^2\rangle}}{2\Delta E_0}\right). \tag{14}$$

So, finally, we obtain

$$t_0 \notin \bigcup_{\alpha\in\Omega}\Delta_\alpha = \left(\frac{-\pi\hbar}{2\Delta E_0}, \frac{\pi\hbar}{2\Delta E_0}\right) \tag{15}$$

which implies (1).

In order to find intelligent states for the bound (1) we define

$$\gamma_\alpha(x) \equiv (x+\alpha)^2 - \frac{\pi^2}{4} + \pi\cos(x+\alpha). \tag{16}$$

Then

$$\gamma_\alpha(x) \geqslant 0 \tag{17}$$

and $\gamma_\alpha(x)=0$ if and only if $x=-\alpha\pm\pi/2$.

Assuming $t_0 = \pi\hbar/2\Delta E_0$ we find from Eqs. (12) and (16)

$$\left\langle \gamma_\alpha\left(\frac{Ht_0}{\hbar}\right)\right\rangle_0 = 0 \quad \text{for } \alpha = \frac{-\pi\langle H\rangle_0}{2\Delta E_0}. \tag{18}$$

Now, due to (17), Eq. (18) implies $c_n \neq 0$ only if $E_n t_0/\hbar = \pi\langle H\rangle_0/2\Delta E_0 \pm \pi/2$. Therefore, $c_n \neq 0$ for at most two levels and $E_{n_1} = \langle H\rangle_0 + \Delta E_0$, $E_{n_2} = \langle H\rangle_0 - \Delta E_0$, which holds provided $|c_{n_1}|^2 = |c_{n_2}|^2 = \frac{1}{2}$. Therefore, the intelligent states are of the form [18]

$$|\chi\rangle = c_1|n_1\rangle + c_2|n_2\rangle, \quad |c_1|^2 = |c_2|^2 = \frac{1}{2}. \tag{19}$$

Finally, let us briefly discuss the general case when no assumption concerning the spectrum of H is made. The spectral theorem [21] allows us to write

$$\langle\Psi(0)|\Psi(t)\rangle = \langle\Psi(0)|e^{-(it/\hbar)H}|\Psi(0)\rangle = \int e^{-(iEt/\hbar)} d\langle\Psi(0)| \times |P_E|\Psi(0)\rangle \tag{20}$$

where P_E is a spectral measure for energy. By assumption $|\Psi(0)\rangle$ belongs to the domain of H, which implies [21]

$$\int E^2 d\langle\Psi(0)|P_E|\Psi(0)\rangle < \infty. \tag{21}$$

Therefore, $\gamma_\alpha(Et/\hbar)$ is integrable and

$$\int \gamma_\alpha\left(\frac{Et}{\hbar}\right) d\langle\Psi(0)|P_E|\Psi(0)\rangle \geqslant 0 \tag{22}$$

which again leads to the estimate (1).

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[1] N. Margolus and L. B. Levitin, Physica D **120**, 188 (1998).
[2] A. Miyake and M. Wadati, Phys. Rev. A **64**, 042317 (2001).
[3] S. Lloyd, Phys. Rev. Lett. **88**, 237901 (2002).
[4] S. Lloyd, e-print quant-ph/9908043.
[5] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. A **67**, 052109 (2003).
[6] V. Giovannetti, S. Lloyd, and L. Maccone, Proc. SPIE **5111**, 1 (2003).
[7] L. Mandelstam and I. Tamm, J. Phys. (USSR) **9**, 249 (1945).
[8] G. N. Fleming, Nuovo Cimento Soc. Ital. Fis., A **16**, 232 (1973).
[9] K. Bhattacharyya, J. Phys. A **16**, 2991 (1983).
[10] D. Home and M. A. B. Whitaker, J. Phys. A **19**, 1847 (1986).
[11] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer, Hingham, 1985).
[12] L. Vaidman, Am. J. Phys. **60**, 182 (1992).
[13] L. Vaidman and O. Belkind, e-print quant-ph/9707060.
[14] P. Pfeifer, Phys. Rev. Lett. **70**, 3365 (1993).
[15] J. Uffink, Am. J. Phys. **61**, 935 (1993).
[16] J. Anandan and Y. Aharonov, Phys. Rev. Lett. **65**, 1697 (1990).
[17] D. C. Brody, J. Phys. A **36**, 5587 (2003).
[18] N. Horesh and A. Mann, J. Phys. A **31**, L609 (1998).
[19] J. Söderholm, G. Björk, T. Tsegaye, and A. Trifonov, Phys. Rev. A **59**, 1788 (1999).
[20] K. Andrzejewski (unpublished).
[21] M. Reed and B. Simon, *Methods of Modern Mathematical Physics* (Academic Press, New York, 1972), Vol. 1.