Approximate Fokker-Planck equation for a single-mode laser driven by quadratic pump noise and quantum noise with cross-correlation between real and imaginary parts of noise

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A single-mode laser noise model driven by quadratic pump noise and quantum noise with cross-correlation between the real and imaginary parts of the noises is proposed. The approximate Fokker-Planck equation (AFPE) of the model for the laser phase and the laser amplitude is derived. It is found that the laser phase is controlled intensively by the correlation between the real and imaginary parts of the pump noise and that of the quantum noise. The correlation between the real and imaginary parts of quantum noise λ_q tends to lead the laser phase to be locked at some values and the correlation between the real and imaginary parts of the pump noise λ_p tends to destroy or confine the laser phase lock. Quantitative results are presented and discussed in detail. As an important application of the above-mentioned results, we take a phase lock approximation to get a Langevin equation for the laser field amplitude and an AFPE of the laser intensity.

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I. INTRODUCTION

The noises present in the single-mode laser model are usually assumed to be uncorrelated. The real and imaginary parts of the complex noise of a single-mode laser are also assumed to be uncorrelated. It was so until 1993 when Zhu investigated theoretically the statistical fluctuations of a single-mode laser with correlation between additive and multiplicative white-noise terms. The mean, variance, and skewness of the steady-state laser intensity are calculated by Zhu through a one-dimensional laser Langevin equation [1]. In Refs. [2,3] we have introduced the phenomenological correlation between the additive and multiplicative noises in a single-mode laser cubic model and studied the stochastic equivalent of various one-dimensional models. Subsequently, we have discussed the mechanisms [4] that make the multiplicative noise correlate with the additive noise in a single-mode laser and studied the noise-induced phenomena for the laser intensity [5] and the laser phase [6]. The interplay of noise cross-correlation and nonlinearity in a system far from equilibrium results in some unusual phenomena. Recently, progress for the noise cross-correlation effects in nonlinear systems has been reported [7–15]. However, there are few previous studies in which the effect of cross-correlation between the real and imaginary parts of quantum noise is considered. In 1996, Zhou, Gao, and Zhu first took into account the cross-correlation between the real and imaginary parts of quantum noise in a single-mode laser [16]. In their research, the amplitude equation was not decoupled from the phase equation, and they dealt with it in the two-dimensional case. Soon afterwards Ke *et al.* [17] also considered the cross-correlation between the real and imaginary parts of the quantum noise, but they adopted the locked phase method to make the amplitude equation decouple from the phase equation. Subsequently, Zhang *et al.* [18] further studied the effect of cross-correlation between the real and imaginary parts of quantum noise in terms of the white-gain-noise laser model: the corresponding stationary and transient properties were calculated and compared with the experimental data. To our knowledge, there are no works in which the effect of cross-correlation between the real and imaginary parts of pump noise is considered. The reason is that in the single-mode laser model used, the cross-correlation coefficient between the real and imaginary parts of pump noise does not appear in the laser-intensity Langevin equation. This is an inherent property of the linear pump-noise laser model. We find that when quadratic pump noise is introduced to the laser field equation, the cross-correlation coefficient of the real and imaginary parts of the pump noise should appear in the decoupled intensity Langevin equation and it strongly alters the statistical properties of the output laser light. In the present paper, our goals are twofold: (i) In comparison with the linear pump-noise laser model, in which the white-noise limit of pump noise can exist, the pump noise in the quadratic pump-noise laser model must be colored in nature because the quadratic Dirac δ function is not well defined. So we must face a non-Markovian process. The first goal of the paper is to make a Markovain approximation of the process. Based on our early work of the approximate Fokker-Planck equation (AFPE) for systems driven by nonlinear external noise [19], we derive an AFPE for the model proposed in this paper. (ii) It has been shown that the pump noise comes from the fluctuation in the pump laser and those in the active medium (for dye laser). So the pump noise is an external noise in nature. That is, randomness in the external conditions entails the parameters of a dynamical system to fluctuate. The extent of these fluctuations is independent of any thermodynamic characteristic of the system in contrast to intrinsic fluctuations, the amplitude of which is proportional to the equilibrium temperature, in accordance with the fluctuation-dissipation theorem. The physical foundation of

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the quadratic pump-noise laser model proposed in this paper is that the fluctuation of the external condition, the pump laser field, makes the net gain fluctuate with quadratic nonlinearity in the noise [20].

The rest of this paper is arranged as follows. In Sec. II, the model is proposed. An approximate Fokker-Planck equation of the model for the laser phase is derived. In Sec. III, the laser phase lock is discussed in detail. It is found that the laser phase is controlled intensively by the correlation between the real and imaginary parts of the pump noise and that of the quantum noise. The quantitative results are discussed in detail. In Sec. IV, as an important application of the above-mentioned results, we take a phase lock approximation to get a Langevin equation for the laser field amplitude. An AFPE of the laser intensity and its solution are given in Sec. V. Section VI contains some concluding remarks.

II. APPROXIMATE FOKKER-PLANCK EQUATION FOR THE LASER PHASE

To investigate the effects of correlation between the real and imaginary parts of the pump noise, we propose a single-mode laser cubic model driven by quadratic pump noise described by the following Langevin equation for the laser field E:

$$\dot{E} = aE - A|E|^2E + \xi^2(t)E + q(t). \tag{1}$$

Equation (1) retains the saturation parameter A of the laser theory. a is the net gain. $q(t)$ denotes the complex quantum noise, $q(t)=q_1(t)+iq_2(t)$, with correlation between its real and imaginary parts [16–18]:

$$\langle q(t)\rangle = 0, \tag{2}$$

$$\langle q_i(t)q_j(t')\rangle = [\delta_{ij} + \lambda_q(1-\delta_{ij})]D\delta(t-t') \quad (i,j=1,2), \tag{3}$$

where λ_q is the correlation coefficient, $-1 \leqslant \lambda_q \leqslant 1$.

In Eq. (1), the quadratic pump noise term $\xi^2(t)E$ is introduced. $\xi(t)$ is a complex pump noise, $\xi(t)=\xi_1(t)+i\xi_2(t)$, and its statistical properties are assumed to be

$$\langle \xi(t)\rangle = 0, \tag{4}$$

$$\langle \xi_i(t)\xi_j(t')\rangle = [\delta_{ij} + \lambda_p(1-\delta_{ij})]\frac{Q}{\tau}e^{-|t-t'|/\tau} \quad (i,j=1,2), \tag{5}$$

$$\langle \xi_i(t)q_j(t')\rangle = 0, \quad (i,j=1,2), \tag{6}$$

where λ_p stands for the correlation coefficient between the real and imaginary parts of the pump noise, $-1 \leqslant \lambda_p \leqslant 1$. We introduce $\eta(t)=\xi^2(t)=\eta_1(t)+i\eta_2(t)$, so the real and imaginary parts of $\eta(t)$ are given by

$$\eta_1(t) = \xi_1^2(t) - \xi_2^2(t), \quad \eta_2(t) = 2\xi_1(t)\xi_2(t). \tag{7}$$

In this, Eq. (1) can be written as

$$\dot{r} = ar - Ar^3 + \eta_1(t)r + \varepsilon_r(t), \tag{8}$$

$$\dot{\varphi} = \varepsilon_\varphi(t)/r + \eta_2(t). \tag{9}$$

Here we have used $E=re^{i\varphi}$ and r and φ are the amplitude and phase of the laser field. $\varepsilon_r(t)$ and $\varepsilon_\varphi(t)$ are defined as

$$\varepsilon_r(t) = q_2(t)\sin\varphi + q_1(t)\cos\varphi, \tag{10}$$

$$\varepsilon_\varphi(t) = q_2(t)\cos\varphi - q_1(t)\sin\varphi. \tag{11}$$

Based on the following analysis for the phase Langevin equation (9), it can be seen that the correlation between the real and imaginary parts of the quantum noise leads to the laser phase being locked at a stationary value, while the correlation between the real and imaginary parts of the pump noise tends to destroy this phase lock. The analysis is based on the small fluctuation approximation in which the laser amplitude r in Eq. (9) can be viewed as a deterministic steady value of $r_s=\sqrt{a/A}$. Since the average values of $\langle\varepsilon_\varphi(t)\rangle$ and $\langle\eta_2(t)\rangle$ are not zero, we write

$$\varepsilon_\varphi(t) = \langle\varepsilon_\varphi(t)\rangle + \widetilde{\varepsilon_\varphi(t)} \tag{12}$$

and

$$\eta_2(t) = \langle\eta_2(t)\rangle + \widetilde{\eta_2(t)}. \tag{13}$$

Using the techniques developed in Ref. [20] to calculate $\langle\varepsilon_\varphi(t)\rangle$, $\langle\eta_2(t)\rangle$, and the noise correlation, we obtain the following equations from Eq. (9):

$$\dot{\varphi} = \langle\varepsilon_\varphi(t)\rangle/r_s + \langle\eta_2(t)\rangle + \widetilde{\varepsilon_\varphi(t)}/r_s + \widetilde{\eta_2(t)}, \tag{14}$$

with

$$\langle\varepsilon_\varphi(t)\rangle = -\frac{D}{2r_s}\lambda_q\cos 2\varphi, \tag{15}$$

$$\langle\eta_2(t)\rangle = 2\lambda_p\frac{Q}{\tau}. \tag{16}$$

Here $\widetilde{\varepsilon_\varphi(t)}$ and $\widetilde{\eta_2(t)}$ are the noises with

$$\langle\widetilde{\varepsilon_\varphi(t)}\rangle = \langle\widetilde{\eta_2(t)}\rangle = 0, \tag{17}$$

$$\langle\widetilde{\varepsilon_\varphi(t)}\widetilde{\varepsilon_\varphi(t')}\rangle = D(1-\lambda_q\sin 2\varphi)\delta(t-t'), \tag{18}$$

$$\langle\widetilde{\eta_2(t)}\,\widetilde{\eta_2(t')}\rangle = 4(1+\lambda_p^2)\left(\frac{Q}{\tau}\right)^2 e^{-2|t-t'|/\tau}. \tag{19}$$

What is new in Eq. (14) with Eqs. (15)–(19) is the presence of nonzero drift terms. They are controlled intensively by the correlations between the real and imaginary parts of complex noises: the quantum noise and the pump noise. Due to the form of Eqs. (14)–(19), the laser phase may be locked at a stationary value. To see this, we need to make a Markovian approximation to the non-Markovian process governed by Eq. (14). This can be done by using a nonlinear external noise theory proposed by the authors [19] to obtain an AFPE for the non-Markovian process (14)–(19). Following Ref. [19], we have

$$\frac{\partial}{\partial t}p(\varphi,t) = -\frac{\partial}{\partial \varphi}A(\varphi)p(\varphi,t) + \frac{\partial^2}{\partial \varphi^2}K_2(\varphi)p(\varphi,t), \quad (20)$$

in which

$$A(\varphi) = -\frac{D}{r_s^2}\lambda_q \cos 2\varphi + 2\lambda_p \frac{Q}{\tau}, \quad (21)$$

$$K_2(\varphi) = \frac{D}{r_s^2}(1 - \lambda_q \sin 2\varphi) + 4(1 + \lambda_p^2)\frac{Q^2}{\tau}. \quad (22)$$

III. PHASE LOCKING

It is known that Eqs. (20)–(22) are equivalent to the Langevin equation [21]

$$\dot{\varphi} = -\frac{D}{2r_s^2}\lambda_q \cos 2\varphi + 2\lambda_p \frac{Q}{\tau} + f(\varphi)\Gamma(t), \quad (23)$$

where $\Gamma(t)$ is a Gaussian white noise with

$$\langle \Gamma(t) \rangle = 0, \quad (24)$$

$$\langle \Gamma(t)\Gamma(t') \rangle = \delta(t - t'). \quad (25)$$

The multiplicative function

$$f(\varphi(t - \Delta t)) = \left(\frac{D}{r_s^2}[1 - \lambda_q \sin 2\varphi(t - \Delta t)] + 4(1 + \lambda_p^2)\frac{Q^2}{\tau}\right)^{1/2} \quad (26)$$

is not a functional of noise $\Gamma(t)$ due to $\Delta t \gtrsim 0$ and thus $\varphi(t - \Delta t)$ is independent of the noise $\Gamma(t)$ at the time t [2].

Now we determine the phase lock value using Eqs. (23)–(26). The lock condition is

$$\langle \dot{\varphi} \rangle = 0. \quad (27)$$

The condition for the phase lock to be stable is [3]

$$\frac{\partial}{\partial \varphi}\langle \dot{\varphi} \rangle - \frac{1}{2}\frac{\partial^2}{\partial \varphi^2}\langle [f(\varphi)\Gamma(t)]^2 \rangle < 0. \quad (28)$$

Condition (27) leads to the lock value φ_0 determined by

$$\cos 2\varphi_0 = J\frac{\lambda_p}{\lambda_q} \quad (29)$$

with $J = 4aQ/AD\tau$. And condition (28) gives

$$\lambda_q \sin 2\varphi_0 > 0. \quad (30)$$

General conclusions are as follows.

(i) The combination of conditions (29) and (30) can be written as a single equation

$$\lambda_q \sin 2\varphi_0 = \left| \lambda_q \sin \arccos J\frac{\lambda_p}{\lambda_q} \right|, \quad \text{with } \lambda_q \neq 0. \quad (31)$$

(ii) If $\lambda_q \neq 0$ and $\lambda_p = 0$, Eq. (31) reduces to $\lambda_q \sin 2\varphi_0 = |\lambda_q|$ and the phase lock is determined by the correlation between the real and imaginary parts of quantum noise only.

(iii) As can be seen from Eq. (31), if both $\lambda_q \neq 0$ and $\lambda_p \neq 0$, for a fixed value of J/λ_q, there exists a critical value of the correlation coefficient λ_p^c determined by $|J\lambda_p^c/\lambda_q| = 1$. When $|\lambda_p| > |\lambda_p^c|$, no phase lock value of φ_0 exists and the laser phase diffuses freely.

(iv) The physical aspect of the appearance of the laser phase lock lies in the presence of a new drift term $(-D/2r_s^2)\lambda_q \cos 2\varphi$ in Eq. (23) which arises from the nonzero correlation coefficient λ_q.

(v) If $\lambda_q = 0$ but $\lambda_p \neq 0$, it can be seen from Eq. (23) that, in addition to the diffusion arising from the noise, a constant force, which comes from the correlation between the real and imaginary parts of the pump noise, $2\lambda_p Q/\tau$, makes the laser phase increase linearly with time at an average speed $V_\varphi = 2\lambda_p Q/\tau$. This feature has a potential application.

(vi) In the case when both $\lambda_q = 0$ and $\lambda_p = 0$, the laser phase diffuses freely as in the usual laser theory.

To sum up, the correlation between the real and imaginary parts of the quantum noise λ_q tends to lead the laser phase to be locked at some values and the correlation between the real and imaginary parts of the pump noise λ_p tends to destroy or confine the laser phase lock.

The key for the experimental realization of various phase-locking possibilities discussed above is to control the correlation between the real and imaginary parts of the complex pump noise λ_p. The control can be experimentally realized in the following way. A real Ornstein-Uhlenbeck (OU) noise is generated using a noise source and is split into two parts. One of the two goes through a linear system and then is combined with the other part as the real and imaginary parts of a complex noise. Note that the input and output of a linear system are correlated and that their correlation coefficients are controllable. Moreover, a linear system preserves the Gaussian nature of the noise. The complex noise so generated is Gaussian, and its real and imaginary parts are correlated with correlation coefficients that are adjustable. This complex noise passes through a square device and then multiplies with the feedback laser obtained by an external mirror [22]. When this is input into the lasing cavity, the square noise term in Eq. (1) is realized [23].

IV. PHASE LOCK APPROXIMATION AND THE LANGEVIN EQUATION OF LASER FIELD AMPLITUDE

As an important application of the phase dynamics discussed above, we take a phase locking approximation to get a Langevin equation for the laser field amplitude and an AFPE of the laser intensity and to obtain an exact steady-state solution for the AFPE.

The Langevin equation (8) with Eqs. (7) and (10) is not closed due to the coupling with the laser phase φ. In order to obtain a closed equation for the laser field amplitude r, we use a phase locking approximation—that is, replace the phase variable φ in Eq. (8) by its stable lock value φ_0 determined by Eqs. (29) and (30). Before doing this, we make a transformation of Eq. (8). Because the average value of $\langle \varepsilon_r(t) \rangle$ is not zero, we write

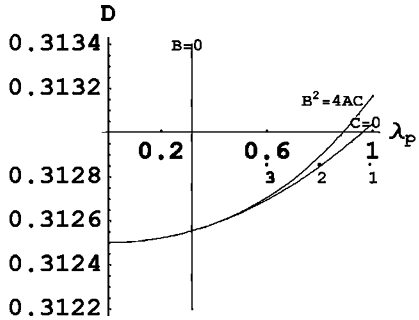

FIG. 1. The phase diagram according to Eq. (43). Below the $C=0$ boundary is the parameter region of a single extremum. In the half plane of $B>0$, the region sandwiched by the $C=0$ and $B^2-4AC=0$ boundaries is where there are two extrema. The left-over region above the $C=0$ boundary is where there is no extremum. The parameters are $a=0.18$, $A=40$, $\tau=0.1$, $Q=0.1$, and $\lambda_q=0.6$.

$$\varepsilon_r(t)=\langle\varepsilon_r(t)\rangle+\widetilde{\varepsilon_r(t)}. \tag{32}$$

Using the techniques developed in Ref. [20] to calculate $\langle\varepsilon_r(t)\rangle$ and the noise correlation, we obtain

$$\langle\varepsilon_r(t)\rangle=\frac{D}{2r}(1-\lambda_q\sin 2\varphi), \tag{33}$$

$$\langle\widetilde{\varepsilon_r(t)}\rangle=0, \tag{34}$$

$$\langle\widetilde{\varepsilon_r(t)}\widetilde{\varepsilon_r(t')}\rangle=D(1+\lambda_q\sin 2\varphi)\delta(t-t'). \tag{35}$$

Now we use a phase locking approximation to replace the phase variable φ in Eqs. (33) and (35) by its stable lock value φ_0. Then Eq. (8) becomes

$$\dot{r}=ar-Ar^3+\frac{D}{2r}(1-|\lambda_q\sin 2\varphi_0|)+\eta_1(t)r+\widetilde{\varepsilon_r(t)}, \tag{36}$$

with

$$\langle\eta_1(t)\rangle=0,\quad\langle\eta_1(t)\eta_1(t')\rangle=2(1-\lambda_p^2)\left(\frac{Q}{\tau}\right)^2e^{-2|t-t'|/\tau} \tag{37}$$

and

$$\langle\widetilde{\varepsilon_r(t)}\widetilde{\varepsilon_r(t')}\rangle=D(1+|\lambda_q\sin 2\varphi_0|)\delta(t-t'). \tag{38}$$

V. APPROXIMATE FOKKER-PLANCK EQUATION AND STEADY-STATE PROBABILITY DISTRIBUTION FOR LASER INTENSITY

We now change the field amplitude Langevin equation (36) to a Langevin equation for the laser intensity I by using the relation $I=r^2$:

$$\dot{I}=2aI-2AI^2+D(1-|\lambda_q\sin 2\varphi_0|)+2I\eta_1(t)+2\sqrt{I}\widetilde{\varepsilon_r(t)}. \tag{39}$$

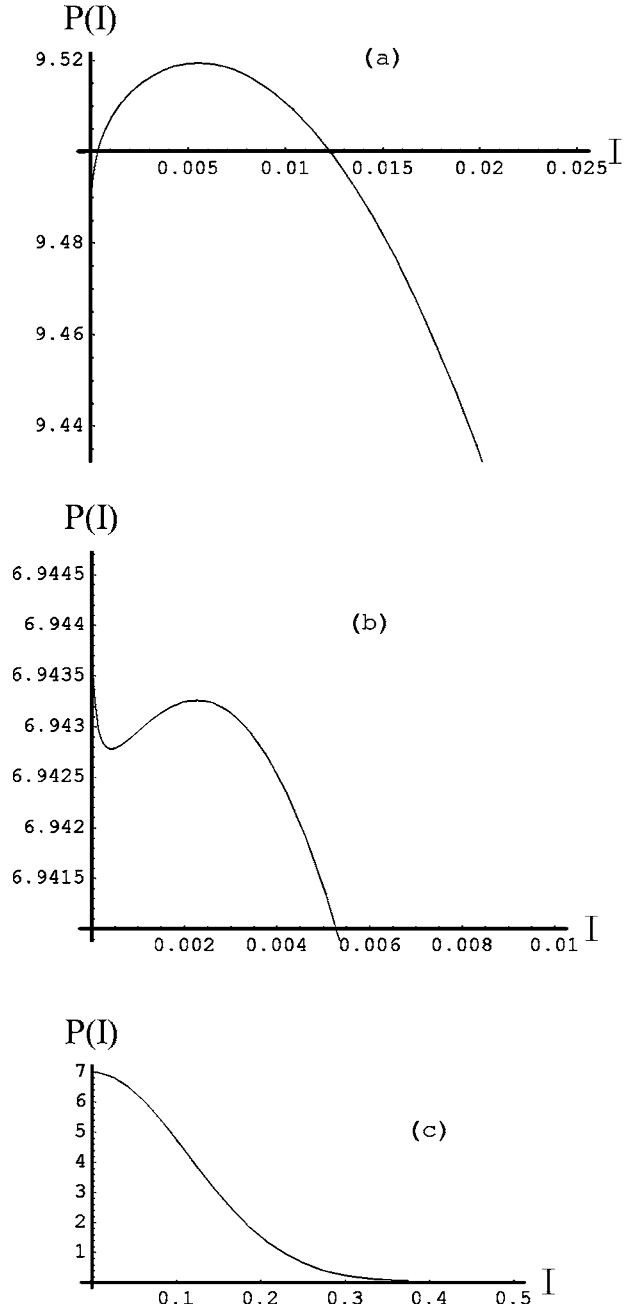

FIG. 2. The $P(I)\sim I$ curves. The curves in (a), (b), and (c) correspond, respectively, to points 1, 2, and 3 in Fig. 1. The values of other parameters are $a=0.18$, $A=40$, $\tau=0.1$, $Q=0.1$, and $\lambda_q=0.6$.

Equation (39) determines a non-Markovian process. We also need to make a Markovian approximation to the non-Markovian process governed by Eq. (39). In accordance with Ref. [19], the AFPE for the non-Markovian process (39) is as follows:

$$\frac{\partial}{\partial t}p(I,t)=-\frac{\partial}{\partial I}D^{(1)}(I)p(I,t)+\frac{\partial^2}{\partial I^2}D^{(2)}(I)p(I,t), \tag{40}$$

in which

$$D^{(1)}(I) = 2aI - 2AI^2 + 2D + 4(1-\lambda_p^2)\frac{Q^2}{\tau}I, \tag{41}$$

$$D^{(2)}(I) = 2D(1 + |\lambda_q \sin 2\varphi_0|)I + 4(1-\lambda_p^2)\frac{Q^2}{\tau}I^2. \tag{42}$$

Under the condition $\sqrt{Q\tau} \ll 1$, the AFPE (40) is a good Markovian approximation of the non-Markovian process (39) [19].

Solving Eqs. (40)–(42) with $(\partial/\partial t)p(I,t)=0$, we get the steady-state probability distribution $p(I)$ [21]:

$$p(I) = \frac{N}{D^{(2)}(I)} \exp\left(\int^I \frac{D^{(1)}(I')}{D^{(2)}(I')}dI'\right) = N\left[2D(1 + |\lambda_q \sin 2\varphi_0|) + 4(1-\lambda_p^2)\frac{Q^2}{\tau}I\right]^{\alpha} I^{\beta} e^{-\gamma I}. \tag{43}$$

In Eq. (43),

$$\alpha = \frac{a}{2(1-\lambda_p^2)\frac{Q^2}{\tau}} + \frac{AD(1 + |\lambda_q \sin 2\varphi_0|)}{\left(2(1-\lambda_p^2)\frac{Q^2}{\tau}\right)^2} - \frac{1}{2D(1 + |\lambda_q \sin 2\varphi_0|)}, \tag{44}$$

$$\beta = \frac{1}{2D(1 + |\lambda_q \sin 2\varphi_0|)} - 1, \tag{45}$$

$$\gamma = \frac{A}{2(1-\lambda_p^2)\frac{Q^2}{\tau}}. \tag{46}$$

N is the normalization constant.

Now we analyze various types of $P(I) \sim I$ curves using our analytical solution (43). Setting $dP(I)/dI=0$, Eq. (43) yields the quadratic equation

$$AI^2 + BI + C = 0, \tag{47}$$

where $A=4(1-\lambda_p^2)Q^2\gamma/\tau$, $B=2D(1+|\lambda_q \sin 2\varphi_0|)\beta-4(1-\lambda_p^2)Q^2(\beta+\alpha)/\tau$, and $C=-2D(1+|\lambda_q \sin 2\varphi_0|)\beta$. On the two-dimensional plane of (λ_p, D), we draw the phase boundaries $C=0$, $B=0$, and $B^2-4AC=0$. The result is the phase diagram in Fig. 1. Below the $C=0$ boundary is the parameter region of a single extremum. In the half plane of $B>0$, the region sandwiched by the $C=0$ and $B^2-4AC=0$ boundaries is where there are two extrema. The leftover region above the $C=0$ boundary is where there is no extremum. The $P(I) \sim I$ curves in Figs. 2(a)–2(c) correspond to points 1, 2, and 3 in Fig. 1, respectively. When $\lambda_p < \lambda_p^0 = \sqrt{1-\tau a/2Q^2}$, the $P(I) \sim I$ curve has zero or one extremum. When $\lambda_p > \lambda_p^0$, the $P(I) \sim I$ curve can have two extrema. From Figs. 1, 2(a), and 2(b), we can draw an interesting conclusion: when the value of λ_p is decreased, the system goes through a first-order phase transition from the single-extremum region to the double-extremum region.

VI. CONCLUSIONS

The important physical conclusions of this paper are summarized as follows.

(i) We derived the phase dynamics equations (20)–(22) for the laser model we proposed in this paper and obtained Eq. (31) satisfied by the phase locking value φ_0.

(ii). Our analysis of Eq. (31) shows that the functions of λ_q and λ_p are as follows: The correlation between the real and imaginary parts of the quantum noise λ_q tends to lead the laser phase to be locked at some values, and the correlation between the real and imaginary parts of the pump noise λ_p tends to destroy or confine the laser phase lock.

(iii) Using the phase locking approximation, we obtained AFPE (40) for the laser-intensity probability distribution and the analytical solution (43) for the stationary-state distribution of the laser intensity.

(iv) Using the extremum condition equation of the laser-intensity distribution (43), we drew the phase diagram on the two-dimensional (λ_p, D) plane in Fig. 1. When λ_p is tuned, we see the first-order phase transition in the shape of the $P(I) \sim I$ curve from single extremum to double extrema.

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