Criterion for Bose-Einstein condensation in a harmonic trap in the case with attractive interactions

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Using a model many-body wave function I analyze the standard criterion for Bose-Einstein condensation and its relation to coherence properties of the system. I pay special attention to an attractive condensate under such a condition that a characteristic length scale of the spatial extension of its center of mass differs significantly from length scales of relative coordinates. I show that although no interference fringes are produced in the two-slit Young interference experiment performed on this system, fringes of a high visibility can be observed in a conditional simultaneous detection of two particles.

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Since the work of Bose and Einstein [1] a picture of the macroscopic occupation of a single-particle state serves as the paradigm of Bose-Einstein condensation. This picture, however, cannot be automatically extended to a real—i.e., interacting—system. The criterion for Bose-Einstein condensation requires a more detailed analysis. To a large extent this goal was completed many years ago by Penrose and Onsager [2,3] and Yang [4] who introduced the concept of off-diagonal long-range order (ODLRO). The existence of a dominant eigenvalue of a one-particle density matrix implies ODLRO. Thus, according to conventional wisdom, a system shows simple Bose-Einstein condensation (BEC) if only one eigenvalue is of the order of 1. In the case when many eigenvalues are large and of comparable size, then the system is a fragmented condensate [5,6].

This conventional definition caused a controversy in the case of the subtle issue of attractive interactions [7,8] between bosonic particles. As shown in the paper of Wilkin *et al.* [7] the lowest excitation of a condensate with attractive forces is associated with the rotation of its center of mass while all relative degrees of freedom remain in their ground state. Analysis of the corresponding one-particle density matrix proves that such a system has many eigenvalues of comparable size; therefore, it is a fragmented Bose-Einstein condensate. Pethick and Pitaevskii [8] pointed out that this conclusion cannot be correct because an atomic cloud being in its ground state but undergoing rigid-body motion as a whole must be a genuine Bose-Einstein condensate. They concluded, therefore, that the generally used criterion for Bose-Einstein condensation can be misleading.

The mathematical criterion of a dominant eigenvalue has well-defined physical meaning—at least in the case of optical lasers. The theory of coherence developed by Glauber [9] allows one to identify these properties which are responsible for the coherence of an electromagnetic field—i.e., for the ability to produce interference fringes: the largest eigenvalue of a two-point correlation function (equivalent to a one-particle density matrix of atomic systems) is equal to a visibility of fringes in the standard two-slit Young experiment. If this eigenvalue is equal to 1, the contrast of fringes equals 100% and the field is coherent at first order—the two-point correlation function factorizes.

In analogy to optical lasers [10] this is evidently the coherence, preferably a higher-order coherence [11–13], which is irrefutable evidence of Bose-Einstein condensation. The question I want to pose here is whether the rotating attractive condensate considered in Ref. [8] can show a large-scale coherence as a genuine condensate should do. If one assumes that the Glauber theory of coherence applies also to atomic condensates, as I shall assume in the following, then the answer is negative: no interference fringes shall be produced in the Young experiment performed on the system because the corresponding one-particle density matrix has no dominant eigenvalue. In turn, following the arguments of Pethick and Pitaevskii [8] the system is a perfect condensate. These two conclusions seem to contradict each other.

Getting ahead, I can say that a solution of this paradox is based on the observation that the properties of a quantum system depend on the measurements performed. Every measurement can be related to an appropriate correlation function [14] depending on the particular method of detection. I am not going to analyze the subtle details of any particular detection scheme but simply assume that the observed signal is proportional to an equal-time sth-order correlation function [15,16], where s is the number of detected particles. I shall consider two different detection techniques: one-particle detection and two-particle conditional measurement. I will show that they lead to different results when applied to a trapped attractive condensate.

To illustrate my point I will give an example which should shed more light on the intriguing issue of a criterion for Bose-Einstein condensation. I consider a model system of N interacting particles in a harmonic trap and described by the following Hamiltonian:

$$H = \frac{1}{2}\sum_{i=1}^{N} (\mathbf{p}_i^2 + \mathbf{x}_i^2) + \frac{(\omega^2 - 1)}{2N}\sum_{i<j} (\mathbf{x}_i - \mathbf{x}_j)^2, \tag{1}$$

where $\mathbf{x}_i$ is the position and $\mathbf{p}_i$ is the momentum of the ith particle. I use oscillatory units—i.e., all frequencies are expressed in units of external trap frequency ω_0—and the unit of length is $\sqrt{\hbar/(m\omega_0)}$, where m is the mass of a particle. The energy of two-body interactions is proportional to the term $(\omega^2 - 1)/N$.

The form of the interaction is nonrealistic due to its infinite range. Nevertheless, for the purposes of the problems discussed in this paper, the model reveals generic features of a BEC trapped in a harmonic potential—this issue will be discussed in more detail later on. I shall use arguments based on the characteristic length scales, and I believe that this line of reasoning puts me on safe ground.

The Hamiltonian (1) can be easily brought to the diagonal form

$$H = \frac{1}{2}\sum_{i=1}^{N-1} (\mathbf{P}_i^2 + \omega^2 \mathbf{q}_i^2) + \frac{1}{2}(\mathbf{P}_N^2 + \mathbf{q}_N^2) \qquad (2)$$

by introducing the collective variables

$$\mathbf{q}_i = \sum_j Q_{ij}\mathbf{x}_j \qquad (3)$$

and corresponding canonical momenta $\mathbf{P}_i = -i\partial/\partial\mathbf{q}_i$. The matrix Q_{ij} is orthogonal and $Q_{Nj} = 1/\sqrt{N}$. These two conditions imply that $\Sigma_j Q_{ij} = 0$ for all $i \neq N$. Only one of the collective coordinates—namely, the center of mass—is uniquely defined:

$$\mathbf{R}_N \equiv \mathbf{q}_N = \frac{1}{\sqrt{N}}\sum_i^N \mathbf{x}_i. \qquad (4)$$

The choice of the remaining coordinates $\{\mathbf{q}_1, \ldots, \mathbf{q}_{N-1}\} = \{\mathbf{q}\}_{N-1}$ is not unique; however, the physically important quantity is the "length" of the relative coordinates r_N in configurational space which do not depend on a particular choice of $\{\mathbf{q}\}_{N-1}$:

$$r_N^2 \equiv \sum_{i=1}^{N-1} \mathbf{q}_i^2 = \sum_{i=1}^{N} \mathbf{x}_i^2 - \mathbf{R}_N^2. \qquad (5)$$

The ground-state wave function of the Hamiltonian depends on the set of variables $\{\mathbf{x}\}_N$ through $\mathbf{R}_N$ and r_N only [17,18]:

$$\Psi_N(\{\mathbf{x}\}_N) = \mathcal{N} \exp\left[\frac{-r_N^2}{2\ell^2}\right]\exp\left[-\frac{\mathbf{R}_N^2}{2}\right], \qquad (6)$$

where $\mathcal{N} = (\pi\ell^2)^{-3(N-1)/4}(\pi)^{-3/4}$ is the normalization constant while $\ell = 1/\sqrt{\omega}$ defines a spreading of relative coordinates. The center-of-mass extension is equal to one because of the choice of units.

The wave function is the product of the center-of-mass ground state of the harmonic trap and the term depending on relative coordinates only. This form reveals general features of all interacting systems in external harmonic potentials. The wave function is symmetric with respect to the variables $\{\mathbf{x}\}_N$. The relative term depends on $3(N-1)$ relative degrees of freedom. All relative eigenmodes are described by the same wave function. Its spatial extension is equal to ℓ. Different relative eigenmodes might be characterized by different length scales in realistic cases. Note that $\ell < 1$ corresponds to attractive while $\ell > 1$ to repulsive forces. In the case of a noninteracting system the center-of-mass and relative length scales are equal, $\ell = 1$, and the N-particle wave function becomes a simple N-fold product of one-particle terms.

Even such a simple wave function shows a variety of different features depending on the length scale ℓ. For future convenience I introduce the interaction strength parameter κ which is defined by the relation

$$\omega = N^{\kappa}. \qquad (7)$$

The main question I want to pose in the following is whether the many-body system being in its ground state, Eq. (6), is Bose condensed. To this end I start with a discussion of possible measurements which can be performed on a many-body system. One can imagine two extreme situations: (a) the detection of one particle (repeated many times in order to collect statistically significant data) or (b) simultaneous detection of all particles. Obviously there are many other possibilities. A measurement in which s out of N particles are simultaneously detected is related to the reduced s-particle density matrix which can be obtained by integrating the N-particle matrix over all but s coordinates:

$$\rho^{(s)}(\{\mathbf{x}\}_s;\{\mathbf{x}'\}_s) = \int d\{\mathbf{y}\}_{N-s}\Psi_N^*(\{\mathbf{x}\}_s,\{\mathbf{y}\}_{N-s}) \times \Psi_N(\{\mathbf{x}'\}_s,\{\mathbf{y}\}_{N-s}), \qquad (8)$$

where $\{\mathbf{x}\}_s$ and $\{\mathbf{y}\}_{N-s}$ denote sets of s and $N-s$ coordinates, respectively. In the case studied this integration can be evaluated explicitly and leads to the following result:

$$\rho^{(s)}(\{\mathbf{x}\}_s;\{\mathbf{x}\}_s') = \rho^{\text{c.m.}}(\mathbf{R}_s,\mathbf{R}_s')\Phi_0^{(s-1)*}(r_s)\Phi_0^{(s-1)}(r_s'), \qquad (9)$$

where $\mathbf{R}_s = (1/\sqrt{s})\Sigma_i^s \mathbf{x}_i$ is the center of mass of an s-particle subsystem and $r_s = \sqrt{\Sigma_i^s \mathbf{x}_i^2 - \mathbf{R}_s^2}$ is the length of the relative coordinate vector defined in $[3(s-1)]$-dimensional space. The wave function $\Phi_0^{(s-1)}(r_s)$ is a Gaussian function depending on $s-1$ independent variables through their length in configurational space only:

$$\Phi_0^{(s-1)}(r_s) = \left(\frac{\omega}{\pi}\right)^{3(s-1)/4} \exp\left[-\frac{r_s^2}{2\ell^2}\right]. \qquad (10)$$

Obviously, if $s=1$, then $\Phi_0^{(s-1)}(r_s) = 1$. The center-of-mass term is

$$\rho^{\text{c.m.}}(\mathbf{R}_s,\mathbf{R}_s') = \left(\frac{\omega_s}{\pi}\right)^{3/2} \times \exp\left[-\tfrac{1}{2}\left(\omega_s + \tfrac{1}{2}\delta_s\right)[\mathbf{R}_s^2 + \mathbf{R}_s'^2] + \tfrac{1}{2}\delta_s \mathbf{R}_s \cdot \mathbf{R}_s'\right], \qquad (11)$$

where the frequencies ω_s and δ_s as well as the parameter γ_s are

$$\gamma_s = 1 - (1-\omega)\frac{s}{N}, \qquad (12)$$

$$\omega_s = \frac{\omega}{\gamma_s}, \qquad (13)$$

$$\delta_s = (1-\omega)^2 \frac{s(N-s)}{N^2 \gamma_s}. \tag{14}$$

The density matrix can be brought to diagonal form (note that it has the form of the temperature density matrix of a harmonic oscillator)

$$\rho^{(s)}(\mathbf{x};\mathbf{x}') = \sum_{\mathbf{n}} \lambda_{\mathbf{n}}^{(s)} \phi_{\mathbf{n}}^*(\mathbf{x}) \phi_{\mathbf{n}}(\mathbf{x}'). \tag{15}$$

Spectral decomposition uniquely determines the set of eigenstates $\phi_{\mathbf{n}}(\mathbf{x})$ and eigenvalues $\lambda_{\mathbf{n}}^{(s)}$ [17,18]. The eigenvectors of the one-particle density matrix are

$$\phi_{\mathbf{n}}(\mathbf{x}) = \Phi_0^{(s-1)}(\mathbf{r}_s) \psi_{\mathbf{n}}(\sqrt{\alpha_s}\mathbf{R}_s), \tag{16}$$

where $\psi_{\mathbf{n}}(\sqrt{\alpha_s}\mathbf{R}_s)$ are the eigenstates of the harmonic oscillator of frequency $\alpha_s = [\omega_s(\omega_s+\delta_s)]^{1/2}$ and $\mathbf{n}=(n_1,n_2,n_3)$ represents a set of three oscillatory quantum numbers. The eigenvalues depend on the prinicpal quantum number $n=n_1+n_2+n_3$ only, $\lambda_{\mathbf{n}}=\lambda_n$. They are equal to the relative populations of one-particle states $\phi_{\mathbf{n}}$ and are of the form

$$\lambda_n^{(s)} = \left(\frac{2\omega_s}{\alpha_s+\omega_s}\right)^3 \left(\frac{\alpha_s-\omega_s}{\alpha_s+\omega_s}\right)^n. \tag{17}$$

I shall discuss the one-particle density matrix first. It is directly related to a repeated single-particle detection. Its largest eigenvalue is

$$\lambda_0^{(1)} = \left(\frac{2}{\sqrt{\varepsilon+1}+1}\right)^3, \tag{18}$$

where $\varepsilon = (N^{\kappa-1}+N^{-\kappa-1}-2N^{-1})$. In the limit of $N \gg 1$, there are two distinct situations.

If $|\kappa|<1$, then $\varepsilon \to 0$ and one-particle subsystem is in a pure state—i.e., $\lambda_0^{(1)} \approx 1$ and $\rho^{(1)}(\mathbf{x};\mathbf{x}') = \phi_0^*(\mathbf{x})\phi_0(\mathbf{x}')$. The two-point correlation function is separable; therefore, the interacting system is indistinguishable (from the point of view of one-particle measurements) from a system of independent particles, all occupying the same state. The wave function of this state—the order parameter—can be evaluated exactly [18]:

$$\phi_0(x) \approx \left(\frac{\alpha_1}{\pi}\right)^{3/4} \exp[-\alpha_1 \mathbf{x}^2/2]. \tag{19}$$

In the limit of large-N frequencies α_1 and ω become equal, $\alpha_1 \approx \omega$. Separability, in the language of coherence theory, signifies the first-order coherence of a system—interference fringes with 100% visibility shall be observed in a two-slit Young interference experiment.

If $|\kappa|>1$—i.e., interactions are strong—then $\varepsilon \to \infty$ and all eigenvalues are negligible (but $\Sigma\lambda_{\mathbf{n}}^{(1)}=1$). The one-particle subsystem is in a mixed state which resembles a temperature state. No long-distance coherence shall be observed at all. The two-point correlation function is not separable, and the particle density has the form

$$\rho^{(1)}(\mathbf{x};\mathbf{x}) = \left(\frac{\omega_1}{\pi}\right)^{3/2} e^{-\omega_1 \mathbf{x}^2/2}. \tag{20}$$

These conclusions are based on the conventional criterion for Bose-Einstein condensation. However, the authors of Ref. [8] introduced a modified criterion. They suggested that the center-of-mass length scale has to be eliminated prior to the determination of the one-particle correlation function. To this end one can find a *relative* density matrix $\rho_{rel}^{(N)}$ by integrating the N-particle density matrix, $\rho^N(\{\mathbf{x}_N\};\{\mathbf{y}_N\}) = \Psi_N^*(\{\mathbf{x}_N\})\Psi_N(\{\mathbf{y}_N\})$, over the center-of-mass variable. Further integration over all but one internal variable leads in our case to a fully separable (in the whole range of the interaction strength) two-point correlation function. Therefore, according to Ref. [8] the system described by Eq. (6) is a perfect Bose-Einstein condensate. The same system, however, will not disclose the large-scale coherence in the two-slit Young interference experiment. It is not clear, therefore, what this particular feature, which can be related to the physical properties conventionally attributed to a Bose-Einstein condensate, is. The reason for this ambiguity is that the relative density matrix introduced in Ref. [8] is useful only in the case when one is interested in observables which are independent of the center of mass, such as the relative distance between two particles, for example. The relative density matrix cannot be used for studying the global properties of a system such as the coherence. This criterion is not related to any feature which is characteristic for a Bose-Einstein condensate.

The key issue in solving the problem of the criterion for Bose-Einstein condensation is to point to reasons of the disappearance of the dominant eigenvalue. In our case, the disappearance of the first-order coherence originates in the existence of two different length scales characterizing the system. The amplitude of zero-point oscillations of the center of mass, $\ell_{\text{c.m.}}=1$, does not depend on the number of particles and the interaction strength, but the spatial extension ℓ of the relative coordinates does. The amplitude of the relative oscillations becomes much larger than the center-of-mass spreading, $\ell \gg \ell_{\text{c.m.}}$, if repulsive forces are strong, $\kappa<-1$. On the contrary, in the case of strong attractive forces, $\kappa>1$, particles are aggregated over relatively small distances while the center-of-mass position is a statistical variable having huge quantum uncertainty, $\ell \ll \ell_{\text{c.m.}}$. The coherence length of an N-particle system is reduced to the smaller of the two scales. Such a situation is basically identical to the one studied in Refs. [7,8].

It is reasonable to expect that the coherence length of a genuine Bose-Einstein condensate is of the order of its size. The size, however, might depend on the detection scheme. The spatial extension observed in repeated single-particle measurements is determined by extension of the one-particle density matrix—i.e., the largest length scale of the problem. On the other hand, if all particles are detected simultaneously, then the observed size of the cloud is determined by extension of the relative coordinates. There is no information about the uncertainty of the center-of-mass position in a single realization of the system.

The last observation provides a hint on how to solve the issue of the criterion for Bose-Einstein condensation. Let us consider a conditional detection: the detection of one particle provided that simultaneously another one—a reference particle—is detected at a given position $\mathbf{x}_0$. The corresponding *conditional* one-particle reduced density matrix reads

$$\rho_{cond}^{(1|1)}(\mathbf{x};\mathbf{x}'|\mathbf{x}_0)=\frac{\rho^{(2)}(\mathbf{x},\mathbf{x}_0;\mathbf{x}',\mathbf{x}_0)}{\rho^{(1)}(\mathbf{x}_0;\mathbf{x}_0)}, \qquad (21)$$

where $\rho^{(2)}$ is a two-particle reduced density matrix. In an example of the model studied here, the matrix $\rho_{cond}^{(1|1)}(\mathbf{x};\mathbf{x}')$ has the form

$$\rho_{cond}^{(1|1)}(\mathbf{x};\mathbf{x}'|\mathbf{x}_0)=\left(\frac{\Omega_1}{\pi}\right)^{3/2}\exp\Big(-\tfrac{1}{2}\big(\Omega_1+\tfrac{1}{2}\Delta_1\big)\times[(\mathbf{x}-\mathbf{y}_1)^2+(\mathbf{x}'-\mathbf{y}_1)^2]+\tfrac{1}{2}\Delta_1(\mathbf{x}-\mathbf{y}_1)\cdot(\mathbf{x}'-\mathbf{y}_1)\Big), \qquad (22)$$

where

$$\Omega_1=\frac{\omega+\omega_2}{2}, \qquad (23)$$

$$\Delta_1=\frac{\delta_2}{2}, \qquad (24)$$

$$\mathbf{y}_1=\frac{\omega-\omega_2}{\omega+\omega_2}\mathbf{x}_0. \qquad (25)$$

The conditional single-particle density is equal to the diagonal elements of $\rho_{cond}^{(1|1)}$:

$$\rho_{cond}^{(1|1)}(\mathbf{x};\mathbf{x}|\mathbf{x}_0)=\left(\frac{\Omega_1}{\pi}\right)^{3/2}e^{-\Omega_1(\mathbf{x}-\mathbf{y}_1)^2}. \qquad (26)$$

It is a Gaussian function centered around $\mathbf{y}_1$. Its width is equal to $\ell_1^{cond}=1/\sqrt{\Omega_1}$. This result should be compared with the standard single-particle density obtained in repeated one-particle measurements, Eq. (20). It is easy to check that in the case of weak interactions, $|\kappa|<1$, as well as in the case of very strong repulsion, $\kappa<-1$, the parameters of Eq. (26) are $\Omega_1\approx\omega_1$ and $\mathbf{y}_1\approx 0$. Therefore the conditional single-particle density matrix is equal to the standard one,

$$\rho_{cond}^{(1|1)}(\mathbf{x};\mathbf{x}|\mathbf{x}_0)=\left(\frac{\omega_1}{\pi}\right)^{3/2}e^{-\omega_1\mathbf{x}^2/2}, \qquad (27)$$

and obviously the conditional and direct measurements are equivalent.

In the case of a very strong attraction ($\kappa\gg 1$—i.e., $\omega=N^{\kappa}\gg N$) the parameters of Eq. (22) are

$$\Omega_1\approx\frac{\omega}{2}, \qquad (28)$$

$$\Delta_1\approx\frac{\omega}{2}, \qquad (29)$$

$$\mathbf{y}_1\approx\mathbf{x}_0, \qquad (30)$$

and Eq. (26) simplifies to

$$\rho_{cond}^{(1|1)}(\mathbf{x};\mathbf{x}|\mathbf{x}_0)=\left(\frac{\omega}{2\pi}\right)^{3/2}e^{-\omega(\mathbf{x}-\mathbf{x}_0)^2/2}. \qquad (31)$$

Conditional and direct detection schemes lead to different results. The spatial extension of the single-particle density given by Eq. (9), $\ell_1=1/\sqrt{\omega_1}$, is much larger than the spatial extension of the single-particle density obtained in the conditional detection, $\ell_1^{cond}=1/\sqrt{\Omega_1}$: i.e.,

$$\frac{\ell_1^{cond}}{\ell_1}=(2N^{\kappa-1})^{1/2}\ll 1. \qquad (32)$$

Moreover, the single-particle density is centered at the minimum of the external trap, $\mathbf{x}=0$, while the conditional density is concentrated around the position of the preselected reference point, $\mathbf{x}=\mathbf{x}_0$. In order to determine experimentally the conditional single-particle density one should perform a series of joint two-particle detections and collect only the data where one particle is observed at the preselected position $\mathbf{x}_0$. The conditional density is a function of the position of the second particle.

The conditional density matrix can be diagonalized in the case of the model studied here. It has the same structure as the one studied above, Eq. (11). If interactions are weak, $|\kappa|<1$, all particles are independent and obviously conditional and ordinary detections are equivalent—the one-particle subsystem is in a pure state. In the case of strong attractive interactions, $\kappa>1$, the conditional density matrix has a single dominant eigenvalue, which can be explicitly calculated:

$$\lambda_0^{(1|1)}=\left(\frac{2}{\sqrt{1+\Delta_1/\Omega_1}+1}\right)^3. \qquad (33)$$

In the large-N limit this eigenvalue is simply

$$\lambda_0^{(1|1)}\approx\left(\frac{2}{\sqrt{2}+1}\right)^3. \qquad (34)$$

Its value is large, but smaller than 1, because the uncertainty of the center-of-mass position can be reduced, but not totally eliminated, in the two-particle measurement.

The explanation of the existence of the dominant eigenvalue is rather simple. Particles in the strongly attractive system are correlated. They are located close to each other; therefore, detection of one of them gives a great deal of information about the center-of-mass position—it ought to be not too far away. In fact, the position of the center of mass is a stochastic variable distributed according to $\rho^{CM}(\mathbf{R}_N,\mathbf{R}_N)$ which is widely spread over a large distance. Detection of the first particle reduces the center-of-mass uncertainty to a small distance of the order of relative coordinate spreading. Simultaneous detection of a second particle is a practical realization of the measurement performed in the center-of-mass frame. Such a conditional measurement automatically eliminates the center-of-mass length scale. The size of the system is therefore reduced to the spatial extension of the relative degrees of freedom which is equal to the coherence length of the cloud. The conditional density matrix has only one large eigenvalue.

In the opposite case of strong repulsive interactions, $\kappa\ll-1$, the largest eigenvalue of the conditional density matrix $\rho_{cond}^{(1|1)}(\mathbf{x};\mathbf{x}')$ is negligible. The system does not show any coherence in the conditional detection, which proves that the system is not in a Bose-condensed phase. Conditional detection does not give any new information on the system as compared to the simple one-particle measurement.

The scheme presented above of conditional measurements can be easily generalized. One can use more reference particles to determine the center-of-mass position in a single realization of the system. This can be done by a joint detection of $s+1$ particles—the obtained data are collected only in such a situation when s particles are found at preselected positions $\{\mathbf{x}_1,\ldots,\mathbf{x}_s\}=\{\mathbf{x}\}_s$. The corresponding conditional one-particle density matrix is

$$\rho^{(1|s)}_{cond}(\mathbf{x};\mathbf{x}'|\{\mathbf{x}\}_s)=\frac{\rho^{(s+1)}(\mathbf{x},\{\mathbf{x}\}_s;\mathbf{x}',\{\mathbf{x}\}_s)}{\rho^{(s)}(\{\mathbf{x}\}_s;\{\mathbf{x}\}_s)}. \tag{35}$$

In the model studied, the conditional one-particle matrix corresponding to joint detection of $s+1$ paricles, Eq. (35), has the same form as $\rho^{(1|1)}_{cond}(\mathbf{x};\mathbf{x}'|\mathbf{x}_0)$, Eq. (22) but Ω_1, Δ_1, and $\mathbf{y}_1$ have to be substituted by

$$\Omega_s=\frac{s\omega+\omega_{s+1}}{s+1}, \tag{36}$$

$$\Delta_s=\frac{\delta_{s+1}}{s+1}, \tag{37}$$

$$\mathbf{y}_s=\frac{(\omega-\omega_{s+1})}{s\omega+\omega_{s+1}}\sum_{i=1}^{s}\mathbf{x}_i. \tag{38}$$

Similarly, the dominant eigenvalue of the $\rho^{(1|s)}_{cond}(\mathbf{x};\mathbf{x}'|\{\mathbf{x}\}_s)$ has the form of Eq. (33) and approaches the value of 1 if $s \gg 1$:

$$\lambda_0^{(1|s)}\approx\left(\frac{2}{\sqrt{1+1/s}+1}\right)^3. \tag{39}$$

This result can be easily understood: the more particles used to determine the center of mass of the N-particle system, the stronger the reduction of the center-of-mass uncertainty and the larger the first-order coherence of the system observed in that kind of measurement.

It is worth adding that the model I used to illustrate the main thesis is certainly artificial. However, this is done solely for the sake of being able to do analytical calculations, starting from a free Hamiltonian. However, the results obtained here are quite general. They do not depend on the particular form of interactions because the effect studied is not of dynamical origin. It is related to simple kinematics only. It relies on the fact that in the harmonic trap the center of mass decouples from the internal dynamics; therefore, the amplitude of its oscillations is completely independent of the size of the system as observed in a single shot of a camera. To disclose the coherence of such a system one should perform detection in the center-of-mass frame.

In conclusion I want to stress that the criterion for Bose-Einstein condensation requires particular attention in the case of attractive systems whose center-of-mass extension significantly exceeds other relevant length scales. In such a case, as suggested in Ref. [8], the criterion of the dominant eigenvalue of the one-particle density matrix might lead to false conclusions. In order to disclose the coherence of the system one should perform a measurement in its center-of-mass frame. This is, however, practically impossible if the center of mass is a quantum variable with a large uncertainty. One can overcome this problem by performing a conditional detection—in the case of attractive systems, detection of one particle quite precisely determines the position of the center of mass. Therefore, in the conditional measurement one might prove a long-range coherence of the system—coherence of the extension of the relative degrees of freedom.

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