

Number-of-particle fluctuations and stability of Bose-Einstein-condensed systems

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In this paper we show that a normal total number-of-particle fluctuation can be obtained consistently from the static thermodynamic relation and dynamic compressibility sum rule. In models using the broken U(1) gauge symmetry, in order to keep the consistency between statics and dynamics, it is important to identify the equilibrium state of the system with which the density response function is calculated, so that the condensate particle number N_0 , the number of thermal depletion particles \tilde{N} , and the number of noncondensate particles N_{NC} can be unambiguously defined. We also show that the chemical potential determined from the Hugenholtz-Pines theorem should be consistent with that determined from the equilibrium equation of state. The $N^{4/3}$ anomalous fluctuation of the number of noncondensate particles is an intrinsic feature of the broken U(1) gauge symmetry. However, this anomalous fluctuation does not imply the instability of the system. Using the random phase approximation, which preserves the U(1) gauge symmetry, such an anomalous fluctuation of the number of noncondensate particles is completely absent.

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I. INTRODUCTION

The number-of-particle fluctuation $\langle \delta N^2 \rangle$ in an equilibrium system is a fundamental statistic problem since its scaling with the number of particles $\langle \delta N^2 \rangle \propto N^\gamma$ relates to the stability of the system. The fluctuation is called normal if $\gamma=1$ and anomalous if $\gamma>1$. In the latter case, it implies the system is unstable, since the isothermal compressibility $\kappa_T \rightarrow \infty$ in the thermodynamic limit [see Eq. (1) below]. For example, for a noninteracting uniform Bose system below the critical temperature, the fluctuation of the number of condensate particles $\langle \delta \hat{N}_0^2 \rangle \propto N^2$, and that of the number of noncondensate particles $\langle \delta \hat{N}_{\text{NC}}^2 \rangle \propto N^{4/3}$ in the grand canonical ensemble, while $\langle \delta \hat{N}_0^2 \rangle = \langle \delta \hat{N}_{\text{NC}}^2 \rangle \propto N^{4/3}$ in the canonical ensemble [1]; all are anomalous since $\gamma>1$. However, for a trapped ideal Bose gas, the fluctuation of the number particles is normal, since the confinement effectively suppresses the thermal fluctuation [2].

Recently the number-of-particle fluctuation of interacting Bose-Einstein-condensed systems has attracted much theoretical attention, but whether or not it is anomalous still has not been resolved, since different methods predict different values of γ [3–12]. Particularly, even for the Bogoliubov approximation, both $\gamma=4/3$ [3,5,7,8,11] and $\gamma=1$ [9,12] scaling laws have been obtained.

In order to see how these controversies arise, it is useful to examine the bases of these model calculations. References [4,6] use an energy functional of the total number of particles N , the number of thermal excited particles N_{ex} , and the single-particle spectrum $\varepsilon_{\vec{k}}$. Using this energy functional, the fluctuations of the number of condensate and noncondensate particles can be calculated using the partition function in either the grand canonical ensemble, canonical ensemble, or microcanonical ensemble, and a $\gamma=1$ scaling law was obtained for both the condensate and noncondensate number-of-particles fluctuations. One important observation, which is essential to obtain the $\gamma=1$ scaling law in this approach, is

that phonon excitations have been excluded from the single-particle spectrum. The $\gamma=1$ scaling law for the condensate fluctuation is also obtained in Ref. [10], in which a single-condensate-mode Hamiltonian is used. In Refs. [3,7,8,11], the spectrum obtained by the Bogoliubov approximation was used and a $\gamma=4/3$ scaling law was obtained for the noncondensate number-of-particle fluctuation. However, using the compressibility sum rule [see Eq. (2)], a $\gamma=1$ scaling law was obtained in Refs. [9,12] for the total number-of-particle fluctuation in the same Bogoliubov approximation. From this brief literature survey, it is understood that the number-of-particle fluctuation in an interacting Bose-Einstein-condensate system is highly model dependent, since the definition of the condensate fraction, the number of noncondensate particles, and the energy spectrum are highly model dependent. But the contradictory results in Refs. [3,7,8,11] with that in Refs. [9,12] within the same Bogoliubov approximation deserve further investigation.

As can be shown that in the grand canonical ensemble, the total number-of-particle fluctuations of any equilibrium system is related to the isothermal compressibility through the static thermodynamic relation

$$\frac{\langle \delta \hat{N}^2 \rangle}{N} = \frac{\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2}{N} = \frac{k_B T}{N} \left. \frac{\partial N}{\partial \mu} \right|_{T, \Omega} = \rho k_B T \kappa_T, \quad (1)$$

where \hat{N} is the particle number operator with expectation value N , $k_B T$ is the temperature, μ is chemical potential, Ω is the volume of the system, and $\rho = N/\Omega$ is the number density. On the other hand, the total number-of-particle fluctuation can be also determined by the following dynamic compressibility sum rule:

$$\frac{\langle \delta \hat{N}^2 \rangle}{N} = - \frac{k_B T}{\rho} \lim_{\vec{q} \rightarrow 0} \chi_{nn}(\vec{q}; \omega = 0), \quad (2)$$

where $\chi_{nn}(\vec{q}; \omega)$ is the density response function. The number-of-particle fluctuations calculated from these two re-

lations must be consistent in any approximation. However, we have seen that the static result $\gamma=4/3$ obtained in Refs. [3,7,8,10] is not consistent with the dynamic result $\gamma=1$ obtained in Refs. [9,12]. This leads to a contradictory conclusion about the stability of the system since, as argued by Yukalov [12], an anomalous fluctuation of the number of noncondensate particles would inevitably lead the system to be unstable when grand canonical ensemble is used, while an interacting Bose-Einstein-condensed system *is* stable.

One might conclude that the stability problem only arises when grand canonical ensemble is used since in both microcanonical and canonical ensembles, the number-of-particle fluctuation is identically zero. However, stability is an intrinsic property of the system itself, independent of the statistic ensemble. For example, for an ideal homogenous Bose-condensate system, it can be shown that the isothermal compressibility κ_T is scaled as $V^{1/3}$ in microcanonical and canonical ensembles and as V in grand canonical ensemble, which implies that the system is unstable in all ensembles. For a trapped ideal Bose gas, as we have mentioned, the confinement potential stabilizes the system. However, there is no link between the compressibility and number-of-particle fluctuations in canonical and microcanonical ensembles, and therefore, those fluctuations cannot tell the stability of the system.

We have seen that the above inconsistency of the statics with the dynamics in the Bogoliubov approximation is related to the separation of the condensate and noncondensate components when the broken Bose U(1) gauge symmetry is used. In this case, the Bose field operator $\hat{\psi}$ is split as

$$\hat{\psi}(\vec{r}) = \Psi(\vec{r}) + \delta\hat{\psi}(\vec{r}), \quad (3)$$

where $\Psi(\vec{r}) \neq 0$ is the Bogoliubov order parameter and $\delta\hat{\psi}(\vec{r})$, usually called the noncondensate field operator, represents both the dynamic excitation and thermal depletion out of the condensate. This subtlety in $\delta\hat{\psi}$ shows that the condensate and noncondensate components are strongly correlated, and the condensate component cannot be treated just as a static reservoir. Instead, the dynamics aspect of Ψ must be taken into account in calculation of the number of condensate and noncondensate particles for the purpose of calculating the number-of-particle fluctuation from Eq. (1). To resolve the inconsistency of statics with dynamics and to treat Ψ as a dynamic quantity, it is crucial to identify the equilibrium state with which the density response function χ_{nn} is determined. Using this equilibrium state as a reference, the number of condensation particles N_0 , the number of noncondensate particles N_{NC} , and the number of thermally depleting particle \tilde{N} can be unambiguously defined.

We will also show that the anomalous fluctuation of the number of noncondensate particles is an intrinsic feature of the broken U(1) gauge symmetry. It is well known that a direct consequence of the above broken U(1) gauge symmetry is that the poles of the single-particle Green function defined with $\delta\hat{\psi}$ and the total density response function are identical. Therefore, the fluctuation of the number of noncondensate particles inevitably follows the $N^{4/3}$ anomalous law,

since the momentum distribution of this noncondensate particle always has a $1/k^2$ singularity in the long-wavelength limit, regardless of at what level the interacting Bose Hamiltonian is truncated. However, we shall show that this anomalous fluctuation does not imply the instability of the interacting Bose system. This is because, as we shall show, the condensate and noncondensate components cannot be treated as linearly independent terms so that an anomalous fluctuation of the number of noncondensate particles does not necessarily indicate that the system is unstable.

One way to avoid such an anomalous fluctuation of the number of noncondensate particles is to work with an ensemble in which the gauge symmetry is *not* broken [13], so that we can avoid the entangling of particle-conserving collective excitations and single-particle excited state in the pole of the noncondensate single-particle Green function. Indeed, using the random phase approximation with inclusion of exchange (RPAE) developed by Minguzzi and Tosi [14], which keeps the U(1) gauge symmetry, we are able to show that while the total number-of-particle fluctuation is normal and consistently determined from statics and dynamics, the anomalous fluctuation of the number of noncondensate particles is completely absent.

This paper is organized as follows. In Sec. II, we briefly summarize the rules to build the noncondensate single-particle Green function and the density response function with the broken U(1) gauge symmetry. In Sec. III, we examine how the consistency between Eqs. (1) and (2) can be obtained in calculating the number-of-particle fluctuation, and interpret the physical meaning of the anomalous fluctuation of the number of noncondensate particles in the Bogoliubov approximation. In Sec. IV, we carry out a calculation in the random phase approximation with inclusion exchange and in the dielectric formalism to support our interpretations presented in Sec. III. The discussions and conclusion are presented in the last section.

II. SINGLE-PARTICLE GREEN FUNCTION AND DENSITY RESPONSE FUNCTION WITH BROKEN U(1) GAUGE SYMMETRY

In this section, we briefly summarize the rules to construct the single-particle Green function and density response function with broken U(1) gauge symmetry. The details can be found in Refs. [15–17].

We start from the Hamiltonian for a homogenous interacting Bose system in the second-quantized form

$$\hat{H} = \sum_{\vec{k}} \mathcal{E}_{\vec{k}} a_{\vec{k}}^\dagger a_{\vec{k}} + \frac{g}{2\Omega} \sum_{\vec{q}, \vec{k}_1, \vec{k}_2} a_{\vec{k}_1+\vec{q}}^\dagger a_{\vec{k}_2-\vec{q}}^\dagger a_{\vec{k}_2} a_{\vec{k}_1}, \quad (4)$$

where a contact two-body potential with strength g is used, $\mathcal{E}_{\vec{k}} = \hbar^2 \vec{k}^2 / 2m - \mu = \varepsilon_{\vec{k}} - \mu$ is the noninteracting single-particle energy with respect to the chemical potential, and $a_{\vec{k}}^\dagger$ and $a_{\vec{k}}$ are creation and annihilation operators for the interacting Bose particles.

Now using Eq. (3) in its momentum space form, i.e., replacing a_0^\dagger and a_0 with $\sqrt{N_0}$, where N_0 is the number of particles condensed onto the ground state $\vec{k}=0$, one obtains the approximated Hamiltonian [17]

$$\begin{aligned} \hat{H} \approx & \frac{gN_0^2}{2\Omega} - \mu N_0 + \sum_{\vec{k} \neq 0} \mathcal{E}_{\vec{k}} a_{\vec{k}}^\dagger a_{\vec{k}} + \frac{g\rho_0}{2} \sum_{\vec{q} \neq 0} \hat{A}_{\vec{q}} \hat{A}_{-\vec{q}} \\ & + \frac{g\sqrt{N_0}}{2\Omega} (\hat{A}_{\vec{q}} \hat{\rho}_{-\vec{q}} + \hat{\rho}_{\vec{q}} \hat{A}_{-\vec{q}}) + \frac{g}{2\Omega} \sum_{\vec{q} \neq 0} \hat{\rho}_{\vec{q}} \hat{\rho}_{-\vec{q}}, \end{aligned} \quad (5)$$

where $\rho = N_0/\Omega$ is the condensate density, $\hat{A}_{\vec{q}} = a_{\vec{q}}^\dagger + a_{-\vec{q}}$ and

$$\hat{\rho}_{\vec{q}} = \sum_{\vec{k} \neq 0, -\vec{q}} a_{\vec{k}}^\dagger a_{\vec{k}+\vec{q}} \quad (6)$$

is the density operator for the noncondensate particles. The total density operator is

$$\hat{\rho}_{\vec{q}} = \sqrt{N_0} \hat{A}_{\vec{q}} + \hat{\rho}_{\vec{q}}. \quad (7)$$

Equation (5) provides the starting point for many approximations in which the broken U(1) gauge symmetry is used.

The single-particle Green function matrix defined with $\delta\hat{\psi}$ is [15,17]

$$G_{\alpha\beta}(\vec{q} \neq 0, \tau) = -\langle T_\tau a_{\vec{q}\alpha}(\tau) a_{\vec{q}\beta}^\dagger \rangle, \quad (8)$$

where

$$a_{\vec{q}\alpha} = \begin{cases} a_{\vec{q}} & \alpha = +, \\ a_{-\vec{q}}^\dagger & \alpha = -. \end{cases} \quad (9)$$

Solving the Dyson equation involving a 2×2 matrix self-energy $\Sigma_{\alpha\beta}$, the single-particle Green function $G_{\alpha\beta}$ has the general form [15,16]

$$G_{\alpha\beta}(k) = \frac{(\alpha i\omega_n + \mathcal{E}_{\vec{k}}) \delta_{\alpha\beta} + \alpha\beta \Sigma_{-\alpha-\beta}(k)}{D(k)}, \quad (10)$$

where

$$\begin{aligned} D(k) = & [i\omega_n - \mathcal{E}_{\vec{k}} - \Sigma_{++}(k)][i\omega_n + \mathcal{E}_{\vec{k}} + \Sigma_{--}(k)] \\ & + \Sigma_{+-}(k) \Sigma_{-+}(k). \end{aligned} \quad (11)$$

Here notation $k = (\vec{k}; i\omega_n)$ is used. Various truncations of the Hamiltonian (5) correspond to select certain types of self-energy diagrams in such a way that the Hugenholtz-Pines theorem [18]

$$\mu = \Sigma_{++}(0) - \Sigma_{--}(0) \quad (12)$$

is satisfied, so that the pole determined by $D(k)=0$ is gapless in the long-wavelength limit.

The density response function defined as

$$\chi_{nm}(\vec{q}; \tau) = -\langle T_\tau \hat{\rho}_{\vec{q}}(\tau) \hat{\rho}_{-\vec{q}}(0) \rangle \quad (13)$$

can be written as [17]

$$\chi_{nm}(q) = \sum_{\alpha\beta} \Lambda_\alpha(q) G_{\alpha\beta}(q) \Lambda_\beta(q) + \chi_{nm}^R(q), \quad (14)$$

where Λ_α is the vertex function describing process of (de) excitations (in) out of the condensate and χ_{nm}^R is the regular part response function. It is not obvious in this approach to identify the equilibrium state with which the density response function χ_{nm} is determined. We shall show in

the next two sections that the identification of such an equilibrium state is important to unambiguously define the equilibrium condensate particle number N_0 , thermal depletion particle number \tilde{N} , and the corresponding thermal depletion single-particle Green function, which are used to build Λ_α , $\Sigma_{\alpha\beta}$, $G_{\alpha\beta}$, and χ^R , and to calculate the noncondensate particle number N_{NC} , so that the consistency between statics and dynamics in calculating the number-of-particle fluctuation can be obtained.

III. NUMBER-OF-PARTICLE FLUCTUATION IN BOGOLIUBOV APPROXIMATION

We now reexamine the number-of-particle fluctuation problem in the Bogoliubov approximation at finite temperature. A comment is deserved: even though we work at finite temperature, there are no thermal depletion particles.

The vertex function is $\Lambda_\alpha = \sqrt{\rho_0}$, and the self-energies are

$$\Sigma_{++}(\vec{q}; i\omega_n) = \Sigma_{--}(\vec{q}; i\omega_n) = 2g\rho_0, \quad (15)$$

$$\Sigma_{+-}(\vec{q}; i\omega_n) = \Sigma_{-+}(\vec{q}; i\omega_n) = g\rho_0, \quad (16)$$

and the chemical potential determined by Hugenholtz-Pines theorem is

$$\mu = \Sigma_{++}(0; 0) - \Sigma_{--}(0; 0) = g\rho_0. \quad (17)$$

Substituting the above $\Sigma_{\alpha\beta}$, Λ_α , and μ into Eq. (10), one gets the corresponding single-particle Green functions for the noncondensate particles [16,17]

$$G_{++}^{BA}(\vec{k}; i\omega_n) = \frac{u_{\vec{k}}^2}{i\omega_n - \omega_{\vec{k}}} - \frac{v_{\vec{k}}^2}{i\omega_n + \omega_{\vec{k}}},$$

$$G_{+-}^{BA}(\vec{k}; i\omega_n) = -u_{\vec{k}} v_{\vec{k}} \left(\frac{1}{i\omega_n - \omega_{\vec{k}}} - \frac{1}{i\omega_n + \omega_{\vec{k}}} \right), \quad (18)$$

where

$$\omega_{\vec{k}}^2 = [\varepsilon_{\vec{k}} - \Delta][\varepsilon_{\vec{k}} + 2g\rho_0 - \Delta] \quad (19)$$

and

$$u_{\vec{k}}^2 = \frac{\varepsilon_{\vec{k}} - \Delta + g\rho_0 + \omega_{\vec{k}}}{2\omega_{\vec{k}}}, \quad (20)$$

$$v_{\vec{k}}^2 = \frac{\varepsilon_{\vec{k}} - \Delta + g\rho_0 - \omega_{\vec{k}}}{2\omega_{\vec{k}}}. \quad (21)$$

Here $\Delta = \mu - g\rho_0$ has been introduced for future convenience, and $\Delta=0$ for temperatures below T_c .

Substituting the vertex functions and the Green function into Eq. (14), one gets the density response function $\chi_{nm}^{BA}(\vec{q}, i\omega_n)$ for the interacting Bose gas

$$\chi_{nm}^{BA}(\vec{q}, i\omega_n) = \frac{\rho_0 \varepsilon_{\vec{q}}}{\omega_{\vec{q}}} \left(\frac{1}{i\omega_n - \omega_{\vec{q}}} - \frac{1}{i\omega_n + \omega_{\vec{q}}} \right). \quad (22)$$

We first calculate the number-of-particle fluctuation from dynamics. Substituting Eq. (22) into Eq. (2), one gets

$$\frac{\langle \delta \hat{N}^2 \rangle^{BA}}{N} = \frac{k_B T}{m c_B^2}, \quad (23)$$

where $c_B = \sqrt{g \rho_0 / m}$ is the Bogoliubov phonon velocity. Therefore, we get a normal number-of-particle fluctuation from the dynamics.

Now let us use Eq. (1) to calculate the number-of-particle fluctuation. The total number of particles is given as

$$N = N_0 + N_{\text{NC}}, \quad (24)$$

where N_{NC} is calculated as

$$N_{\text{NC}} = -\frac{1}{\beta} \sum_{n, \vec{k}} G_{++}^{BA}(\vec{k}; i\omega_n) = \sum_{\vec{k} \neq 0} [(u_{\vec{k}}^2 + v_{\vec{k}}^2) n_{\vec{k}} + v_{\vec{k}}^2], \quad (25)$$

where $n_{\vec{k}} = (e^{\beta \omega_{\vec{k}}} - 1)^{-1}$. Direct calculation shows that

$$\begin{aligned} \left. \frac{k_B T}{N} \frac{\partial N_{\text{NC}}}{\partial \mu} \right|_{\mu=g\rho_0} &= \left. \frac{k_B T}{N} \frac{\partial N_{\text{NC}}}{\partial \Delta} \right|_{\Delta=0} \\ &= \frac{1}{N} \sum_{\vec{k} \neq 0} \left[(u_{\vec{k}}^2 + v_{\vec{k}}^2)^2 n_{\vec{k}} (n_{\vec{k}} + 1) \right. \\ &\quad \left. + 4u_{\vec{k}}^2 v_{\vec{k}}^2 \frac{k_B T}{\omega_{\vec{k}}} \left(n_{\vec{k}} + \frac{1}{2} \right) \right]. \quad (26) \end{aligned}$$

The anomalous $N^{4/3}$ behavior of this equation can be seen, since in the long-wavelength limit $u_{\vec{k}}^2 \sim v_{\vec{k}}^2 \sim 1/k$ and $n_{\vec{k}} \sim 1/k$, so the integrand has a $1/k^4$ singularity.

On the other hand, using the chemical potential $\mu = g\rho_0$, one gets

$$\left. \frac{k_B T}{N} \frac{\partial N_0}{\partial \mu} \right|_{\mu=g\rho_0} = \left. \frac{k_B T}{N} \frac{\partial N_0}{\partial \Delta} \right|_{\Delta=0} = \frac{k_B T}{m c_B^2}. \quad (27)$$

The number-of-particle fluctuation is the sum of Eqs. (26) and (27), which is clearly not consistent with Eq. (23). This is the inconsistency of the result in Refs. [3,7,8,11] with that in Refs. [9,12]. We should remind ourselves that Eq. (1) is a thermodynamic relation for an equilibrium system which is described by a set of equations of state. Also, according to finite-temperature linear-response theory, the density response function is calculated from an equilibrium state. Of course the equilibrium states that are used in Eqs. (1) and (2) should be the same. So what is the equilibrium state for the Bogoliubov approximation at finite temperature? To find the answer, we notice that Eq. (27) is identical to Eq. (23) and we get it from relation $\mu = g\rho_0$. Therefore, the equilibrium state in the Bogoliubov approximation is identified to be a state that all particles occupy in the $\vec{k}=0$ level and its equation of state is given by $\mu = g\rho_0 = g\rho$. This identification is sound since the relation $\mu = g\rho_0$ is exactly the time-independent Gross-Pitaevskii equation for a uniform system without thermal depletion particles. Indeed, as proved by Leggett [19], the Bogoliubov Hamiltonian can be obtained from the Hamiltonian (4) by keeping the terms that have nonzero expectation values in a subclass of states built from Gross-Pitaevskii ground state. If we adopt this identification we find the following.

(i) The number-of-particle fluctuation from statics is given by Eq. (27), which is now completely consistent with Eq. (23). Both are normal, and therefore the system is proved to be stable, as it should be.

(ii) N_{NC} is the number of particles excited out of the condensate due to its oscillation, i.e., it is the depletion of the Gross-Pitaevskii equilibrium state [19]. N_0 in Eq. (24) should be replaced by N'_0 ,

$$N'_0 = N_0 - N_{\text{NC}}, \quad (28)$$

the number of particles remaining in the condensate after N_{NC} particles are dynamically excited out of the condensate. The single-particle Green function $G_{\alpha\beta}$ describes the dynamic process of the oscillation of the condensate. With the broken U(1) gauge symmetry, the oscillation of the condensate is interpreted as ejecting particles, which become the noncondensate particles. This interpretation of the single-particle Green's function gives a reasonable explanation of the density response function given by Eq. (22). It also explains the physical meaning of the result given by Eq. (26). We argue that, the N_{NC} dynamically excited particles form a noninteracting system with chemical potential given by $\Delta = 0$. Equation (26) then represents the number-of-particle fluctuation of this noninteracting system. In order to see this, we calculate the density response function of this noninteracting system [17]

$$\begin{aligned} \chi_{mn, \text{NC}}^{BA}(\vec{q}; i\omega_n) &= -\frac{1}{\Omega \beta} \sum_{m\vec{k} \neq 0} [G_{++}^{BA}(\vec{k}; i\omega_m) G_{++}^{BA}(\vec{k} + \vec{q}; i\omega_m \\ &\quad + i\omega_n) + G_{-+}^{BA}(\vec{k}; i\omega_m) G_{-+}^{BA}(\vec{k} + \vec{q}; i\omega_m + i\omega_n)]. \quad (29) \end{aligned}$$

A similar result with Eq. (29) is obtained by Meier and Zwinger [5] by calculating the phase fluctuation of the order parameter $\Psi(\vec{r})$ of Eq. (3). It is important to point out the difference of the physical meanings between Eqs. (22) and (29). According to Eq. (2), the number-of-particle fluctuation of the noninteracting system is

$$\begin{aligned} \frac{\langle \delta \hat{N}_{\text{NC}}^2 \rangle}{N} &= -\frac{k_B T}{\rho} \lim_{\vec{q} \rightarrow 0} \chi_{mn, \text{NC}}^{BA}(\vec{q}; 0) \\ &= \frac{1}{N} \sum_{\vec{k} \neq 0} \{ [(u_{\vec{k}}^2 + v_{\vec{k}}^2)^2 + 4u_{\vec{k}}^2 v_{\vec{k}}^2] n_{\vec{k}} (n_{\vec{k}} + 1) + u_{\vec{k}}^2 v_{\vec{k}}^2 \}. \quad (30) \end{aligned}$$

This is exactly the same as Eq. (7) in Ref. [3] obtained by Giorgini *et al.* for the fluctuation of noncondensate particles $\langle \delta N_{\text{NC}}^2 \rangle / N$. We notice that the leading order terms of this result is identical to those of Eq. (26) in $\vec{k} \sim 0$ region, where the anomalous behavior arises, since $k_B T / \omega_{\vec{k}} \approx n_{\vec{k}}$. Therefore, for this noninteracting system, the number-of-particle fluctuations obtained from statics and dynamics are also consistent, even though they are anomalous. However, this anomalous fluctuation is not an implication of instability of the interacting Bose gas, since we have proved from both statics and dynamics that the total number-of-particle fluctuation is normal. This can also be seen by substituting Eq. (24) into

Eq. (1) but replacing N_0 with N'_0 ; the anomalous fluctuation due to the fact that N_{NC} is completely canceled out. This calculation clearly shows the importance of the dynamic aspect of the condensate reservoir.

(iii) There is a new consistency. The chemical potential as a functional of the total number of particles N and the equilibrium number of particles N_0 in condensate can be determined both dynamically from the Hugenholtz-Pines theorem (17) and statically from the equilibrium equation of state. These two must be consistent with each other. However, there is a deeper physical meaning of this consistency. The equilibrium state described by the equation of state has a definite number of particles (here this is N_0). Therefore, this new consistency shows that the Hugenholtz-Pines theorem is to restore the conservation of the number of particles. Indeed, it is well known that the Hugenholtz-Pines theorem is required by the continuity equation [20,21].

In the next section, we shall show that these interpretations about the number-of-particle fluctuation, N'_0 , N_{NC} , and the single-particle Green function $G_{\alpha\beta}$, as well as the Hugenholtz-Pines theorem in the Bogoliubov approximation remain valid at the level of approximation in which all the terms in Eq. (5) are kept. As examples, we consider the random-phase approximation with inclusion of exchange (RPAE) developed by Minguzzi and Tosi [14] and the dielectric formalism by Fliesser *et al.* [22]. We shall not give the detailed derivations, since they are available in the literature. The steps presented here highlight the physics at hand.

IV. RANDOM-PHASE APPROXIMATION AND DIELECTRIC APPROACH WITH INCLUSION EXCHANGE

In the RPAE, the equilibrium equations of state of the Bose-Einstein-condensed system are the time-independent finite-temperature Gross-Pitaevskii equation for the condensate and static Hartree-Fock equation for the thermal depletion particles. For a homogenous system, they are

$$\mu = g\rho_0 + 2g\tilde{\rho}, \quad (31)$$

$$h_{\text{HF}}\psi_{\vec{k}}(\vec{r}) = \varepsilon_{\vec{k}}^{\text{HF}} \psi_{\vec{k}}(\vec{r}), \quad \vec{k} \neq 0, \quad (32)$$

where

$$h_{\text{HF}} = -\frac{\nabla^2}{2m} + 2g\rho - \mu, \quad (33)$$

$$\varepsilon_{\vec{k}}^{\text{HF}} = \varepsilon_{\vec{k}} + 2g\rho - \mu, \quad (34)$$

are the static Hartree-Fock Hamiltonian and single-particle energy with respect to the chemical potential for the thermal depletion particles. Here $\rho_0 = N_0/\Omega$, $\tilde{\rho} = \tilde{N}/\Omega$, and $\rho = N/\Omega$ are the equilibrium condensate, thermal depletion, and total densities with N_0 , \tilde{N} and N the corresponding equilibrium condensate, thermal depletion, and total number of particles. We emphasize again that the number of particles in this equilibrium system is conserved. By neglecting the thermal depletion \tilde{N} , we arrive at the equilibrium equation of state for

the Bogoliubov approximation. We notice that single-particle orbits for the condensate and thermal depletion particles are governed by two different Hamiltonians, and therefore, they are not generally orthogonal. However, for a uniform system, these single-particle orbits are simple orthogonal plane waves. We also notice that there is a gap in the single-particle spectrum

$$\lim_{\vec{k} \rightarrow 0} \varepsilon_{\vec{k}}^{\text{HF}} = g\rho_0. \quad (35)$$

This gap has important effects on many properties of a Bose-Einstein-condensed system [23]. We will come back this issue in the last section.

We define a thermal depletion single-particle Green function for the static h_{HF}

$$\tilde{G}_{\text{HF}}(\vec{k}; i\omega_n) = \frac{1}{i\omega_n - \varepsilon_{\vec{k}}^{\text{HF}}}. \quad (36)$$

The number of thermal depletion particles is found to be

$$\tilde{N} = -\frac{1}{\beta} \sum_{n, \vec{k} \neq 0} \tilde{G}_{\text{HF}}(\vec{k}; i\omega_n) = \frac{\Omega}{\lambda_T^3} g_{3/2}(z), \quad (37)$$

where $\lambda_T = \sqrt{2\pi/mk_B T}$ and $z = e^{\beta(\mu - 2g\rho)} = e^{-\beta g\rho_0}$ and $g_\gamma(z)$ is the Bose function. Equation (37) is the equation of state equivalent to Eq. (32) for the thermal depletion particles. The self-consistent relations among N , N_0 , \tilde{N} , and μ are given by Eq. (31) and

$$N(\mu) = N_0(\mu) + \tilde{N}(\mu) = N_0(\mu) + \frac{\Omega}{\lambda_T^3} g_{3/2}[z(\mu)]. \quad (38)$$

The number-of-particle fluctuation can be calculated by substituting Eq. (38) into Eq. (1). Using the equations of states (31) and (37), we find

$$\begin{aligned} \frac{\partial N_0}{\partial \mu} &= -\frac{\Omega}{g} + 2\frac{\partial N}{\partial \mu}, \\ \frac{\partial \tilde{N}}{\partial \mu} &= \frac{\beta}{\lambda_T^3} g_{1/2}(z) \left(\Omega - 2g \frac{\partial N}{\partial \mu} \right), \end{aligned} \quad (39)$$

and as a consequence

$$\frac{\langle \delta \hat{N}^2 \rangle}{N} = k_B T \left. \frac{\partial N}{\partial \mu} \right|_T = \frac{\rho_0 k_B T}{\rho m c_B^2} \frac{1 + g\tilde{P}_0}{1 + 2g\tilde{P}_0}, \quad (40)$$

where \tilde{P}_0 is defined as

$$\tilde{P}_0 = -\frac{\beta}{\lambda_T^3} g_{1/2}(e^{-\beta g\rho_0}). \quad (41)$$

Equation (40) reduces to Eq. (27), obtained in Bogoliubov approximation when one sets $\tilde{P}_0 = 0$ and $\rho_0 = \rho$.

We must also point out that $k_B T (\partial N_0 / \partial \mu) \neq \langle \delta N_0^2 \rangle$, $k_B T (\partial \tilde{N} / \partial \mu) \neq \langle \delta \tilde{N}^2 \rangle$. Because of the ensemble used in the RPAE [24], $\langle \delta N_0^2 \rangle$ and $\langle \delta \tilde{N}^2 \rangle$ are actually

$$\langle \delta N_0^2 \rangle \equiv 0, \quad (42)$$

$$\langle \delta \tilde{N}^2 \rangle = \sum_{\vec{k}} \tilde{n}_{\vec{k}} (\tilde{n}_{\vec{k}} + 1) = z \frac{\partial \tilde{N}}{\partial z} = \frac{\Omega}{\lambda_T^3} g_{1/2}(z). \quad (43)$$

These two results are clearly not the same those given by Eq. (39). Therefore,

$$\langle \delta N^2 \rangle \neq \langle \delta N_0^2 \rangle + \langle \delta \tilde{N}^2 \rangle. \quad (44)$$

This shows that, even in the mean-field level approximation, the condensate and the thermal depletion components are strongly correlated. This is not surprising, since below the critical temperature, the presence of the condensate pins down the chemical potential to be $\mu = 2g\tilde{\rho} + g\rho_0$ and Eq. (38) is a self-consistent relation between N and μ . Even in the Bogoliubov approximation, it is this self-consistent relation that predicts a number-of-particle fluctuation given by Eq. (27) consistent with Eq. (23) while the fluctuation of the condensate itself is identically zero. Similar calculations for the RPA without the exchange show that $\langle \delta \tilde{N}^2 \rangle$ follows the $N^{4/3}$ anomalous scaling law, but the total number-of-particle fluctuation is

$$\frac{\langle \delta \hat{N}^2 \rangle}{N} = k_B T \left. \frac{\partial N}{\partial \mu} \right|_T = \frac{k_B T}{mc_B^2}, \quad (45)$$

which is normal.

We now calculate the density response function around the above equilibrium state and calculate the number-of-particle fluctuation from dynamics. The linearized equations for the density fluctuation have the matrix form [14]

$$\begin{pmatrix} \delta \rho_0 \\ \delta \tilde{\rho} \end{pmatrix} = \begin{pmatrix} \chi_{cc} & \chi_{c\tilde{n}} \\ \chi_{\tilde{n}c} & \chi_{\tilde{n}\tilde{n}} \end{pmatrix} \begin{pmatrix} \delta U^c \\ \delta U^{\tilde{n}} \end{pmatrix}, \quad (46)$$

where δU^c and $\delta U^{\tilde{n}}$ are the spatially and time-varying external potentials for the condensate and thermal depletion components. We emphasize here that the matrix form (46) of the density response function is simply a result by splitting the system into a condensate and a thermal depletion component, in which the number of particles is conserved, instead of a

result of using the broken U(1) gauge symmetry (3) as claimed by Minguzzi and Tosi [14]. The total density response function is then given by

$$\chi_{nn} = \chi_{cc} + \chi_{c\tilde{n}} + \chi_{\tilde{n}c} + \chi_{\tilde{n}\tilde{n}}. \quad (47)$$

On the other hand, according to the linear response theory,

$$\delta \rho_0 = \chi_c^0 \delta U_{\text{HF}}^c = \chi_c^0 (\delta U^c + g \delta \rho_0 + 2g \delta \tilde{\rho}),$$

$$\delta \tilde{\rho} = \chi_{\tilde{n}}^0 U_{\text{HF}}^{\tilde{n}} = \chi_{\tilde{n}}^0 (\delta U^{\tilde{n}} + 2g \delta \rho_0 + 2g \delta \tilde{\rho}), \quad (48)$$

where χ_c^0 and $\chi_{\tilde{n}}^0$ are the density response functions of the condensate and the thermal depletion around the equilibrium state, respectively. From the above two equations, the four components are found

$$\begin{aligned} \chi_{cc} &= (1 - 2g\chi_{\tilde{n}}^0) D^{-1} \chi_c^0, & \chi_{\tilde{n}\tilde{n}} &= (1 - g\chi_c^0) D^{-1} \chi_{\tilde{n}}^0, \\ \chi_{c\tilde{n}} &= 2g\chi_c^0 D^{-1} \chi_{\tilde{n}}^0, & \chi_{\tilde{n}c} &= 2g\chi_{\tilde{n}}^0 D^{-1} \chi_c^0, \end{aligned} \quad (49)$$

where

$$D = (1 - g\chi_c^0)(1 - 2g\chi_{\tilde{n}}^0) - 4g^2\chi_c^0\chi_{\tilde{n}}^0. \quad (50)$$

For homogenous systems, the density response functions of the condensate and of the thermal depletion component can be obtained by linearizing the time-dependent Gross-Pitaevskii equation for the condensate around Eq. (31), and the time-dependent Hartree-Fock equation for the noncondensate around Eq. (32), respectively. They are given by

$$\chi_c^0(\vec{q}; i\omega_n) = \frac{2\rho_0 \varepsilon_{\vec{q}}}{(i\omega_n)^2 - \varepsilon_{\vec{q}}^2}, \quad (51)$$

$$\chi_{\tilde{n}}^0(\vec{q}; i\omega_n) = \frac{1}{\Omega} \sum_{\vec{k} \neq 0, -\vec{q}} \frac{\tilde{n}_{\vec{q}+\vec{k}} - \tilde{n}_{\vec{k}}}{i\omega_n + \varepsilon_{\vec{q}+\vec{k}}^{\text{HF}} - \varepsilon_{\vec{k}}^{\text{HF}}} \quad (52)$$

where $\tilde{n}_{\vec{k}} = (z^{-1} e^{\beta \varepsilon_{\vec{k}}^{\text{HF}}} - 1)^{-1}$ is the occupation number of the static Hartree-Fock single-particle level of the thermal depletion particles. Therefore, the total density response function for the homogenous Bose-condensed system is

$$\chi_{nn}(\vec{q}; i\omega_n) = \frac{[(i\omega_n)^2 - \varepsilon_{\vec{q}}^2] \chi_{\tilde{n}}^0(\vec{q}; i\omega_n) + 2\rho \varepsilon_{\vec{q}} [1 + g\chi_{\tilde{n}}^0(\vec{q}; i\omega_n)]}{[(i\omega_n)^2 - \varepsilon_{\vec{q}}^2] [1 - 2g\chi_{\tilde{n}}^0(\vec{q}; i\omega_n)] - 2\rho \varepsilon_{\vec{q}} [1 + 2g\chi_{\tilde{n}}^0(\vec{q}; i\omega_n)]}. \quad (53)$$

Now substituting Eq. (53) into Eq. (2), one gets the total number-of-particle fluctuation from dynamics

$$\frac{\langle \delta \hat{N}^2 \rangle}{N} = \frac{\rho_0 k_B T}{\rho mc_B^2} \frac{1 + g\tilde{P}_0}{1 + 2g\tilde{P}_0}. \quad (54)$$

Here we have made use of the fact that $\lim_{|\vec{q}| \rightarrow 0} \chi_{\tilde{n}}^0(\vec{q}, 0) = -(\beta/\lambda_T^3) g_{1/2}(e^{-\beta g \rho_0}) = \tilde{P}_0$ as given by Eq. (41). One can see

that Eq. (54) is exactly the same as Eq. (40).

We have shown the total number-of-particle fluctuation is normal and consistent between statics and dynamics in the RPAE, and therefore, the interacting Bose system is proved to be stable. In the above derivations, the number of particles is conserved and there is not any anomalous number-of-particle fluctuation. This is because in this RPAE, the numbers of particles in the condensate and thermal depletion component are not time dependent [19] and do not change

with the external potential because of entropy conservation [25]. Therefore, they always take the equilibrium values. The above results can be derived in a more general time-dependent Hartree-Fock scheme which preserves the U(1) gauge symmetry [23].

Now in order to see how the anomalous fluctuation arises when the broken U(1) gauge symmetry is used, we can follow the steps in Ref. [22] to build the vertex function Λ_α ,

$$G_{++}(\vec{q}; i\omega_n) = G_{--}(\vec{q}; -i\omega_n) = \frac{[i\omega_n - \varepsilon_{\vec{q}}][1 - 2g\chi_n^0(\vec{q}; i\omega_n)] + g\rho_0[1 + g\chi_n^0(\vec{q}; i\omega_n)]}{[(i\omega_n)^2 - \varepsilon_{\vec{q}}^2][1 - 2g\chi_n^0(\vec{q}; i\omega_n)] - 2\rho_0\varepsilon_{\vec{q}}[1 + 2g\chi_n^0(\vec{q}; i\omega_n)]},$$

$$G_{+-}(\vec{q}; i\omega_n) = G_{-+}(\vec{q}; i\omega_n) = -\frac{g\rho_0[1 + g\chi_n^0(\vec{q}; i\omega_n)]}{[(i\omega_n)^2 - \varepsilon_{\vec{q}}^2][1 - 2g\chi_n^0(\vec{q}; i\omega_n)] - 2\rho_0\varepsilon_{\vec{q}}[1 + 2g\chi_n^0(\vec{q}; i\omega_n)]}. \quad (55)$$

and the density response function $\chi_{mn}(\vec{q}; i\omega_n)$, which is same as Eq. (53). Since both the equilibrium state and the density response function are the same as in RPAE, therefore, one gets the same consistent normal number-of-particle fluctuations from statics and dynamics in this dielectric formalism as those in the RPAE [22]. The chemical potential from the Hugenholtz-Pines theorem in this approximation turns out to be

$$\mu_{\text{HP}} = \Sigma_{++}(0;0) - \Sigma_{--}(0;0) = g\rho_0 + 2g\tilde{\rho}, \quad (56)$$

which is the exactly the same as Eq. (31).

Using $G_{++}(\vec{q}; i\omega_n)$, the number noncondensate particles N_{NC} and its fluctuation $\langle \delta N_{\text{NC}}^2 \rangle$ are found to be the same as Eqs. (25), (29), and (30). For a dilute gas $\rho a^3 \ll 1$, where $a = gm/4\pi$, χ_n^0 is usually very small because of the single-particle gap [23]. It is thus straightforward to show that the single-particle Green function $G_{\alpha\beta}$ given by Eq. (55) has the similar form as that in Bogoliubov approximation for small \vec{k} . For example, the pole is given by $\omega_{\vec{k}} \approx c_B k [1 + 2g\chi_n^0(\vec{k}, \omega = c_B k)]$. Therefore, following the steps as in the Bogoliubov approximation, one can show that $\langle \delta N_{\text{NC}} \rangle$ follows the $\gamma = 4/3$ scaling law.

Since \tilde{N} from Eq. (37) is the number of thermal depletion particles, the difference

$$\delta N_0 = N_{\text{NC}} - \tilde{N} \quad (57)$$

can be interpreted as the number of particles excited out of the condensate by the oscillation of the whole system induced by the external time-dependent potential. Indeed, in the Bogoliubov approximation $\delta N_0 = N_{\text{NC}}$ since the depletion of the condensate is completely caused by the dynamic collective excitation. Therefore, the single-particle Green functions (55) can be interpreted as dynamic ones comparing to the thermal depletion \tilde{G}_{HF} defined by Eq. (36). As in Bogoliubov approximation, this interpretation is allowed only be-

self-energy $\Sigma_{\alpha\beta}$, and the single-particle Green function $G_{\alpha\beta}$ by using Eqs. (31) and (32) as the reference. This means that the equilibrium condensate N_0 , thermal depletion \tilde{N} , and Eq. (36) should be used to build up Λ_α , $\Sigma_{\alpha\beta}$, and regular χ^R , not the as yet to be determined N'_0 , $G_{\alpha\beta}$, and N_{NC} . Here we skip those steps and just cite the final results for the single-particle Green function matrix below

cause of the broken U(1) gauge symmetry. The total number of particles is expressed as

$$N = N'_0 + N_{\text{NC}}, \quad (58)$$

where $N'_0 = N_0 - \delta N_0$. When substituting Eq. (58) into Eq. (1) to calculate the number-of-particle fluctuation, the anomalous fluctuation due to N_{NC} is exactly canceled out by the one from N'_0 , so that the total number-of-particle fluctuation is normal, which is given by Eq. (40).

We can see that the anomalous fluctuation of the number of noncondensate particles $\langle \delta N_{\text{NC}}^2 \rangle$ is solely due to the single-particle Green functions defined by Eq. (10) whose poles entangle the single-particle and particle-conserving collective excitations, a directly consequence of the U(1) symmetry breaking rather than an implication of the instability of the Bose system since the total number-of-particle fluctuation is normal and consistent from statics and dynamics. More than thirty years ago, Straley advised caution [26] in using such a single-particle Green function to describe the zero-sound characteristic spectrum of the superfluid ^4He . Leggett also argued [19] that there are no circumstances in which Eq. (3) is physically correct.

V. DISCUSSIONS AND CONCLUSION

We have shown that the anomalous fluctuation of the number of noncondensate particles is an intrinsic feature of the broken U(1) gauge symmetry and is completely absent in the RPAE in which the U(1) gauge symmetry is preserved. This may be just related to the different interpretations of the dynamic process of the condensate of these models, as we point out in previous sections. But since this anomalous fluctuation of the number of noncondensate particles has not any physical significance, we can safely say that it is just a by-product of using the broken U(1) gauge symmetry.

Our calculation can be applied to trapped or closed Bose systems. The procedure is still to identify the equilibrium

state with definite number of particles. Using the spectrum of such a state, one can calculate the canonical partition function, from which fluctuations of the number of condensate and noncondensate particles can be obtained. The calculations following this line, which is appropriate for trapped systems, show that both fluctuations of the number of condensate and noncondensate particles are normal [4,6]. Using the Bogoliubov spectrum, one again obtains an anomalous fluctuations of the number of noncondensate particles [3,7,8,11]. However, it is not appropriate to conclude that the fluctuation of the number of condensate particles is also anomalous since one can not use canonical ensemble in Bogoliubov approximation.

By using the broken U(1) gauge symmetry, the boundary of the single-particle spectrum and the collective mode are entangled together if the poles of $G_{\alpha\beta}$ are interpreted as the single-particle excitations. But as far as the dielectric formalism in Ref. [22] concerned, there is a gapped single-particle spectrum as same as that in the RPAE for the equilibrium reference. In this sense, a gapped single-particle spectrum and a gapless collective mode do coexist even in the dielectric formalism. This is another point to identify the equilibrium state with which the density response function is calculated.

As pointed out by Meier and Zwerger [5], the anomalous fluctuation of the noncondensate particles is related to the gapless mode in the superfluid Bose system. Our analysis

shows this is the case if one uses the broken Bose U(1) gauge symmetry. In the RPAE, which preserves this gauge symmetry, there is no such an anomalous number-of-particle fluctuation related to the gapless superfluid mode.

In conclusion, we have shown that in models using the broken U(1) gauge symmetry, the number-of-particle fluctuation is normal and can be calculated consistently from the static thermodynamic relation and dynamic compressibility sum rule if the equilibrium states are identified. We also show that the chemical potential determined from the Hugenholtz-Pines theorem should also be consistent with that determined from the equilibrium equation of state. The $N^{4/3}$ anomalous fluctuation of the number of noncondensate particles is an intrinsic feature of the broken U(1) gauge symmetry. However, this anomalous fluctuation does not imply the instability of the system. Using the RPAE, which preserves the U(1) gauge symmetry, such an anomalous fluctuation of the number of noncondensate particles is completely absent.

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