

Dynamics of a trapped Fermi gas in the BCS phase

Michael Urban and Peter Schuck

Institut de Physique Nucléaire, F-91406 Orsay Cédex, France

(Received 19 September 2005; published 20 January 2006)

We derive semiclassical transport equations for a trapped atomic Fermi gas in the BCS phase at temperatures between zero and the superfluid transition temperature. These equations interpolate between the two well-known limiting cases of superfluid hydrodynamics at zero temperature and the Vlasov equation at the critical one. The linearized version of these equations, valid for small deviations from equilibrium, is worked out and applied to two simple examples where analytical solutions can be found: a sound wave in a uniform medium and the quadrupole excitation in a spherical harmonic trap. In spite of some simplifying approximations, the main qualitative results of quantum mechanical calculations are reproduced, which are the different frequencies of the quadrupole mode at zero and the critical temperature and strong Landau damping at intermediate temperatures. In addition we suggest a numerical method for solving the semiclassical equations without further approximations.

DOI: [10.1103/PhysRevA.73.013621](https://doi.org/10.1103/PhysRevA.73.013621)

PACS number(s): 03.75.Ss, 03.75.Kk, 67.40.Bz

I. INTRODUCTION

Due to improved cooling techniques, current experiments with trapped fermionic atoms like ^6Li or ^{40}K reach very low temperatures of the order of $T \approx 0.03T_F$ [1], where $T_F = \epsilon_F/k_B$ denotes the degeneracy temperature. The main motivation for these experiments is to study the so-called BEC-BCS crossover by tuning the magnetic field around a Feshbach resonance, thus changing the atom-atom scattering length a from the repulsive side ($a > 0$) through the unitary limit ($a \rightarrow \infty$) to the attractive side ($a < 0$). On the BEC side, where the system forms a Bose-Einstein condensate (BEC) of tightly bound molecules, as well as on the BCS side of the crossover, where the atoms form Cooper pairs that are very large compared with the mean distance between atoms, one expects that the system is superfluid, provided the temperature lies below a certain critical temperature T_c . Until now the experiments have concentrated on the crossover region, but the low temperatures that are currently reached suggest that future experiments will also be able to study the superfluid BCS phase ($a < 0$ and $k_F|a| \ll 1$), although its critical temperature will be extremely low.

In order to find signals for superfluidity, some recent experiments with ultracold trapped Fermionic atoms looked at dynamical observables like the expansion of the atom cloud after the trap has been switched off [2], or collective oscillations of the cloud [1,3]. The theoretical interpretation of such experiments is usually based on a theory called “superfluid hydrodynamics” [4–6], which is valid for a superfluid at zero temperature if the trap potential is sufficiently smooth to justify a local-density approximation. Apart from the latter condition, which is not necessarily fulfilled in the experiments [7], it is also clear that experiments are not done at zero temperature.

Very recently, Landau’s two-fluid hydrodynamics has been used to describe collective modes in trapped superfluid gases at a nonzero temperature [8]. In this approach, in the temperature range $0 < T < T_c$, a certain fraction of the atoms is not superfluid, but forms a normal-fluid component with

density ρ_n , while the remaining atoms with density $\rho_s = \rho - \rho_n$ still behave like a superfluid. It is also assumed that the atoms undergo enough collisions to be always in local equilibrium. This condition, however, cannot be taken for granted. In Ref. [9] it was found that even in the unitary limit, where the scattering length a diverges, the collision rate might be too low to ensure hydrodynamic behavior of the normal phase. This is certainly true in the BCS phase, where $k_F|a|$ and the temperature are so small that one can safely assume that the system is in the so-called collisionless regime. Collisionless means in this context that the collision rate $1/\tau$ is much smaller than the trap frequency Ω , i.e., an atom performs several oscillations in the trap before colliding with another atom. Since the frequencies of the collective oscillations are of the order of the trap frequency Ω , this implies that it is impossible to reach local equilibrium during the oscillation.

Nevertheless, the idea of a two-fluid model is useful in the collisionless regime, too. It has been developed for this case in the theory of superconductivity [10–13]. Similar approaches to describe liquid ^3He should be mentioned as well, although they are more complicated because of the spin structure of the order parameter [14,15]. Recently the two-fluid model has also been applied to the case of trapped fermionic atoms in the BCS phase [16,17]. Because of the possibility of Fermi-surface deformations, the normal component of a collisionless gas does not behave hydrodynamically, but more like an elastic body. A semiclassical method for treating the Fermi surface deformation in a normal Fermi gas is given by the Vlasov equation. The latter was used with great success in nuclear physics, e.g., in order to describe giant resonances in atomic nuclei [18], and recently it was also applied to trapped atomic Fermi gases in order to predict the frequencies of collective modes in the collisionless regime [4]. Contrary to hydrodynamical equations, where all quantities are local (i.e., functions of the spatial coordinate \mathbf{r} only), the Vlasov equation requires a phase-space description (i.e., the quantities are functions of \mathbf{r} and \mathbf{p}). Our aim in the present article is to derive a hydrodynamical equation for the superfluid component coupled to a Vlasov equation for the

normal component, interpolating between superfluid hydrodynamics at $T=0$ and the usual Vlasov equation at $T=T_c$. In principle, as it was done in the theory of liquid ^3He [14], one could also think of including a collision term into this equation, in order to treat systems that are neither collisionless nor hydrodynamical, but somewhere in between. However, in the present article we will restrict ourselves to the collisionless case.

Like the semiclassical description of the ground state (Thomas-Fermi approximation), the semiclassical description of the dynamics of the system can be expected to become more and more accurate if the number of atoms in the trap increases. This was the main motivation for us to develop the semiclassical approach presented here. A fully quantum-mechanical description of the collective modes of a trapped Fermi gas can be obtained, e.g., by the quasiparticle random-phase approximation (QRPA), corresponding to the linearization of the time-dependent Bogoliubov-de Gennes equations around equilibrium. The latter are also known as time-dependent Hartree-Fock-Bogoliubov (TDHFB) equations, especially in nuclear physics. QRPA calculations become tremendously difficult and time consuming if the number of particles increases. At present, they are restricted to systems of $\sim 10^4$ atoms [7,19,20], while the numbers of atoms in the experiments are at least ten times larger. In addition, all present QRPA calculations are done for the case of spherically symmetric traps, while the traps used in the experiments are generally not spherical. The numerical solution of the QRPA equations without spherical symmetry seems to be almost unfeasible, unless one reduces drastically the number of particles. Therefore semiclassical approaches are at the moment the only way to perform calculations for large numbers of atoms in realistic trap geometries.

Our article is organized as follows. In Sec. II we will present the formalism. Having derived a quasiparticle transport equation in Sec. II A, an important point will be to work out the linearized version of this equation in order to apply it to oscillations around the equilibrium state. This is done in Sec. II B. In Sec. II C we will show explicitly that our equations indeed reproduce superfluid hydrodynamics and the Vlasov equation in the limits of zero and critical temperature, respectively. The next part, Sec. III is devoted to two simple examples for which our equations can be solved more or less analytically. The first example, discussed in Sec. III A, is a sound wave in a uniform gas. The second one, described in Sec. III B, concerns a quadrupole oscillation of a harmonically trapped gas with some additional simplifications. Finally, in Sec. IV we will summarize and draw our conclusions.

II. FORMALISM

A. Derivation of a quasiparticle transport equation

In this section we will derive a quasiparticle transport equation for a superfluid gas of trapped fermionic atoms in the BCS phase. Throughout this article we will assume that the two spin states \uparrow and \downarrow are equally populated, which allows us to remove the spin degree of freedom from the beginning. However, the generalization to include the spin,

which in fact would be necessary, e.g., in order to describe spin waves or systems with unequal populations, is straightforward. In order to be in the BCS phase, the atoms must have an attractive interaction, i.e., a negative scattering length $a < 0$, which, on the other hand, must be weak enough for the BCS approximation to be valid.

Let us start by writing down the TDHFB equations [18]. To that end we define the nonlocal normal and anomalous density matrices,

$$\varrho(\mathbf{r}, \mathbf{r}') = \langle \psi_{\uparrow}^{\dagger}(\mathbf{r}') \psi_{\uparrow}(\mathbf{r}) \rangle = \langle \psi_{\downarrow}^{\dagger}(\mathbf{r}') \psi_{\downarrow}(\mathbf{r}) \rangle, \quad (1)$$

$$\kappa(\mathbf{r}, \mathbf{r}') = \langle \psi_{\uparrow}(\mathbf{r}') \psi_{\downarrow}(\mathbf{r}) \rangle = -\langle \psi_{\downarrow}(\mathbf{r}') \psi_{\uparrow}(\mathbf{r}) \rangle, \quad (2)$$

where ψ is the field operator. The single-particle Hamiltonian (minus the chemical potential μ) reads as

$$h = -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}) + g\rho(\mathbf{r}) - \mu, \quad (3)$$

where m is the atomic mass, $V_{\text{ext}}(\mathbf{r})$ is the potential of the trap, and $g\rho(\mathbf{r})$ is the mean-field potential. The coupling constant g is related to the atom-atom scattering length a by $g = 4\pi\hbar^2 a/m$ and the density per spin state $\rho(\mathbf{r})$ is just equal to the local part of the density matrix,

$$\rho(\mathbf{r}) = \varrho(\mathbf{r}, \mathbf{r}). \quad (4)$$

According to the usual regularization prescription [21], the pairing gap is related to the anomalous density by

$$\Delta(\mathbf{r}) = -g \lim_{s \rightarrow 0} \frac{d}{ds} s \kappa \left(\mathbf{r} + \frac{\mathbf{s}}{2}, \mathbf{r} - \frac{\mathbf{s}}{2} \right). \quad (5)$$

Combining all quantities in the 2×2 matrices,

$$\mathcal{H} = \begin{pmatrix} h & \Delta \\ -\Delta^{\dagger} & -\bar{h} \end{pmatrix}, \quad \mathcal{R} = \begin{pmatrix} \varrho & -\kappa \\ -\kappa^{\dagger} & 1 - \bar{\varrho} \end{pmatrix}, \quad (6)$$

where $\bar{\varrho}$ and \bar{h} denote the time-reversed operators to ϱ and h , respectively, the TDHFB equation can be written in the compact form [18]

$$i\hbar \dot{\mathcal{R}} = [\mathcal{H}, \mathcal{R}]. \quad (7)$$

In analogy to the derivation of the Vlasov equation in the normal phase from the Hartree-Fock equation [18], it is useful to introduce the Wigner transform of the density matrix,

$$\varrho(\mathbf{r}, \mathbf{p}) = \int d^3s e^{-i\mathbf{p}\cdot\mathbf{s}/\hbar} \varrho \left(\mathbf{r} + \frac{\mathbf{s}}{2}, \mathbf{r} - \frac{\mathbf{s}}{2} \right). \quad (8)$$

It is appealing, although strictly speaking not correct, to interpret the function $\varrho(\mathbf{r}, \mathbf{p})$ as a distribution function of particles in phase space. In a completely analogous way we define the Wigner transform of the anomalous density matrix, $\kappa(\mathbf{r}, \mathbf{p})$, and the Wigner transform of the Hamiltonian, $h(\mathbf{r}, \mathbf{p})$, which is equal to the classical Hamiltonian,

$$h(\mathbf{r}, \mathbf{p}) = \frac{p^2}{2m} + V_{\text{ext}}(\mathbf{r}) + g\rho(\mathbf{r}) - \mu. \quad (9)$$

(For the sake of readability we are using the same symbol for the operators and their Wigner transforms, but whenever

there is a risk of confusion we will write down the arguments.) Equations (4) and (5) can be written in terms of the Wigner transformed quantities as follows:

$$\rho(\mathbf{r}) = \int \frac{d^3p}{(2\pi\hbar)^3} \varrho(\mathbf{r}, \mathbf{p}), \quad (10)$$

$$\Delta(\mathbf{r}) = -g \int \frac{d^3p}{(2\pi\hbar)^3} \left(\kappa(\mathbf{r}, \mathbf{p}) - \frac{\Delta(\mathbf{r})}{p^2/m} \right). \quad (11)$$

We also need the Wigner transforms of the time-reversed operators $\bar{\varrho}$ and \bar{h} , and the Wigner transforms of the adjoint operators κ^\dagger and Δ^\dagger . To that end we recall the general relations

$$\bar{A}(\mathbf{r}, \mathbf{p}) = A(\mathbf{r}, -\mathbf{p}), \quad [A^\dagger](\mathbf{r}, \mathbf{p}) = A^*(\mathbf{r}, \mathbf{p}), \quad (12)$$

which are valid for an arbitrary operator A . The usefulness of the Wigner transform lies in the fact that, to first order in an expansion into powers of \hbar , the Wigner transform of the product of two operators A and B can be obtained according to

$$[AB](\mathbf{r}, \mathbf{p}) \approx A(\mathbf{r}, \mathbf{p})B(\mathbf{r}, \mathbf{p}) + \frac{i\hbar}{2} \{A(\mathbf{r}, \mathbf{p}), B(\mathbf{r}, \mathbf{p})\}, \quad (13)$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket,

$$\{A, B\} = \sum_{i=x,y,z} \left(\frac{\partial A}{\partial r_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial r_i} \right). \quad (14)$$

Applying this product rule to the Wigner transform of the TDHFB equation (7), one obtains four coupled equations:

$$i\hbar \dot{\varrho} = i\hbar \{h, \varrho\} + 2i \text{Im}(\Delta^* \kappa) - i\hbar \text{Re}\{\Delta^*, \kappa\}, \quad (15a)$$

$$\begin{aligned} i\hbar \dot{\kappa} &= (h + \bar{h})\kappa + \frac{i\hbar}{2} \{h - \bar{h}, \kappa\} \\ &+ \Delta(\varrho + \bar{\varrho} - 1) - \frac{i\hbar}{2} \{\Delta, \varrho - \bar{\varrho}\}, \end{aligned} \quad (15b)$$

$$\begin{aligned} i\hbar \dot{\kappa}^* &= -(h + \bar{h})\kappa^* + \frac{i\hbar}{2} \{h - \bar{h}, \kappa^*\} \\ &- \Delta^*(\varrho + \bar{\varrho} - 1) - \frac{i\hbar}{2} \{\Delta^*, \varrho - \bar{\varrho}\}, \end{aligned} \quad (15c)$$

$$i\hbar \dot{\bar{\varrho}} = -i\hbar \{\bar{h}, \bar{\varrho}\} + 2i \text{Im}(\Delta^* \kappa) + i\hbar \text{Re}\{\Delta^*, \kappa\}. \quad (15d)$$

In order to proceed further, it is useful to separate the collective superfluid motion from the dynamics due to quasiparticle excitations. This can be achieved by a gauge transformation, $\tilde{\psi}(\mathbf{r}) = \psi(\mathbf{r}) \exp[i\phi(\mathbf{r})]$ [13,14]. According to their definitions, the normal and anomalous density matrices behave very differently under this transformation. The corresponding Wigner transforms are given by

$$\bar{\varrho}(\mathbf{r}, \mathbf{p}) = \varrho[\mathbf{r}, \mathbf{p} - \hbar \nabla \phi(\mathbf{r})], \quad (16)$$

$$\bar{\kappa}(\mathbf{r}, \mathbf{p}) = \kappa(\mathbf{r}, \mathbf{p}) e^{2i\phi(\mathbf{r})}. \quad (17)$$

If the Hamiltonian and the gap are changed according to

$$\tilde{h}(\mathbf{r}, \mathbf{p}) = \frac{[\mathbf{p} - \hbar \nabla \phi(\mathbf{r})]^2}{2m} - \hbar \dot{\phi}(\mathbf{r}) + V(\mathbf{r}) - \mu, \quad (18)$$

$$\tilde{\Delta}(\mathbf{r}) = \Delta(\mathbf{r}) e^{2i\phi(\mathbf{r})}, \quad (19)$$

the equation of motion of the gauge-transformed quantities looks exactly like Eq. (7). The superfluid velocity is proportional to the gradient of the phase of the gap. Hence, if we choose the gauge transformation such that the transformed gap $\tilde{\Delta}$ is real, we have completely separated the collective motion of the superfluid component from the motion due to quasiparticle excitations. A formal argument for the necessity of this choice of the gauge is given in Ref. [14].

From now on we will suppose that $\tilde{\Delta}$ is real. Splitting $\bar{\varrho}$ and \bar{h} into time-even and time-odd parts,

$$\bar{\varrho}_{ev} = \frac{1}{2}(\bar{\varrho} + \bar{\bar{\varrho}}), \quad \bar{\varrho}_{od} = \frac{1}{2}(\bar{\varrho} - \bar{\bar{\varrho}}), \quad (20)$$

$$\tilde{h}_{ev} = \frac{1}{2}(\tilde{h} + \bar{\tilde{h}}) = \frac{p^2}{2m} + \frac{(\hbar \nabla \phi)^2}{2m} + V - \mu - \hbar \dot{\phi}, \quad (21)$$

$$\tilde{h}_{od} = \frac{1}{2}(\tilde{h} - \bar{\tilde{h}}) = -\frac{\hbar}{m} \mathbf{p} \cdot \nabla \phi, \quad (22)$$

and $\bar{\kappa}$ into real and imaginary parts,

$$\bar{\kappa}_{re} = \text{Re} \bar{\kappa}, \quad \bar{\kappa}_{im} = \text{Im} \bar{\kappa}, \quad (23)$$

one can rewrite the gauge transformed version of the system of Eqs. (14) as follows:

$$\hbar \dot{\bar{\varrho}}_{ev} = \hbar \{\tilde{h}_{ev}, \bar{\varrho}_{od}\} + \hbar \{\tilde{h}_{od}, \bar{\varrho}_{ev}\} + 2 \tilde{\Delta} \bar{\kappa}_{im}, \quad (24a)$$

$$\hbar \dot{\bar{\varrho}}_{od} = \hbar \{\tilde{h}_{ev}, \bar{\varrho}_{ev}\} + \hbar \{\tilde{h}_{od}, \bar{\varrho}_{od}\} - \hbar \{\tilde{\Delta}, \bar{\kappa}_{re}\}, \quad (24b)$$

$$\hbar \dot{\bar{\kappa}}_{re} = 2 \tilde{h}_{ev} \bar{\kappa}_{im} + \hbar \{\tilde{h}_{od}, \bar{\kappa}_{re}\} - \hbar \{\tilde{\Delta}, \bar{\varrho}_{od}\}, \quad (24c)$$

$$\hbar \dot{\bar{\kappa}}_{im} = -2 \tilde{h}_{ev} \bar{\kappa}_{re} + \tilde{\Delta} (1 - 2 \bar{\varrho}_{ev}) + \hbar \{\tilde{h}_{od}, \bar{\kappa}_{im}\}. \quad (24d)$$

For a semiclassical \hbar expansion it seems disturbing that these equations mix different orders in \hbar . However, it is possible to decouple the equations of motion for the leading-order quantities from those of the higher-order ones. In order to show this, we expand $\bar{\varrho}$ and $\bar{\kappa}$ into powers of \hbar . Since Eqs. (24) themselves are only valid up to order \hbar , it does not make sense to go beyond the first order in this series. From Eqs. (24a) and (24c) it is evident that $\bar{\kappa}_{im}$ must be suppressed by one power of \hbar with respect to the other quantities. We therefore write

$$\bar{\varrho}_{ev,od} = \bar{\varrho}_{ev,od}^{(0)} + \hbar \bar{\varrho}_{ev,od}^{(1)} + \dots, \quad (25)$$

$$\bar{\kappa}_{re} = \bar{\kappa}_{re}^{(0)} + \hbar \bar{\kappa}_{re}^{(1)} + \dots, \quad (26)$$

$$\tilde{\kappa}_{im} = \hbar \tilde{\kappa}_{im}^{(1)} + \dots \quad (27)$$

Inserting these expansions into Eqs. (24a)–(24d) and retaining only the leading order in each equation [order \hbar in the case of Eqs. (24a)–(24c); order 1 in the case of Eq. (24d)], one obtains

$$\dot{\tilde{\varrho}}_{ev}^{(0)} = \{\tilde{h}_{ev}, \tilde{\varrho}_{od}^{(0)}\} + \{\tilde{h}_{od}, \tilde{\varrho}_{ev}^{(0)}\} + 2\tilde{\Delta} \tilde{\kappa}_{im}^{(1)}, \quad (28a)$$

$$\dot{\tilde{\varrho}}_{od}^{(0)} = \{\tilde{h}_{ev}, \tilde{\varrho}_{ev}^{(0)}\} + \{\tilde{h}_{od}, \tilde{\varrho}_{od}^{(0)}\} - \{\tilde{\Delta}, \tilde{\kappa}_{re}^{(0)}\}, \quad (28b)$$

$$\dot{\tilde{\kappa}}_{re}^{(0)} = 2\tilde{h}_{ev} \tilde{\kappa}_{im}^{(1)} + \{\tilde{h}_{od}, \tilde{\kappa}_{re}^{(0)}\} - \{\tilde{\Delta}, \tilde{\varrho}_{od}^{(0)}\}, \quad (28c)$$

$$2\tilde{h}_{ev} \tilde{\kappa}_{re}^{(0)} = \tilde{\Delta} (1 - 2\tilde{\varrho}_{ev}^{(0)}). \quad (28d)$$

Only one of the higher-order quantities, namely $\tilde{\kappa}_{im}^{(1)}$, appears in these equations, but it can be expressed in terms of the leading-order quantities, e.g., with the help of Eq. (28a):

$$\tilde{\kappa}_{im}^{(1)} = \frac{1}{2\tilde{\Delta}} (\dot{\tilde{\varrho}}_{ev}^{(0)} - \{\tilde{h}_{ev}, \tilde{\varrho}_{od}^{(0)}\} - \{\tilde{h}_{od}, \tilde{\varrho}_{ev}^{(0)}\}). \quad (29)$$

By taking a linear combination of Eqs. (28a) and (28c), one can eliminate $\tilde{\kappa}_{im}^{(1)}$. The resulting equation reads as

$$\tilde{h}_{ev} \dot{\tilde{\varrho}}_{ev}^{(0)} - \tilde{\Delta} \dot{\tilde{\kappa}}_{re}^{(0)} = E_{ev} \{E_{ev}, \tilde{\varrho}_{od}^{(0)}\} + \tilde{h}_{ev} \{\tilde{h}_{od}, \tilde{\varrho}_{ev}^{(0)}\} - \tilde{\Delta} \{\tilde{h}_{od}, \tilde{\kappa}_{re}^{(0)}\}, \quad (30)$$

where we have introduced the abbreviation

$$E_{ev} = \sqrt{\tilde{h}_{ev}^2 + \tilde{\Delta}^2}. \quad (31)$$

Equations (28b), (28d), and (30) form a system of three coupled equations for the three leading-order quantities $\tilde{\varrho}_{ev}^{(0)}$, $\tilde{\varrho}_{od}^{(0)}$, and $\tilde{\kappa}_{re}^{(0)}$.

From now on we will suppress the index “(0)” and simply write $\tilde{\varrho}_{ev}$, $\tilde{\varrho}_{od}$, and $\tilde{\kappa}_{re}$ instead of $\tilde{\varrho}_{ev}^{(0)}$, $\tilde{\varrho}_{od}^{(0)}$, and $\tilde{\kappa}_{re}^{(0)}$. The next step is to exploit Eq. (28d) in order to reduce the number of unknown functions. To that end we introduce a new phase-space function $\nu_{ev}(\mathbf{r}, \mathbf{p})$, the so-called “quasiparticle distribution function,” which is defined in such a way that the two members of Eq. (28d) are equal to $\tilde{h}_{ev} \tilde{\Delta} (1 - 2\nu_{ev}) / E_{ev}$. In other words, $\tilde{\varrho}_{ev}$ and $\tilde{\kappa}_{re}$ can be expressed in terms of this function ν_{ev} as follows:

$$\tilde{\varrho}_{ev} = \frac{1}{2} - \frac{\tilde{h}_{ev}}{2E_{ev}} (1 - 2\nu_{ev}), \quad (32)$$

$$\tilde{\kappa}_{re} = \frac{\tilde{\Delta}}{2E_{ev}} (1 - 2\nu_{ev}). \quad (33)$$

In fact, the definition of ν_{ev} has been chosen such that these relations resemble the well-known expressions for ϱ and κ in equilibrium, where ν_{ev} has to be replaced by the Fermi distribution function for quasiparticles, $f(E)$ (see Sec. II B). With the help of Eqs. (32) and (33), the remaining two equations, (28b) and (30), take the rather simple form

$$\dot{\tilde{\varrho}}_{od} = \{E_{ev}, \nu_{ev}\} + \{\tilde{h}_{od}, \tilde{\varrho}_{od}\}, \quad (34a)$$

$$\dot{\tilde{\nu}} = \{E_{ev}, \tilde{\varrho}_{od}\} + \{\tilde{h}_{od}, \nu_{ev}\}. \quad (34b)$$

Since the first of these equations is purely time odd while the second one is purely time even, we can add both equations without any loss of information. The result can be written as

$$\dot{\nu} = \{E, \nu\}, \quad (35)$$

where we have introduced the new functions

$$\nu = \nu_{ev} + \tilde{\varrho}_{od}, \quad E = E_{ev} + \tilde{h}_{od}. \quad (36)$$

Equation (35) resembles very much the usual Vlasov equation for the normal Fermi gas, which can be written as $\dot{\varrho} = \{h, \varrho\}$. One just has to replace the distribution function ϱ by the quasiparticle distribution function ν and the Hamiltonian h by the quasiparticle energy E . It should be mentioned that Eq. (35) or similar kinetic equations have already been derived in the literature several times. Probably for the first time it was given by Betbeder-Matibet and Nozières [13] in a linearized form for small deviations from equilibrium. In order to be self-contained, we gave here our own way to arrive at Eq. (35).

In order to obtain a closed system of equations, Eq. (35) must be complemented by an equation for the so-far unknown phase ϕ . As stated above, the phase is fixed by the requirement that the gauge-transformed gap $\tilde{\Delta}$ is real, i.e., $\text{Im} \tilde{\Delta} = 0$. With the help of the relation (29) and of the gap equation (11), this can be rewritten as

$$\int \frac{d^3 p}{(2\pi\hbar)^3} (\dot{\tilde{\varrho}}_{ev} - \{\tilde{h}_{ev}, \tilde{\varrho}_{od}\} - \{\tilde{h}_{od}, \tilde{\varrho}_{ev}\}) = 0. \quad (37)$$

As we will see in a moment, this is nothing but the continuity equation. This observation confirms earlier statements in the literature that the continuity equation should be used for the determination of the phase [13]. In order to derive the continuity equation from Eq. (37), we write down explicitly the Poisson brackets and integrate by parts. In this way we obtain

$$\int \frac{d^3 p}{(2\pi\hbar)^3} \left(\dot{\tilde{\varrho}} + \nabla \cdot \tilde{\varrho} \frac{\mathbf{p} - \hbar \nabla \phi}{m} \right) = 0. \quad (38)$$

Using Eq. (16) and changing the integration variable according to $\mathbf{p} \rightarrow \mathbf{p} + \hbar \nabla \phi$, this can be transformed into the usual continuity equation,

$$\dot{\rho}(\mathbf{r}) + \nabla \cdot \mathbf{j}(\mathbf{r}) = 0, \quad (39)$$

with

$$\mathbf{j}(\mathbf{r}) = \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{\mathbf{p}}{m} \varrho(\mathbf{r}, \mathbf{p}). \quad (40)$$

B. Linearization around equilibrium

From now on we will assume that the external potential V_{ext} can be written as

$$V_{ext} = V_{0\ ext} + V_{1\ ext}, \quad (41)$$

where $V_{0\ ext}$ is time independent and $V_{1\ ext}$ can be considered a small perturbation. The equilibrium quantities corresponding to the static potential $V_{0\ ext}$ will be marked by an index "0," e.g.,

$$\nu_0(\mathbf{r}, \mathbf{p}) = f[E_0(\mathbf{r}, \mathbf{p})], \quad (42)$$

$$\phi_0(\mathbf{r}) = 0, \quad (43)$$

where $f(E)$ denotes the Fermi function

$$f(E) = \frac{1}{e^{E/(k_B T)} + 1} \quad (44)$$

and

$$E_0(\mathbf{r}, \mathbf{p}) = \sqrt{h_0^2(\mathbf{r}, \mathbf{p}) + \Delta_0^2(\mathbf{r})}, \quad (45)$$

$$h_0(\mathbf{r}, \mathbf{p}) = \frac{p^2}{2m} + V_{0\ ext}(\mathbf{r}) + g\rho_0(\mathbf{r}) - \mu, \quad (46)$$

etc. Our aim is to calculate the small deviations from equilibrium induced by the perturbation $V_{1\ ext}$, which will be marked by an index "1." To that end we linearize the equation of motion (35) for the quasiparticle distribution function:

$$\dot{\nu}_1 - \{E_0, \nu_1\} = f'(E_0)\{E_1, E_0\}, \quad (47)$$

where $f'(E_0) = df/dE_0$. We also linearize the continuity equation (38):

$$\dot{\rho}_1(\mathbf{r}) + \nabla \cdot \mathbf{j}_{1\nu}(\mathbf{r}) - \frac{\hbar}{m} \nabla \cdot \rho_0(\mathbf{r}) \nabla \phi_1(\mathbf{r}) = 0, \quad (48)$$

with

$$\mathbf{j}_{1\nu}(\mathbf{r}) = \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{\mathbf{p}}{m} \nu_1(\mathbf{r}, \mathbf{p}). \quad (49)$$

In order to have a closed system of equations, we must express $E_1(\mathbf{r}, \mathbf{p})$ and $\rho_1(\mathbf{r})$ in terms of equilibrium quantities, the perturbation $V_{1\ ext}(\mathbf{r})$, and the unknown functions $\nu_1(\mathbf{r}, \mathbf{p})$ and $\phi_1(\mathbf{r}, \mathbf{p})$. Linearizing $E(\mathbf{r}, \mathbf{p})$, one obtains

$$E_1 = \frac{h_0}{E_0} \tilde{h}_{1ev} + \frac{\Delta_0}{E_0} \tilde{\Delta}_1 + \tilde{h}_{1od}, \quad (50)$$

with

$$\tilde{h}_{1ev}(\mathbf{r}, \mathbf{p}) = V_{1\ ext}(\mathbf{r}) + g\rho_1(\mathbf{r}) - \hbar\dot{\phi}_1(\mathbf{r}), \quad (51)$$

$$\tilde{h}_{1od}(\mathbf{r}, \mathbf{p}) = -\frac{\hbar}{m} \mathbf{p} \cdot \nabla \phi_1(\mathbf{r}). \quad (52)$$

The most difficult part is to derive the expressions for $\rho_1(\mathbf{r})$ and $\tilde{\Delta}_1(\mathbf{r})$. We start by linearizing Eqs. (32) and (33):

$$\begin{aligned} \tilde{\nu}_{1ev} &= \frac{h_0}{E_0} \nu_{1ev} + \frac{1-2f(E_0)}{2E_0^3} \\ &\times [-\Delta_0^2(V_{1\ ext} + g\rho_1 - \hbar\dot{\phi}_1) + h_0\Delta_0\tilde{\Delta}_1], \end{aligned} \quad (53)$$

$$\begin{aligned} \tilde{\kappa}_{1re} &= -\frac{\Delta_0}{E_0} \nu_{1ev} - \frac{1-2f(E_0)}{2E_0^3} [h_0\Delta_0(V_{1\ ext} + g\rho_1 - \hbar\dot{\phi}_1) \\ &+ \Delta_0^2\tilde{\Delta}_1] + \frac{1-2f(E_0)}{2E_0} \tilde{\Delta}_1. \end{aligned} \quad (54)$$

According to Eqs. (10) and (11), ρ_1 and $\tilde{\Delta}_1$ can be obtained by integrating Eqs. (53) and (54) over \mathbf{p} . This gives a coupled system of two linear equations,

$$\begin{aligned} \rho_1(\mathbf{r}) &= \rho_{1\nu}(\mathbf{r}) - A(\mathbf{r})[V_{1\ ext}(\mathbf{r}) + g\rho_1(\mathbf{r}) - \hbar\dot{\phi}_1(\mathbf{r}) \\ &+ B(\mathbf{r})\tilde{\Delta}_1(\mathbf{r})], \end{aligned} \quad (55a)$$

$$\begin{aligned} \tilde{\Delta}_1(\mathbf{r}) &= \Delta_{1\nu}(\mathbf{r}) + gB(\mathbf{r})[V_{1\ ext}(\mathbf{r}) + g\rho_1(\mathbf{r}) - \hbar\dot{\phi}_1(\mathbf{r}) \\ &+ [gA(\mathbf{r}) + 1]\tilde{\Delta}_1(\mathbf{r})], \end{aligned} \quad (55b)$$

where the gap equation (11) for the equilibrium case has been used in the derivation of the last term, and the following abbreviations have been introduced:

$$\rho_{1\nu}(\mathbf{r}) = \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{h_0(\mathbf{r}, \mathbf{p})}{E_0(\mathbf{r}, \mathbf{p})} \nu_{1ev}(\mathbf{r}, \mathbf{p}), \quad (56)$$

$$\Delta_{1\nu}(\mathbf{r}) = g \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{\Delta_0(\mathbf{r})}{E_0(\mathbf{r}, \mathbf{p})} \nu_{1ev}(\mathbf{r}, \mathbf{p}), \quad (57)$$

$$A(\mathbf{r}) = \Delta_0^2(\mathbf{r}) \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{1-2f[E_0(\mathbf{r}, \mathbf{p})]}{2E_0^3(\mathbf{r}, \mathbf{p})}, \quad (58)$$

$$B(\mathbf{r}) = \Delta_0(\mathbf{r}) \int \frac{d^3 p}{(2\pi\hbar)^3} h_0(\mathbf{r}, \mathbf{p}) \frac{1-2f[E_0(\mathbf{r}, \mathbf{p})]}{2E_0^3(\mathbf{r}, \mathbf{p})}. \quad (59)$$

Later we will show that the coefficient B is negligible compared with the coefficient A . In the limit $B \rightarrow 0$ the two equations (55a) and (55b) are decoupled and can immediately be solved for ρ_1 and $\tilde{\Delta}_1$:

$$\rho_1(\mathbf{r}) = \frac{\rho_{1\nu}(\mathbf{r}) - A(\mathbf{r})[V_{1\ ext}(\mathbf{r}) - \hbar\dot{\phi}_1(\mathbf{r})]}{1 + gA(\mathbf{r})}, \quad (60)$$

$$\tilde{\Delta}_1(\mathbf{r}) = \frac{\Delta_{1\nu}(\mathbf{r})}{gA(\mathbf{r})}. \quad (61)$$

We will now calculate the coefficients A and B for the case that both $\Delta_0(\mathbf{r})$ and $k_B T$ are small compared with the local Fermi energy,

$$\epsilon_F(\mathbf{r}) = \frac{p_F^2(\mathbf{r})}{2m} = \mu - V_{0\ ext}(\mathbf{r}) - g\rho_0(\mathbf{r}). \quad (62)$$

[Note that in the case of a trapped system, the condition $\Delta_0(\mathbf{r}) \ll \epsilon_F(\mathbf{r})$ is automatically fulfilled everywhere in the trap if it is valid at the center.] In this case, the relevant contributions to the integrals (58) and (59) come from momenta near the Fermi surface. As usual, the integrals over p can be simplified by transforming them into integrals over the energy variable $\xi = p^2/2m - \epsilon_F(\mathbf{r})$ and approximating the density

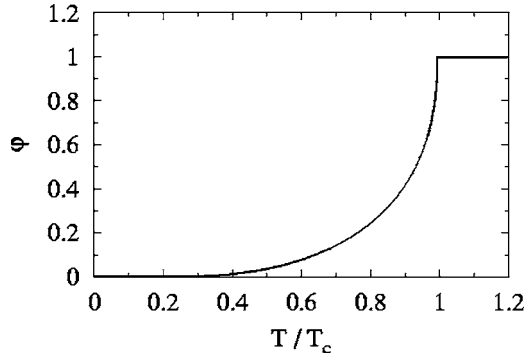


FIG. 1. Temperature dependence of the function φ defined in Eq. (64).

of states by its value at the Fermi energy, i.e., $p^2 dp \approx mp_F(\mathbf{r})d\xi$. For the coefficient A , one obtains in this way

$$A(\mathbf{r}) = \frac{mp_F(\mathbf{r})}{2\pi^2\hbar^3} [1 - \varphi(\mathbf{r})], \quad (63)$$

where the function φ describes the temperature dependence:

$$\varphi(\mathbf{r}) = - \int d\xi \frac{\xi^2}{E^2} f'(E) \Big|_{E=\sqrt{\xi^2+\Delta_0^2(\mathbf{r})}}. \quad (64)$$

One can show that $\varphi=0$ for $T=0$ and $\varphi=1$ for $\Delta_0=0$. In all other cases, the function φ must be evaluated numerically. From its definition one can see that φ depends on \mathbf{r} only through the dimensionless parameter $T/T_c(\mathbf{r})$, where $T_c(\mathbf{r}) = 0.57\Delta_0(\mathbf{r}; T=0)/k_B$ is the local critical temperature. [It is evident that φ is a function of $\Delta_0(\mathbf{r})/(k_B T)$, but $\Delta_0(\mathbf{r})$ can, in turn, be written as $k_B T_c(\mathbf{r})$ times a universal function of $T/T_c(\mathbf{r})$.] For an illustration, the numerical result for φ as a function of this parameter is shown in Fig. 1.

If one applies the same method to the coefficient B , one obtains $B=0$. This is because the integrand in Eq. (59) is odd in ξ if one neglects the energy dependence of the density of states. Already from this argument one can conclude that the coefficient B must be suppressed by at least one power of Δ_0/ϵ_F or T/ϵ_F . Indeed, after a rather lengthy and delicate analysis, one finds

$$B(\mathbf{r}) = \frac{\Delta_0(\mathbf{r})}{2\epsilon_F(\mathbf{r})} \left(\frac{mp_F(\mathbf{r})}{2\pi^2\hbar^3} [2 + \varphi(\mathbf{r})] - \frac{1}{g} \right). \quad (65)$$

This is the justification for neglecting the coefficient B when solving Eqs. (55a) and (55b).

Finally, let us put everything together and give a concise summary of the system of equations that has to be solved. First of all, there is the equation of motion (47) for the quasiparticle distribution function. After the Poisson bracket on the rhs has been written down explicitly, it can be transformed into

$$\begin{aligned} \dot{\nu}_1 - \{E_0, \nu_1\} = & - \frac{f'(E_0)}{m} \left[- \mathbf{p} \cdot \nabla \frac{V_{1\text{ext}} + g\rho_{1\nu} - \hbar\dot{\phi}_1}{1 + gA} \right. \\ & + \frac{\Delta_0}{E_0^2} \mathbf{p} \cdot \nabla \frac{\Delta_0(V_{1\text{ext}} + g\rho_{1\nu} - \hbar\dot{\phi}_1)}{1 + gA} \\ & + \frac{h_0}{E_0^2} \mathbf{p} \cdot \nabla \frac{\Delta_0\Delta_{1\nu}}{gA} + \frac{\hbar h_0}{mE_0} (\mathbf{p} \cdot \nabla)^2 \phi_1 \\ & \left. - \hbar \left(\frac{h_0}{E_0} \nabla (V_{0\text{ext}} + g\rho_0) + \frac{\Delta_0}{E_0} \nabla \Delta_0 \right) \cdot \nabla \phi_1 \right]. \end{aligned} \quad (66)$$

The second equation is the continuity equation,

$$\begin{aligned} \frac{\dot{\rho}_{1\nu}(\mathbf{r}) - A(\mathbf{r})[\dot{V}_{1\text{ext}}(\mathbf{r}) - \hbar\dot{\phi}_1(\mathbf{r})]}{1 + gA(\mathbf{r})} + \nabla \cdot \mathbf{j}_{1\nu}(\mathbf{r}) \\ - \frac{\hbar}{m} \nabla \cdot \rho_0(\mathbf{r}) \nabla \phi_1(\mathbf{r}) = 0. \end{aligned} \quad (67)$$

The definitions of $\rho_{1\nu}$, $\Delta_{1\nu}$, and $\mathbf{j}_{1\nu}$ in terms of ν_1 are given by Eqs. (56), (57), and (49).

C. Limiting cases

We are now going to check that our equations reproduce superfluid hydrodynamics and the Vlasov equation in the cases $T=0$ and $T \geq T_c$, respectively. In the limit of zero temperature, Eq. (66) becomes extremely simple since $f(E)=0$, and therefore the rhs of Eq. (66) vanishes identically. The corresponding solution is of course $\nu_1=0$, which implies $\rho_{1\nu}=\Delta_{1\nu}=\mathbf{j}_{1\nu}=0$. (This trivial solution is unique if we assume that the system was in equilibrium at the moment when the time-dependent perturbation was switched on.) As a consequence, the continuity equation (67) reduces to

$$\frac{\dot{V}_{1\text{ext}}(\mathbf{r}) - \hbar\dot{\phi}_1(\mathbf{r})}{\frac{2\pi^2\hbar^3}{mp_F(\mathbf{r})} + g} + \frac{\hbar}{m} \nabla \cdot \rho_0(\mathbf{r}) \nabla \phi_1(\mathbf{r}) = 0. \quad (68)$$

Here we have used the explicit expression for $A(\mathbf{r})$ and the fact that $\varphi=0$ at zero temperature.

How does Eq. (68) compare to superfluid hydrodynamics? The continuity and Euler equations of superfluid hydrodynamics can be written as [5]

$$\dot{\rho}(\mathbf{r}) + \nabla \cdot \rho(\mathbf{r})\mathbf{v}(\mathbf{r}) = 0, \quad (69)$$

$$\dot{\mathbf{v}}(\mathbf{r}) = - \nabla \left(\frac{\mathbf{v}^2(\mathbf{r})}{2} + \frac{V_{\text{ext}}(\mathbf{r})}{m} + \frac{\mu_{\text{loc}}(\mathbf{r})}{m} \right), \quad (70)$$

where $\mathbf{v}(\mathbf{r})$ denotes the velocity field and $\mu_{\text{loc}}(\mathbf{r})$ is the local chemical potential, which in the BCS phase ($\Delta \ll \epsilon_F$) is related to the density $\rho(\mathbf{r})$ by the Thomas-Fermi relation

$$\mu_{\text{loc}}(\mathbf{r}) = \frac{p_F^2(\mathbf{r})}{2m} + g\rho(\mathbf{r}), \quad (71)$$

with

$$p_F(\mathbf{r}) = \hbar[6\pi^2\rho(\mathbf{r})]^{1/3}. \quad (72)$$

Writing the irrotational velocity field in the form

$$\mathbf{v}(\mathbf{r}) = -\frac{\hbar}{m}\nabla\phi(\mathbf{r}) \quad (73)$$

and linearizing Eqs. (69) and (70) around equilibrium, one obtains

$$\dot{\rho}_1(\mathbf{r}) - \frac{\hbar}{m}\nabla\cdot\rho_0(\mathbf{r})\nabla\phi_1(\mathbf{r}) = 0, \quad (74)$$

$$\hbar\dot{\phi}(\mathbf{r}) = V_{1\text{ ext}}(\mathbf{r}) + \left(\frac{2\pi^2\hbar^3}{m p_F(\mathbf{r})} + g\right)\rho_1(\mathbf{r}). \quad (75)$$

Solving Eq. (75) for ρ_1 and inserting the result into Eq. (74), one reproduces exactly Eq. (68). This can be seen as an alternative to the recent derivation of superfluid hydrodynamics from the underlying microscopic theory in Ref. [22].

The analysis of the other limit, $T \geq T_c$, is more difficult. In this limit, the gap Δ_0 vanishes and, consequently,

$$E_0(\mathbf{r}, \mathbf{p}) = |h_0(\mathbf{r}, \mathbf{p})|, \quad (76)$$

$$\nu_{1ev}(\mathbf{r}, \mathbf{p}) = \text{sgn}[p - p_F(\mathbf{r})]\tilde{\varrho}_{1ev}(\mathbf{r}, \mathbf{p}). \quad (77)$$

In addition, one has $\varphi(\mathbf{r})=1$, $A(\mathbf{r})=0$, $\rho_1(\mathbf{r})=\rho_{1\nu}(\mathbf{r})$, and $\tilde{\Delta}_1(\mathbf{r})=\Delta_{1\nu}(\mathbf{r})=0$. Using these relations, and considering separately the two cases $p < p_F$ (i.e., $h_0 < 0$) and $p > p_F$ (i.e., $h_0 > 0$), one can convince oneself that Eqs. (66) and (67) reduce to

$$\begin{aligned} \dot{\tilde{\varrho}}_1 - \{h_0, \tilde{\varrho}_1\} = & \frac{f'(h_0)}{m} \left(-\mathbf{p} \cdot \nabla (V_{1\text{ ext}} + g\rho_1 - \hbar\dot{\phi}_1) \right. \\ & \left. + \frac{\hbar}{m}(\mathbf{p} \cdot \nabla)^2 \phi_1 - \hbar[\nabla(V_{0\text{ ext}} + g\rho_0)] \cdot \nabla \phi_1 \right) \end{aligned} \quad (78)$$

and

$$\dot{\rho}_1(\mathbf{r}) + \nabla \cdot \mathbf{j}_{1\nu}(\mathbf{r}) - \frac{\hbar}{m}\nabla \cdot \rho_0(\mathbf{r})\nabla\phi_1(\mathbf{r}) = 0. \quad (79)$$

As we will see in a moment, these two equations are not independent of each other. Hence, they do not allow us to determine $\tilde{\varrho}_1(\mathbf{r}, \mathbf{p})$ and $\phi_1(\mathbf{r})$ in a unique way. This is, in fact, very reasonable since the condition $\text{Im}\tilde{\Delta}=0$ fixing the phase ϕ becomes meaningless above T_c , where $\tilde{\Delta}=0$, and therefore the function ϕ should be completely arbitrary in this case. The relevant physical quantity, which of course should be unique, is $\varrho_1(\mathbf{r}, \mathbf{p})$. Linearizing Eq. (16) and using $\varrho_0(\mathbf{r}, \mathbf{p})=f[h_0(\mathbf{r}, \mathbf{p})]$, we can express $\tilde{\varrho}_1(\mathbf{r}, \mathbf{p})$ in terms of $\varrho_1(\mathbf{r}, \mathbf{p})$ as follows:

$$\tilde{\varrho}_1(\mathbf{r}, \mathbf{p}) = \varrho_1(\mathbf{r}, \mathbf{p}) - \frac{\hbar}{m}f'[h_0(\mathbf{r}, \mathbf{p})]\mathbf{p} \cdot \nabla \phi_1. \quad (80)$$

If we insert this into Eq. (78), all terms containing the phase ϕ_1 drop out, and we are left with

$$\dot{\varrho}_1 - \{h_0, \varrho_1\} = f'(h_0)\frac{\mathbf{p}}{m} \cdot \nabla (V_{1\text{ ext}} + g\rho_1). \quad (81)$$

This is nothing but the linearized form of the Vlasov equation,

$$\dot{\varrho}_1 - \{h_0, \varrho_1\} = \{h_1, \varrho_0\}, \quad (82)$$

with $h_1=V_{1\text{ ext}}+g\rho_1$. It remains to check that the continuity equation (79) is satisfied for arbitrary functions ϕ_1 , if ϱ_1 fulfills Eq. (81). To that end, we multiply Eq. (81) by \mathbf{p} and integrate over \mathbf{p} , which leads to the usual continuity equation

$$\dot{\rho}_1(\mathbf{r}) + \nabla \cdot \mathbf{j}_1(\mathbf{r}) = 0. \quad (83)$$

With the help of Eq. (80) the current \mathbf{j}_1 can be written as

$$\mathbf{j}_1(\mathbf{r}) = \mathbf{j}_{1\nu}(\mathbf{r}) - \frac{\hbar}{m}\rho_0(\mathbf{r})\nabla\phi_1(\mathbf{r}). \quad (84)$$

Inserting this into Eq. (83), we indeed recover Eq. (79). Since we did not make any assumptions about the function $\phi_1(\mathbf{r})$, we conclude that it is completely arbitrary, as it should be.

III. SIMPLE EXAMPLES

A. Sound wave in a uniform system

In this section we are considering a particularly simple excitation, namely a sound wave traveling through a uniform medium. This case has already been studied by Leggett [12] many years ago (except for the numerical evaluation of the integrals) by using the standard techniques of normal and anomalous Green's functions. Our purpose in the present section is therefore to check that our apparently very complicated equations (66) and (67) correctly interpolate between the limits of zero and critical temperature.

Since the medium is assumed to be uniform, the equilibrium quantities do not depend on \mathbf{r} . We consider an excitation operator of the form

$$V_{1\text{ ext}}(\mathbf{r}; t) = \hat{V}_{1\text{ ext}} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}. \quad (85)$$

As usual, in order to ensure that the perturbation vanishes for $t \rightarrow -\infty$, one can assume that ω has an infinitesimal positive imaginary part. From translational invariance it is clear that all quantities describing the deviations from equilibrium will also have the form of a plane wave, with the same wave vector \mathbf{k} and frequency ω as the excitation. Like $\hat{V}_{1\text{ ext}}$, the amplitudes will be marked by a hat over the corresponding symbol. Concerning the phase ϕ_1 , it turns out to be convenient to parametrize it in the form

$$\phi_1(\mathbf{r}; t) = \hat{\phi}_1 \frac{i}{\omega} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}. \quad (86)$$

The Poisson bracket on the lhs of Eq. (66) now becomes

$$\{E_0, \nu_1\} = -i \frac{h_0}{E_0} \frac{\mathbf{p} \cdot \mathbf{k}}{m} \hat{\nu}_1 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}, \quad (87)$$

and Eq. (66) can easily be solved for $\hat{\nu}_1$:

$$\hat{v}_1 = \frac{-f'(E_0) \frac{\mathbf{p} \cdot \mathbf{k}}{m\omega} \frac{h_0}{E_0} \left(\frac{h_0 \hat{V}_{1 \text{ ext}} - \hbar \hat{\phi}_1 + g \hat{\rho}_{1\nu}}{1 + gA} - \frac{\Delta_0 \hat{\Delta}_{1\nu}}{E_0 gA} + \frac{\mathbf{p} \cdot \mathbf{k}}{m\omega} \hbar \hat{\phi}_1 \right)}{1 - \frac{h_0 \mathbf{p} \cdot \mathbf{k}}{E_0 m\omega}}. \quad (88)$$

Of course, the quantities $\hat{\rho}_{1\nu}$ and $\hat{\Delta}_{1\nu}$ on the rhs depend themselves on \hat{v}_1 . Therefore the next step consists in inserting this expression for \hat{v}_1 into Eqs. (56) and (57). The integrals over the angle between \mathbf{p} and \mathbf{k} can be evaluated in closed form. For the remaining integrals over p , we will again exploit the fact that the gap and the temperature are much smaller than the Fermi energy, as we did already in Sec. II B. We thus replace $p^2 dp$ by $mp_F d\xi$, and in the integrand we replace p by p_F , except for h_0 and E_0 , which must be replaced by ξ and $\sqrt{\xi^2 + \Delta_0^2}$, respectively. Like the coefficient B in Sec. II B, the integrals that lead to the coupling between the equations for $\hat{\rho}_{1\nu}$ and $\hat{\Delta}_{1\nu}$ are zero within this approximation, i.e., they are of higher order in Δ/ϵ_F or T/ϵ_F and can be neglected. The resulting equation for $\hat{\rho}_{1\nu}$ reads as

$$\hat{\rho}_{1\nu} = -\frac{mp_F}{2\pi^2 \hbar^3} \left(\frac{\hat{V}_{1 \text{ ext}} - \hbar \hat{\phi}_1 + g \hat{\rho}_{1\nu}}{1 + gA} I_2(s) + \hbar \hat{\phi}_1 I_0(s) \right). \quad (89)$$

Here we have introduced the abbreviation

$$I_n(s) = -\int d\xi f'(E) \left(\frac{\xi}{E} \right)^n \left[1 - \frac{sE}{\xi} \operatorname{arctanh} \left(\frac{\xi}{sE} \right) \right], \quad (90)$$

where $E = \sqrt{\xi^2 + \Delta_0^2}$, and s denotes the dimensionless ratio of the sound velocity $c = \omega/k$ and the Fermi velocity $v_F = p_F/m$,

$$s = \frac{c}{v_F} = \frac{m\omega}{p_F k}. \quad (91)$$

Although not marked explicitly, $I_n(s)$ depends not only on s but also on the ratio T/T_c (analogously to the function φ). Note that the integrals $I_n(s)$ have a branch cut along the real axis from $s = -1$ to $s = 1$. The infinitesimal imaginary part of ω , i.e., of s , fixes the sign of the imaginary part of $I_n(s)$.

Until now we have one equation for two unknown quantities, $\hat{\rho}_{1\nu}$ and $\hat{\phi}_1$. The second equation can be obtained from the continuity equation (67). It is evident that the current $\mathbf{j}_{1\nu}$ flows in the longitudinal direction, such that it can be written in the form

$$\mathbf{j}_{1\nu}(\mathbf{r}; t) = \hat{j}_{1\nu} \frac{\mathbf{k}}{k} e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t}. \quad (92)$$

We will now express $\hat{j}_{1\nu}$ in terms of $\hat{V}_{1 \text{ ext}}$, $\hat{\phi}_1$, and $\hat{\rho}_{1\nu}$ by inserting Eq. (88) into Eq. (49). The integration over \mathbf{p} is done as explained above for the case of $\hat{\rho}_{1\nu}$, and the result reads as

$$\hat{j}_{1\nu} = -\frac{mcp_F}{2\pi^2 \hbar^3} \left(\frac{\hat{V}_{1 \text{ ext}} - \hbar \hat{\phi}_1 + g \hat{\rho}_{1\nu}}{1 + gA} I_0(s) + \hbar \hat{\phi}_1 I_{-2}(s) \right) - \frac{\rho_n \hbar \hat{\phi}_1}{mc}. \quad (93)$$

In the last term, we have introduced the ‘‘normal density’’ of the system, ρ_n , which is given by

$$\rho_n = \rho_0 - \rho_s = -\rho_0 \int d\xi f'(E_0). \quad (94)$$

Correspondingly, ρ_s is the ‘‘superfluid density.’’ Note that the ratios ρ_n/ρ_0 and ρ_s/ρ_0 depend only on one parameter, namely T/T_c . The numerical results for ρ_n/ρ_0 and ρ_s/ρ_0 as functions of T/T_c are shown in Fig. 2.

Inserting Eq. (93) into the continuity equation (67), one obtains the second equation that is needed for determining $\hat{\rho}_{1\nu}$ and $\hat{\phi}_1$:

$$\frac{\hat{\rho}_{1\nu} - A(\hat{V}_{1 \text{ ext}} - \hbar \hat{\phi}_1)}{1 + gA} + \frac{mp_F}{2\pi^2 \hbar^3} \left(\frac{\hat{V}_{1 \text{ ext}} - \hbar \hat{\phi}_1 + g \hat{\rho}_{1\nu}}{1 + gA} I_0(s) + \hbar \hat{\phi}_1 I_{-2}(s) \right) - \frac{\rho_s \hbar \hat{\phi}_1}{mc^2} = 0. \quad (95)$$

In principle, we could now solve Eqs. (89) and (95) for the two unknown variables $\hat{\rho}_{1\nu}$ and $\hat{\phi}_1$. However, it is more transparent to use the amplitude of the total density oscillations, $\hat{\rho}_1$, as a variable instead of the auxiliary quantity $\rho_{1\nu}$. Expressing $\hat{\rho}_{1\nu}$ in terms of $\hat{\rho}_1$ with the help of Eq. (60), we rewrite Eqs. (89) and (95) as

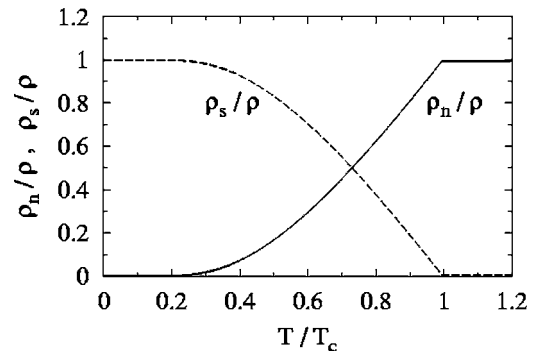


FIG. 2. Temperature dependence of ρ_n/ρ_0 (solid line) and ρ_s/ρ_0 (dashed line).

$$\left(1 + \frac{gmp_F}{2\pi^2\hbar^3}[1 - \varphi + I_2(s)]\right)\hat{\rho}_1 - \frac{mp_F}{2\pi^2\hbar^3}[1 - \varphi + I_2(s) - I_0(s)]\hbar\hat{\phi}_1 = -\frac{mp_F}{2\pi^2\hbar^3}[1 - \varphi + I_2(s)]\hat{V}_1 \text{ ext}, \quad (96a)$$

$$\left(1 + \frac{gmp_F}{2\pi^2\hbar^3}I_0(s)\right)\hat{\rho}_1 + \frac{mp_F}{2\pi^2\hbar^3}\left(I_{-2}(s) - I_0(s) - \frac{1}{3s^2}\frac{\rho_s}{\rho_0}\right)\hbar\hat{\phi}_1 = -\frac{mp_F}{2\pi^2\hbar^3}I_0(s)\hat{V}_1 \text{ ext}. \quad (96b)$$

It is straightforward to solve this 2×2 system of equations for $\hat{\rho}_1$. Let us introduce the response function Π , defined such that

$$\hat{\rho}_1 = \frac{mp_F}{2\pi^2\hbar^3}\Pi(s; T/T_c; k_F a)\hat{V}_1. \quad (97)$$

The first term has been factored out in order to make Π dimensionless. From the system of equations (96) one can see that Π is a function of s and the two parameters T/T_c and

$$k_F a = \frac{gmp_F}{4\pi\hbar^3}. \quad (98)$$

The explicit expression for Π can most conveniently be expressed in the form

$$\Pi(s; T/T_c; k_F a) = \frac{\Pi_0(s; T/T_c)}{1 - \frac{2k_F a}{\pi}\Pi_0(s; T/T_c)}, \quad (99)$$

where Π_0 is the response function in the limit $k_F a \rightarrow 0$:

$$\Pi_0 = \frac{(1 - \varphi + I_2)\left(\frac{1}{3s^2}\frac{\rho_s}{\rho_0} - I_{-2}\right) + I_0^2}{1 - \frac{1}{3s^2}\frac{\rho_s}{\rho_0} - \varphi + I_2 + I_{-2} - 2I_0}. \quad (100)$$

Note that these expressions coincide exactly with the quantum mechanical result in the long-wavelength and low-frequency limit, as given by Eqs. (68) and (69) of Ref. [12]. In order to see this, it is sufficient to observe that after integration over the solid angle the quantities α , ζ , and η defined in Eq. (65) of Ref. [12] can be expressed in terms of our integrals as $\alpha = (1 - \varphi + I_2 - I_0)/2$, $\zeta = [\rho_s/(3\rho_0) + s^2(I_0 - I_{-2})]/2$, and $\eta = -I_0$. In our case of a pure s -wave interaction, the Landau parameters in Eq. (68) of Ref. [12] are given by $F_0 = 2k_F a/\pi$ and $F_1 = 0$. Then the quantities K_1 and Q of Ref. [12] correspond to our Π and Π_0 , respectively. As stated in Ref. [12], the long-wavelength and low-frequency limit is valid if $\hbar\omega, v_F\hbar k \ll \Delta$. Since our semiclassical result coincides with this limit of the quantum mechanical result, we conclude that this is the condition for the validity of our semiclassical theory. A calculation of the response function beyond the long-wavelength and low-frequency limit can be found in Ref. [23].

The excitation spectrum of the system is characterized by the imaginary part of Π , which is plotted in Fig. 3 for $k_F a = -0.25$ and several temperatures between $0.8T_c$ and T_c . One

can see that at $0.8T_c$ the excitation spectrum exhibits a peak near $s \approx 0.5$, which becomes broader and finally disappears when the temperature approaches T_c .

In order to interpret this behavior, let us again consider the two limits $T \rightarrow 0$ and $T \rightarrow T_c$. In the zero-temperature case, all integrals containing the term $f'(E_0)$ in the integrand vanish, i.e., $\varphi = \rho_n/\rho_0 = I_n(s) = 0$, and the response function reduces to

$$\Pi(s; 0; k_F a) = \frac{1}{3s^2 - 1 - 2k_F a/\pi}. \quad (101)$$

This means that the excitation spectrum is a δ function at

$$s = \sqrt{\frac{1}{3} + \frac{2k_F a}{3\pi}}, \quad (102)$$

corresponding to the hydrodynamic speed of sound.

In the other limit, $T \rightarrow T_c$, one has $\varphi = \rho_n/\rho_0 = 1$. The integrals $I_n(s)$ reduce to

$$I_n(s; T/T_c \geq 1) = 1 - s \operatorname{arctanh} \frac{1}{s}, \quad (103)$$

independent of n , since the factors ξ/E in the integrand of Eq. (90) can be replaced by 1. As a consequence, the two equations of the system (96) become identical and the coefficients in front of $\hat{\phi}_1$ vanish, in accordance with the more general arguments of Sec. II C. Solving for $\hat{\rho}_1$ gives

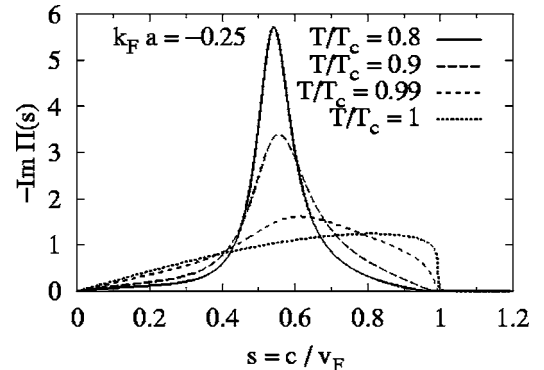


FIG. 3. Excitation spectrum $-\operatorname{Im} \Pi$ of a uniform medium as function of the reduced sound velocity $s = c/v_F$ for different temperatures.

$$\Pi(s; T/T_c \geq 1; k_F a) = \frac{-\left(1 - s \operatorname{arctanh} \frac{1}{s}\right)}{1 + \frac{2k_F a}{\pi} \left(1 - s \operatorname{arctanh} \frac{1}{s}\right)}, \quad (104)$$

in agreement with the usual result of Landau's Fermi-liquid theory for the case of a pure s -wave interaction. If the interaction was repulsive ($a > 0$), Eq. (104) would have a pole at $s > 1$, corresponding to the propagation of zero sound. However, here we are considering the case of an attractive interaction, where zero sound does not exist. Instead there is a continuous spectrum of particle-hole excitations ranging from $s=0$ to $s=1$.

Our numerical results shown in Fig. 3 can be interpreted as follows. At zero temperature, there exists a collective hydrodynamic sound, which is undamped (at least within the present theoretical treatment). As the temperature increases, a normal component consisting of thermally excited quasiparticles builds up. However, at temperatures where ρ_n is already considerably different from zero, the hydrodynamic sound is still practically undamped. The reason for this is that all thermally excited quasiparticles contribute equally to ρ_n , whereas only those quasiparticles whose velocity $dE/dp \approx v_F \xi/E$ is at least equal to the sound velocity c contribute to the Landau damping. At sufficiently high temperature, the Landau damping becomes very strong and the hydrodynamic sound ceases to exist. What remains is a continuum of particle-hole excitations, and the interaction manifests itself only in the rounded edge near $s=1$.

B. Quadrupole mode in a spherical trap

Our main motivation for developing the present semiclassical approach was to apply it to the case of trapped atomic Fermi gases. The simplest example that comes to our mind is the quadrupole oscillation of a Fermi gas in a spherical trap. Even in this case, the \mathbf{r} dependence of the equilibrium quantities induced by the trap potential makes our equations very complicated. Since in this first investigation we are interested in problems that can be solved analytically, we will apply two additional simplifying approximations, which allow us to obtain explicit solutions. A numerical method for solving our equations without additional approximations will be proposed at the end of this section.

Let us start with the linearized equation (66), which has the form

$$\dot{\nu}_1(\mathbf{r}, \mathbf{p}; t) - \{E_0(\mathbf{r}, \mathbf{p}), \nu_1(\mathbf{r}, \mathbf{p}; t)\} = F(\mathbf{r}, \mathbf{p}; t). \quad (105)$$

For its solution we adopt the Green's function method used in Ref. [24] to solve the linearized Vlasov equation for nuclear giant resonances, which is formally very similar to our problem. The starting point is to write the solution of Eq. (105) in the form

$$\nu_1(\mathbf{r}, \mathbf{p}; t) = \int dt' \int d^3 r' \int d^3 p' G(\mathbf{r}, \mathbf{p}, \mathbf{r}', \mathbf{p}'; t-t') \times F(\mathbf{r}', \mathbf{p}'; t'), \quad (106)$$

where G is the Green's function of the differential operator on the lhs of Eq. (105), satisfying

$$\left[\frac{\partial}{\partial t} - \sum_{i=xyz} \left(\frac{\partial E_0}{\partial r_i} \frac{\partial}{\partial p_i} - \frac{\partial E_0}{\partial p_i} \frac{\partial}{\partial r_i} \right) \right] G(\mathbf{r}, \mathbf{p}, \mathbf{r}', \mathbf{p}'; t) = \delta(t) \delta(\mathbf{r} - \mathbf{r}') \delta(\mathbf{p} - \mathbf{p}'). \quad (107)$$

Denoting by $\mathbf{R}(\mathbf{r}, \mathbf{p}; t)$ and $\mathbf{P}(\mathbf{r}, \mathbf{p}; t)$ the solutions of the classical equations of motion,

$$\dot{R}_i = \frac{\partial E_0(\mathbf{R}, \mathbf{P})}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial E_0(\mathbf{R}, \mathbf{P})}{\partial R_i}, \quad (108)$$

satisfying the initial conditions $\mathbf{R}(\mathbf{r}, \mathbf{p}; 0) = \mathbf{r}$ and $\mathbf{P}(\mathbf{r}, \mathbf{p}; 0) = \mathbf{p}$, one can show that

$$G(\mathbf{r}, \mathbf{p}, \mathbf{r}', \mathbf{p}'; t) = \theta(t) \delta[\mathbf{r} - \mathbf{R}(\mathbf{r}', \mathbf{p}'; t)] \delta[\mathbf{p} - \mathbf{P}(\mathbf{r}', \mathbf{p}'; t)] \quad (109)$$

fulfills Eq. (107). Due to time-reversal symmetry and Liouville's theorem, this Green's function can be rewritten as

$$G(\mathbf{r}, \mathbf{p}, \mathbf{r}', \mathbf{p}'; t) = \theta(t) \delta[\mathbf{r}' - \mathbf{R}(\mathbf{r}, -\mathbf{p}; t)] \delta[\mathbf{p}' + \mathbf{P}(\mathbf{r}, -\mathbf{p}; t)]. \quad (110)$$

The latter form renders the phase-space integrals in Eq. (106) trivial. Changing the time integration variable according to $\tau = t - t'$, one obtains

$$\nu_1(\mathbf{r}, \mathbf{p}; t) = \int_0^\infty d\tau F[\mathbf{R}(\mathbf{r}, -\mathbf{p}; \tau), -\mathbf{P}(\mathbf{r}, -\mathbf{p}; \tau); t - \tau]. \quad (111)$$

In the case of a harmonic perturbation,

$$V_{1 \text{ ext}}(\mathbf{r}; t) = \hat{V}_{1 \text{ ext}}(\mathbf{r}) e^{-i\omega t}, \quad (112)$$

the time dependence of ν_1 as well as the explicit time dependence of F will be harmonic, too. We will denote the amplitudes by a hat over the corresponding symbols. Multiplying Eq. (111) by $e^{i\omega t}$, one finds that $\hat{\nu}_1$ is given by the Fourier integral

$$\hat{\nu}_1(\mathbf{r}, \mathbf{p}) = \int_0^\infty d\tau e^{i\omega\tau} \hat{F}[\mathbf{R}(\mathbf{r}, -\mathbf{p}; \tau), -\mathbf{P}(\mathbf{r}, -\mathbf{p}; \tau)]. \quad (113)$$

For the purpose of illustration we want to discuss a simple case in which the classical trajectories are analytically known. We make two approximations: First, we replace the \mathbf{r} -dependent gap $\Delta_0(\mathbf{r})$ by a constant Δ_0 . This approximation implies that ν_{1ev} is odd in ξ and $\tilde{\Delta}_{1\nu}$ can be neglected, as was the case in the preceding section. Second, we will neglect effects from the Hartree mean field as well in the equilibrium state as in the deviations from equilibrium. The second approximation, which is by far not as unrealistic as the first

one, amounts to neglecting all $g\rho_0$ and $g\rho_{1\nu}$ terms and replacing the denominators $1+gA$ in Eq. (66) by 1. The trap potential is assumed to be a spherical harmonic oscillator,

$$V_{0\text{ ext}} = \frac{1}{2}m\Omega^2 r^2. \quad (114)$$

It is evident that the equations of motion (108) conserve E_0 . However, if Δ_0 is a constant, this implies that h_0 is conserved, too, and the solutions of Eqs. (108) are closely related to the those of the ordinary harmonic oscillator. Indeed, it is straightforward to show that the trajectories are given by

$$\mathbf{R}(\mathbf{r}, \mathbf{p}; t) = \mathbf{r} \cos \frac{h_0 \Omega t}{E_0} + \frac{\mathbf{p}}{m\Omega} \sin \frac{h_0 \Omega t}{E_0}, \quad (115a)$$

$$\mathbf{P}(\mathbf{r}, \mathbf{p}; t) = \mathbf{p} \cos \frac{h_0 \Omega t}{E_0} - m\Omega \mathbf{r} \sin \frac{h_0 \Omega t}{E_0}. \quad (115b)$$

Since h_0 and E_0 are constants of the motion, they can likewise be evaluated at (\mathbf{r}, \mathbf{p}) or (\mathbf{R}, \mathbf{P}) .

Due to our approximations, the function F [given by the rhs of Eq. (66)] reduces to

$$\hat{F} = -f'(E_0) \left[-\frac{h_0^2}{E_0^2 m} \mathbf{p} \cdot \nabla (\hat{V}_{1\text{ ext}} + i\hbar\omega\hat{\phi}_1) + \frac{h_0}{E_0} \left(\frac{\mathbf{p}}{m} \cdot \nabla \right)^2 \hat{\phi}_1 - \Omega^2 \frac{h_0}{E_0} \mathbf{r} \cdot \nabla \hat{\phi}_1 \right]. \quad (116)$$

As excitation we choose the quadrupole operator

$$\hat{V}_{1\text{ ext}}(\mathbf{r}) = \alpha m \Omega^2 (\mathbf{r} \otimes \mathbf{r})_{20}, \quad (117)$$

where, explicitly,

$$(\mathbf{v} \otimes \mathbf{w})_{20} = \sum_{\mu\nu} (1\mu 1\nu | 20) v_\mu w_\nu = \frac{2v_z w_z - v_x w_x - v_y w_y}{\sqrt{6}}. \quad (118)$$

The prefactor in Eq. (117) has been chosen such that the coefficient α is dimensionless. Due to the spherical symmetry of the trap, the angular dependence of $\hat{\phi}_1$ must be of the same quadrupolar form as that of $\hat{V}_{1\text{ ext}}$, but the radial dependence could, in principle, be different. Here we make the ansatz that $\hat{\phi}_1$ is proportional to $\hat{V}_{1\text{ ext}}$, i.e.,

$$\hat{\phi}_1(\mathbf{r}) = \beta \frac{m\Omega}{\hbar} (\mathbf{r} \otimes \mathbf{r})_{20}, \quad (119)$$

and we will show afterward that with this ansatz for $\hat{\phi}$ the continuity equation can be satisfied by an appropriate choice of the coefficient β . Quadratic ansätze like Eq. (119), corresponding to a superfluid velocity field that is linear in the coordinates, have frequently been used (see, e.g., Ref. [5]) for the calculation of the frequencies of collective modes in the limit of superfluid hydrodynamics ($T=0$).

Inserting Eqs. (117) and (119) into Eq. (116) and using the explicit form of the trajectories, Eq. (115), we can evaluate the Fourier integral in Eq. (113), with the result

$$\hat{v}_1 = -2f'(E_0) \left[\left(\frac{(\mathbf{p} \otimes \mathbf{p})_{20}}{m} - m\Omega^2 (\mathbf{r} \otimes \mathbf{r})_{20} \right) \frac{h_0}{E_0} \frac{i\beta \frac{\omega}{\Omega} \frac{\Delta_0^2}{E_0^2} - \alpha \frac{h_0^2}{E_0^2}}{\frac{\omega^2}{\Omega^2} - 4 \frac{h_0^2}{E_0^2}} + \Omega (\mathbf{r} \otimes \mathbf{p})_{20} \frac{h_0^2}{E_0^2} \left(\beta - \frac{4\beta \frac{\Delta_0^2}{E_0^2} + i\alpha \frac{\omega}{\Omega}}{\frac{\omega^2}{\Omega^2} - 4 \frac{h_0^2}{E_0^2}} \right) \right]. \quad (120)$$

Now we have to calculate the corresponding current $\hat{\mathbf{j}}$ and density oscillations $\hat{\rho}_{1\nu}$. As detailed in the preceding section, this is accomplished by integrating $\mathbf{p}\hat{v}_1/m$ and \hat{v}_1 , respectively, over \mathbf{p} . Replacing in the integrals $p^2 dp$ by $mp_F(\mathbf{r})d\xi$, p by $p_F(\mathbf{r})$, h_0 by ξ , and E_0 by $\sqrt{\xi^2 + \Delta_0^2}$, we obtain

$$\hat{\mathbf{j}}_{1\nu}(\mathbf{r}) = \rho_0(\mathbf{r})\Omega(\beta[\varphi - 4I_{22}(z)] - i\alpha z I_{20}(z)) \nabla (\mathbf{r} \otimes \mathbf{r})_{20}, \quad (121)$$

$$\hat{\rho}_{1\nu}(\mathbf{r}) = \frac{m^2 \Omega^2 p_F(\mathbf{r})}{\pi^2 \hbar^3} [\alpha I_{40}(z) - i\beta z I_{22}(z)] (\mathbf{r} \otimes \mathbf{r})_{20}, \quad (122)$$

with the abbreviations

$$z = \frac{\omega}{\Omega}, \quad (123)$$

$$I_{ij}(z) = - \int d\xi f'(E) \frac{\xi^i \Delta_0^j}{E^{i+j}} \frac{1}{z^2 - 4\xi^2/E^2}, \quad (124)$$

where $E = \sqrt{\xi^2 + \Delta_0^2}$. From its definition it is evident that $I_{40} = I_{20} - I_{22}$, such that it is sufficient to evaluate two of these integrals numerically. The functions $I_{ij}(z)$ have a branch cut along the real axis from $z=-2$ to $z=2$. Remember that ω , and therefore also z , is assumed to have an infinitesimal positive imaginary part, fixing the sign of the imaginary part of $I_{ij}(z)$.

As stated above, the coefficient β must be determined by

the continuity equation (67). Due to our approximation to neglect the Hartree field, the denominator $1+gA(\mathbf{r})$ in the first term of Eq. (67) can be replaced by 1, and the Fermi momentum can be given in closed form:

$$p_F(\mathbf{r}) = \sqrt{2m\left(\mu - \frac{1}{2}m\Omega^2 r^2\right)}. \quad (125)$$

Inserting the results for $\hat{\mathbf{J}}_{1\nu}$ and $\hat{\rho}_{1\nu}$ into the continuity equation, one finds that the ansatz (119) indeed allows us to satisfy the continuity equation, and the corresponding solution for the coefficient β reads as

$$\beta = iz\alpha \frac{1 - \varphi + 2I_{22}(z)}{(1 - \varphi)(z^2 - 2) + 2I_{22}(z)(z^2 - 4)}. \quad (126)$$

This expression can be used to obtain $\hat{\rho}_{1\nu}$. Here we will immediately give the result for the amplitude of the total density oscillations, i.e., $\hat{\rho}_1 = \hat{\rho}_{1\nu} - A(\hat{V}_{1\text{ext}} + i\hbar\omega\hat{\phi}_1)$, which we write in the form

$$\hat{\rho}_1(\mathbf{r}) = \alpha \frac{m^2\Omega^2 p_F(\mathbf{r})}{\pi^2\hbar^3} (\mathbf{r} \otimes \mathbf{r})_{20} \Pi(z), \quad (127)$$

with

$$\Pi(z) = I_{20}(z) + \frac{[1 - \varphi + 2I_{22}(z)][1 - \varphi + 4I_{22}(z)]}{(1 - \varphi)(z^2 - 2) + 2I_{22}(z)(z^2 - 4)}. \quad (128)$$

Before discussing numerical results, let us again study the two extreme cases $T=0$ and $T \geq T_c$. In the zero-temperature limit, all integrals φ and I_{ij} are zero, and hence the response function becomes

$$\Pi(z; T/T_c = 0) = \frac{1}{z^2 - 2}, \quad (129)$$

i.e., it has a single pole at the hydrodynamical frequency

$$\omega = \sqrt{2}\Omega. \quad (130)$$

In the case $T \geq T_c$, i.e., in the normal phase, we have $\varphi=1$, and in the definition (124) we can replace Δ_0 and E_0 by 0 and ξ , respectively, such that we obtain

$$I_{20}(z; T/T_c \geq 1) = \frac{1}{z^2 - 4}, \quad (131a)$$

$$I_{22}(z; T/T_c \geq 1) = 0. \quad (131b)$$

Thus, the response function reduces to

$$\Pi(z; T/T_c \geq 1) = \frac{1}{z^2 - 4}. \quad (132)$$

Like in the zero-temperature case, we have a single pole, but now at a frequency that is higher by a factor of $\sqrt{2}$. The reason for the difference of the two frequencies is as follows. In the superfluid phase, the momentum distribution stays spherical during the oscillation. In contrast to this, in the normal phase, the momentum distribution is deformed in the opposite direction as the density in coordinate space. This

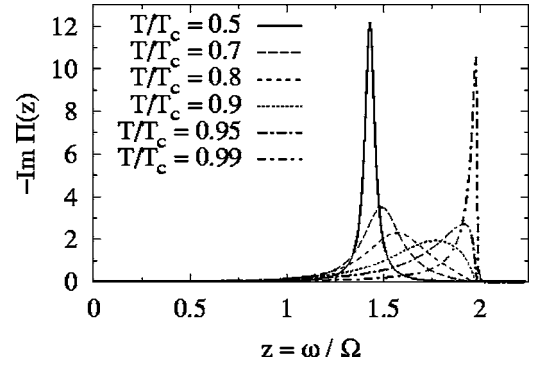


FIG. 4. Quadrupole excitation spectrum $-\text{Im} \Pi$ of a harmonically trapped gas as a function of the excitation frequency (in units of the trap frequency) for different temperatures. The spatial dependence of the gap as well as the Hartree mean field have been neglected.

deformation of the Fermi sphere costs kinetic energy, which increases the restoring force and thereby the frequency of the oscillation.

At intermediate temperatures $0 < T < T_c$, the excitation spectrum is continuous and it is characterized by the imaginary part of $\Pi(z)$, which is shown in Fig. 4 for a set of temperatures between $0.5T_c$ and T_c . At $0.5T_c$, the spectrum exhibits a sharp peak at the hydrodynamic frequency $z = \sqrt{2}$, the weak broadening being due to Landau damping. With increasing temperature, the Landau damping becomes more important, and at the same time the centroid of the distribution moves to higher frequencies. Above $0.8T_c$, however, the width of the peak does not increase any more with temperature, but it decreases. Finally, when the temperature approaches T_c , the peak becomes again very sharp and, not surprisingly, it lies at the frequency $z=2$ predicted by the Vlasov equation.

One might ask the question whether the approximations made in this section are justified or not. Let us therefore compare our results with those of a QRPA calculation [7], where, apart from the mean-field approximation leading to the Bogoliubov-de Gennes equations, no approximations are made. Qualitatively our semiclassical results show the main features of this quantum-mechanical calculation: the hydrodynamic mode at zero temperature, its damping at intermediate temperatures, and the subsequent reappearance of an undamped collective mode with a higher frequency in the normal phase. That our frequency in the normal phase is exactly equal to $z=2$ is a consequence of neglecting the Hartree mean field. However, in the range of validity of our theory ($k_F|a| \ll 1$), the Hartree mean field cannot shift the frequency very much (in Ref. [7], e.g., the frequency is shifted from 2 to ≈ 2.2). We therefore believe that this effect is not very important. More problematic is the constant-gap approximation that we needed for the analytical solution of the equations of motion (108). Because of this approximation, there are no quasiparticles having energies below Δ_0 , and, as a consequence, the Landau damping sets in at rather high temperatures. In the full calculation, however, the lowest-lying quasiparticles are those whose wave functions are localized near the surface, where the gap is small, and

that have much smaller energies than the central value of the gap. Therefore within the full calculation the Landau damping is already quite important at very low temperatures. In the semiclassical formalism these low-lying quasiparticles can be understood as quasiparticles bouncing back and forth between the potential wells created by the trap potential and the spatially varying gap (Andreev reflection) [20,25]. The inclusion of this effect would require a numerical solution of the equations of motion (108).

This leads us to a possible numerical method for solving even the original (i.e., not linearized) kinetic equation (35). In nuclear physics, the Vlasov equation (usually complemented by a collision term) is routinely solved by the so-called test-particle method, e.g., in order to simulate heavy-ion collisions [26]. Recently this method has also been applied to the solution of the Vlasov equation with collision term for trapped atomic Fermi gases [27] and of a Vlasov-like equation for trapped fermion-boson mixtures [28], and it seems to be straightforward to generalize it to our case. The basic idea of the method is as follows. Instead of calculating the time evolution of the continuous quasiparticle distribution function $\nu(\mathbf{r}, \mathbf{p}; t)$, one can use a finite number of “test quasiparticles” and follow their motion in phase space by solving numerically the equations of motion,

$$\dot{R}_i = \frac{\partial E(\mathbf{R}, \mathbf{P})}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial E(\mathbf{R}, \mathbf{P})}{\partial R_i}, \quad (133)$$

for all test quasiparticles simultaneously. Of course, the quasiparticle energy $E(\mathbf{r}, \mathbf{p}; t)$ contains the mean fields $g\rho(\mathbf{r}; t)$ and $\tilde{\Delta}(\mathbf{r}; t)$, which must be calculated at each time step from the actual quasiparticle distribution. At the same time, the phase ϕ must be calculated at each time step from the continuity equation. This seems to be a tractable task, which will be addressed in a subsequent publication.

IV. SUMMARY AND CONCLUSIONS

In the first part of the present article, we derived a set of semiclassical equations describing the dynamics of a collisionless superfluid Fermi gas by taking the $\hbar \rightarrow 0$ limit of the TDHFB or time-dependent Bogoliubov-de Gennes equations. In the limits of zero and critical temperature, these equations reproduce superfluid hydrodynamics and the Vlasov equation, respectively. At intermediate temperatures, there is a complicated interplay between the dynamics of the superfluid component of the system, governed by the function $\phi(\mathbf{r})$, which is related to the phase of the gap $\Delta(\mathbf{r})$, and the dynamics of the normal component, which is described by the quasiparticle distribution function $\nu(\mathbf{r}, \mathbf{p})$. The dynamical equation for ν formally corresponds to the usual Vlasov equation with ϱ and h replaced by ν and E , respectively, while the function ϕ is determined by the continuity equation. The latter point can be seen most easily in the linearized version of the equations for small deviations from equilibrium.

In the second part, we gave an illustration of our equations by applying them to two simple cases, where analytical

solutions could be found. The first example we studied was a sound wave traveling in a uniform system. In this case we could reproduce the usual hydrodynamic speed of sound at zero temperature. At nonzero temperatures below T_c , the sound wave suffers strong Landau damping because of its coupling to thermally excited quasiparticles. For $T \rightarrow T_c$ the excitation spectrum continuously goes over into that of the usual particle-hole continuum (with RPA corrections) that is found above T_c .

The second example was the quadrupole mode of a Fermi gas in a spherical trap. Applying the approximation of a constant gap and neglecting the Hartree field, we were able to solve the linearized quasiparticle kinetic equation exactly also for this case. We could qualitatively reproduce the most important results of quantum mechanical QRPA calculations: At zero temperature, there is an undamped collective mode at the hydrodynamic frequency $\omega = \sqrt{2}\Omega$, which becomes strongly damped at low temperatures ($0 < T \ll T_c$). At a certain temperature the damping rate reaches a maximum, and above that temperature it decreases until at $T = T_c$ an undamped collective mode reappears at the frequency predicted by the Vlasov equation, which is higher than the hydrodynamic frequency. However, quantitatively the agreement with the QRPA calculation is not yet satisfactory, essentially because we replaced the gap by an \mathbf{r} -independent constant. We suggested to use the test-particle method for the numerical solution of the equations in the case of a spatially varying gap $\Delta(\mathbf{r})$, which at the same time would allow us to include the Hartree field and to treat strong deviations from equilibrium, like the expansion of the gas after the trap is switched off.

As long as the gas is close to equilibrium, i.e., as long as the Fermi surface can be regarded as spherical, the effect of collisions is very small due to Pauli blocking of the final states. However, in the case of strong deviations from equilibrium, like during the expansion of the cloud when the trap is switched off, the deformation of the Fermi sphere can lead to rather important collisional effects [29]. Therefore, in this case it is necessary to include the collision term into the theory. This is an interesting problem that should be addressed in a future investigation. For normal-fluid trapped Fermi gases there exist already some calculations that take the collision term into account [27,29]. The more complicated case of paired Fermi systems with collisions has been considered, e.g., in the context of superfluid ^3He [14,15].

Since our equations were obtained as the $\hbar \rightarrow 0$ limit of the TDHFB equations, only the leading gradient terms are included. This can be seen, e.g., in our results for the sound wave, which in fact are the long-wavelength and low-frequency limit of the quantum mechanical result that can be obtained by diagrammatic techniques. For the system in a trap this means in particular that the gradients of the trapping potential, which are related to the trap frequency Ω , must not be too strong. But what is “too strong”? In the normal phase ($\Delta = 0$) the $\hbar \rightarrow 0$ limit works extremely well if $\hbar\Omega \ll \mu$, which is always the case in the experimental situations. In the superfluid phase ($\Delta \neq 0$) the relevant condition reads $\hbar\Omega \ll \Delta$ [7,16], which is much more difficult to satisfy, especially for the radial trap frequency, which is usually much

larger than the axial one. In spite of this limitation, we believe that the semiclassical approach presented here will be useful, since at present no quantum mechanical calculation is able to describe the dynamics of systems with more than $\approx 10^4$ atoms, especially in the case of deformed traps.

ACKNOWLEDGMENTS

We are grateful to E. Balbutsev for numerous discussions. P. S. wants to thank F. Gulminelli for an earlier collaboration on dynamical aspects of superfluidity.

-
- [1] M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, C. Chin, J. Hecker Denschlag, and R. Grimm, *Phys. Rev. Lett.* **92**, 203201 (2004).
 - [2] K. M. O'Hara, S. L. Hemmer, M. E. Gehm, S. R. Granade, and J. E. Thomas, *Science* **298**, 2179 (2002).
 - [3] J. Kinast, S. L. Hemmer, M. E. Gehm, A. Turlapov, and J. E. Thomas, *Phys. Rev. Lett.* **92**, 150402 (2004); J. Kinast, A. Turlapov, and J. E. Thomas, *Phys. Rev. A* **70**, 051401(R) (2004).
 - [4] C. Menotti, P. Pedri, and S. Stringari, *Phys. Rev. Lett.* **89**, 250402 (2002).
 - [5] M. Cozzini and S. Stringari, *Phys. Rev. Lett.* **91**, 070401 (2003).
 - [6] S. Stringari, *Europhys. Lett.* **65**, 749 (2004).
 - [7] M. Grasso, E. Khan, and M. Urban, *Phys. Rev. A* **72**, 043617 (2005).
 - [8] E. Taylor and A. Griffin, e-print cond-mat/0509040.
 - [9] P. Massignan, G. M. Bruun, and H. Smith, *Phys. Rev. A* **71**, 033607 (2005).
 - [10] J. R. Schrieffer, *Theory of Superconductivity* (Benjamin, New York, 1964).
 - [11] A. J. Leggett, *Phys. Rev.* **140**, A1869 (1965).
 - [12] A. J. Leggett, *Phys. Rev.* **147**, 119 (1966).
 - [13] O. Betbeder-Matibet and P. Nozières, *Ann. Phys. (N.Y.)* **51**, 392 (1969).
 - [14] J. W. Serene and D. Rainer, *Phys. Rep.* **101**, 221 (1983).
 - [15] D. Vollhardt and P. Wölfle, *The Superfluid Phases of Helium 3* (Taylor & Francis, London, 1990).
 - [16] M. Urban and P. Schuck, *Phys. Rev. A* **67**, 033611 (2003); M. Urban, *ibid.* **71**, 033611 (2005).
 - [17] N. Nygaard, Ph.D. thesis, University of Maryland, 2004.
 - [18] P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer-Verlag, Berlin, 1980).
 - [19] G. M. Bruun and B. R. Mottelson, *Phys. Rev. Lett.* **87**, 270403 (2001).
 - [20] Y. Ohashi and A. Griffin, e-print cond-mat/0503641.
 - [21] G. Bruun, Y. Castin, R. Dum, and K. Burnett, *Eur. Phys. J. D* **7**, 433 (1999).
 - [22] G. Tonini and Y. Castin, e-print cond-mat/0504612.
 - [23] A. Minguzzi, G. Ferrari, and Y. Castin, *Eur. Phys. J. D* **17**, 49 (2001).
 - [24] H. Kohl, P. Schuck, and S. Stringari, *Nucl. Phys. A* **459**, 265 (1986).
 - [25] G. M. Bruun and H. Heiselberg, *Phys. Rev. A* **65**, 053407 (2002).
 - [26] G. F. Bertsch and S. Das Gupta, *Phys. Rep.* **160**, 190 (1988).
 - [27] F. Toschi, P. Vignolo, S. Succi, and M. P. Tosi, *Phys. Rev. A* **67**, 041605(R) (2003); F. Toschi, P. Capuzzi, S. Succi, P. Vignolo, and M. P. Tosi, *J. Phys. B* **37**, S91 (2004).
 - [28] T. Maruyama, H. Yabu, T. Suzuki, e-print cond-mat/0505357.
 - [29] B. Jackson, P. Pedri, and S. Stringari, *Europhys. Lett.* **67**, 524 (2004).