Quantum control limited by quantum decoherence

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We describe quantum controllability under the influences of the quantum decoherence induced by the quantum control itself. It is shown that, when the controller is considered as a quantum system, it will entangle with its controlled system and then cause quantum decoherence in the controlled system. In competition with this induced decoherence, the controllability will be limited by some uncertainty relation in a well-armed quantum control process. In association with the phase uncertainty and the standard quantum limit, a general model is studied to demonstrate the possibility of realizing a decoherence-free quantum control with a finite energy within a finite time. It is also shown that if the operations of quantum control are to be determined by the initial state of the controller, then due to the decoherence which results from the quantum control itself, there exists a low bound for quantum controllability.

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I. INTRODUCTION

Generally an external field can be utilized to manipulate the time evolution of a quantum system from an arbitrary initial state to reach any wanted target state. If the external field is classical and can be artificially controlled to be time dependent, then we refer to this kind of manipulation as a classical control [1]. In quantum computations [2], the quantum logic gate operations can be regarded as classical controls in most cases where the controller is essentially classical and the control can be turned on or off classically at certain instants.

In this paper we consider quantum control in which the controller is quantized and obeys the laws of quantum mechanics. It is shown that the back action of the controlled system should be considered, which may have a negative side effect on the controllability. There are two motivations for our investigations.

First, it is exciting to explore the finiteness of human beings' abilities to control nature and a "down-to-earth" starting point for this exploration in physics should be a concrete model even though it is oversimplified. With some reasonable models one could demonstrate how the fundamental laws of physics impose limits on controllability in principle. These refer to some basic issues in physics, such as the energy bound, the basic precision of measurement [or standard quantum limit (SQL) [3]]. It is emphasized that quantum decoherence may result from *the control itself* when the controller is essentially considered as a quantum subsystem.

Second, though the physical implementation of quantum computation seems to be difficult, the huge power of quantum computation has been demonstrated by some quantum algorithms in principle. The limit of quantum control can bring a physical limit to quantum computation architecture since it is based on complete quantum blocks including the controller. Lloyd discussed how the physical constants impose a limit on the power and memory in the quantum computer [4], while Ozawa [5] and Gea-Banacloche [6] considered the conservation law and the minimum energy requirement for quantum computation, respectively. Our present study can also be regarded as a part of the growing body of the explorations in this direction.

In Sec. II, we start with a model with a single-mode field as a controller and a two-level system (qubit) as the controlled system. We found that it is possible to implement some phase gate controls without inducing decoherence to the controlled system. However, the single-mode example is far from practical cases, and thus we further study quantum control in a more general case in Sec. III. In Sec. IV the control-induced decoherence is explained as a phase uncertainty by associating it with the SQL. In Sec. V the obtained results are highlighted as the complementarity of controllability and control-induced decoherence. An inequality similar to the Heisenberg uncertainty relation is presented as the accurate bound of quantum gates under quantum control.

II. EXACTLY SOLUBLE MODEL FOR QUANTUM CONTROL

To have a clear picture of quantum control, let us first start with a simple model. The total system that we concern is closed, which consists of the controller C with Hamiltonian H_c and the controlled system Q with Hamiltonian H_q. The system is in the initial states $|\psi_c(0,R)\rangle$ and $|\psi_q(0)\rangle = \Sigma_n c_n|n\rangle$, respectively, where R represents the controlling parameters. For a given target state $|\psi_t\rangle$ of Q, quantum control is described as a factorized evolution

$$|\psi_q(0)\rangle \otimes |\psi_c(0,R)\rangle \rightarrow |\psi_q(T)\rangle \otimes |\psi_c(T)\rangle \qquad (1)$$

of the total system driven by the interaction Hamiltonian H_{qc} within the time duration $(0,T)$. If one could choose an appropriate initial state and the corresponding parameters R such that the partial wave function $|\psi_q(T)\rangle \equiv U_q(T)|\psi_q(0)\rangle$ is

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just the target one $|\psi_t\rangle$, where a global phase difference is allowed, then we could say that an ideal quantum control is realized. Usually $U_q(T)$ defines a quantum logic operation in quantum computation.

We now consider an exactly soluble example, where the controlled system is a qubit with two basis states $|0\rangle$ and $|1\rangle$ and the controller is a single-mode boson field with free Hamiltonian $H_c=\hbar\omega a^\dagger a$; here, $a^\dagger$ (a) is the creation (annihilation) operator. The interaction

$$H_{qc} = |1\rangle\langle 1| \otimes V \equiv 1\rangle\langle 1| \otimes (ga + g^* a^\dagger) \tag{2}$$

between them is of nondemolition [3]—i.e., $[H_{qc},H_c]\neq 0$ and $[H_{qc},H_q]=0$. Since H_q is conserved during the evolution, we take $H_q=0$ without loss of generality. In the interaction picture the time-dependent potential

$$V_I(t) = gae^{-i\omega t} + \text{H.c.} \tag{3}$$

acts only on the state $|1\rangle$, but not on $|0\rangle$. This Hamiltonian originates from the atom-field system in the large detuning limit, but the problem is greatly simplified for convenience [7].

Now we explore the possibility of automatically creating a phase gate operation

$$|\psi_q(0)\rangle = c_0|0\rangle + c_1|1\rangle \to |\psi_q(t)\rangle = c_0|0\rangle + c_1 e^{i\phi}|1\rangle \tag{4}$$

driven by H_{qc}. Essentially, the phase gate operation is supposed to generate a relative phase ϕ between $|0\rangle$ and $|1\rangle$ and the total system experiences a factorized evolution

$$(c_0|0\rangle + c_1|1\rangle) \otimes |\psi_c(0)\rangle \to (c_0|0\rangle + c_1 e^{i\phi}|1\rangle) \otimes |\psi_c(T)\rangle. \tag{5}$$

We will show that only a class of phase gates with special phases depending on the global parameters, such as the coupling coefficients g and the gate operation time T, can be implemented precisely, while the other phase gates definitely result in a decoherence in the qubit system and can only be implemented in an inaccurate way.

Obviously the Hamiltonian $H=H_{qc}+H_c$ describes a typical conditional dynamics [8]. Let the total system be initially in a superposition of

$$|\Psi(0)\rangle = (c_0|0\rangle + c_1|1\rangle) \otimes |\alpha\rangle, \tag{6}$$

where the boson field is in a coherent state $|\alpha\rangle$. The total system will evolve into an entangled state

$$|\Psi(t)\rangle = c_0|0\rangle \otimes |\alpha\rangle + c_1|1\rangle \otimes e^{i\vec{\Phi}}|\alpha\rangle, \tag{7}$$

where

$$e^{i\vec{\Phi}} = \hat{T}\exp\left(-i\int_0^t V_I(t')dt'\right) \tag{8}$$

is a time-ordered integral. A formal phase operator can be explicitly calculated as

$$\vec{\Phi}(t) = \eta(t)a + \text{H.c.} + \varphi(t) + i\xi(t), \tag{9}$$

where the time-dependent coefficients

$$\eta(t) = i\frac{g}{\omega}(1 - e^{-i\omega t}),$$

$$\varphi(t) = \frac{|g|^2}{\omega^2}(\omega t - \sin\omega t),$$

$$\xi(t) = \frac{|g|^2}{\omega^2}(1 - \cos\omega t) \tag{10}$$

are obtained through the Wei-Norman algebraic technique [9]. Then we can write down the total wave function as an entangled state

$$|\Psi(t)\rangle = c_0|0\rangle \otimes |\alpha\rangle + e^{i\varphi(t)-\xi(t)}c_1|1\rangle \otimes |\alpha + \eta(t)\rangle. \tag{11}$$

Obviously, at the special instants

$$t = T = 2k\pi/\omega, \tag{12}$$

where $k\in\mathbb{Z}$ and both the decay factor $\xi(t)$ and the displacement $\eta(t)$ in the coherent state $|\alpha+\eta(t)\rangle$ vanish. And a real phase

$$\varphi(T) = \phi_s = \frac{|g|^2}{\omega}T \tag{13}$$

occurs in the above entanglement state. Thus we realize a phase gate operation, Eq. (4), of a certain phase ϕ_s, which is induced by the factorized evolution

$$|\Psi(0)\rangle \to |\Psi(T)\rangle = (c_0|0\rangle + c_1 e^{i\phi_s}|1\rangle) \otimes |\alpha\rangle. \tag{14}$$

It defines the reduced density matrix of a pure state,

$$\rho_q = |c_0|^2|0\rangle\langle 0| + |c_1|^2|1\rangle\langle 1| + c_1c_0^* e^{i\phi_s}|1\rangle\langle 0| + \text{H.c.}, \tag{15}$$

for the qubit system.

If the evolution time is not just at the instant $t=T$, the reduced density matrix

$$\rho_r = |c_0|^2|0\rangle\langle 0| + |c_1|^2|1\rangle\langle 1| + c_1c_0^* D(t)|1\rangle\langle 0| + \text{H.c.} \tag{16}$$

is not of a pure state due to the decoherence factor

$$D(t) = \langle\alpha|W(t)|\alpha\rangle = e^{i\phi(t)-\xi(t)}, \tag{17}$$

where

$$\phi(t) = 2\,\text{Im}\left[\frac{(1-e^{-i\omega t})g\alpha}{\omega}\right] + \frac{(\omega t - \sin\omega t)|g|^2}{\omega^2} \tag{18}$$

and

$$\xi(t) = \frac{|g|^2}{\omega^2}(1 - \cos\omega t).$$

The difference between ρ_q and ρ_r can be characterized by the control fidelity $F(t)=\text{Tr}(\rho_q\rho_r)$, which is defined as the overlap of the target state ρ_q and the final state ρ_r. By a straightforward calculation, we have

$$\begin{aligned}F(t) &= 1 - 2|c_0|^2|c_1|^2\{1 - \text{Re}[D(t)e^{i\phi_s}]\} \\ &= 1 - 2|c_0|^2|c_1|^2\{1 - e^{-\xi(t)}\cos[\phi(t) - \phi_s]\}.\end{aligned}$$

In Fig. 1 we plot the curve $F(t)$, where $g=0.1$, $\omega=1$, and

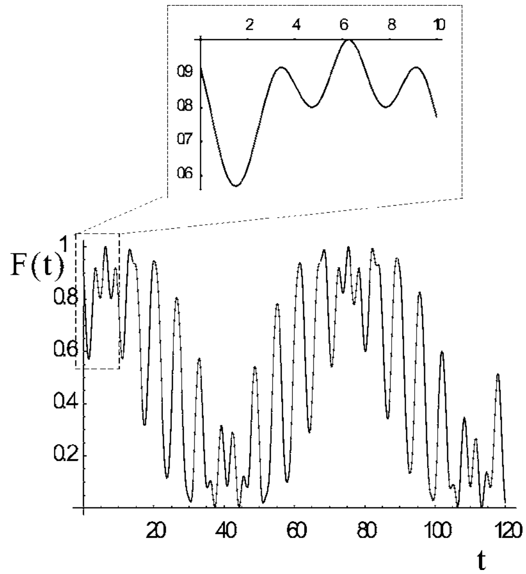

FIG. 1. The control fidelity $F(t)$ defined as the overlap of ρ_q and ρ_r, which varies with t. The above inset is a close-up of the curve, which indicates the collapses and revivals of the controlled system.

$\alpha=1.5$. For convenience, we have taken $|c_0|^2=|c_1|^2=1/2$ and then

$$F(t) = 1 - \frac{1}{2}[1 - e^{-\xi(t)} \cos(\phi(t) - \phi_s)]. \tag{19}$$

It can be seen that $F(t)$ is a periodic function with unity as the maximum value. As a functional, the period $T=f[F]$ is a function of the function $F(t)$. $f[F]$ is determined by the system parameters g and ω. When $t=f[F(t)]$, the control fidelity takes its maximum $F(t)=1$ and then we realized an ideal phase gate operation with the phase $\phi_s=|g|^2T/\omega$.

In order to realize a real control we require that the effective interaction $\langle V_I(t)\rangle$ should be automatically switched on and off at time 0 and T; i.e., the controllable condition (CABC)

$$\langle V_I(0)\rangle = \langle V_I(T)\rangle = 0 \tag{20}$$

is satisfied for $\langle V_I(t)\rangle=\langle\psi_c(t)|V_I(t)|\psi_c(t)\rangle$. For the above example, this requirement means

$$\mathrm{Re}(g\alpha) = 0, \tag{21}$$

$$\mathrm{Im}[g\alpha]\sin \omega T = -\omega\xi(T), \tag{22}$$

for $\sin \omega T \neq 0$. When there is no loss of qubit coherence at the instance $t=T$ $[\xi(T)=0]$, the requirement, Eqs. (21) and (22), for an ideal quantum control is just $g\alpha=0$. It is absurd and impracticable. However, there exist the situations $(\sin \omega T=0)$ satisfying the requirement for quantum control: $\mathrm{Re}[g\alpha]=0$ and $\omega T=2k\pi$, $k \in \mathbb{Z}$, which is reasonable in principle since a pure imaginary number $g\alpha=i|g\alpha|$ does not vanish even though it has a vanishing real part. Therefore some target states are obtained as the superposition states of $|0\rangle$ and $|1\rangle$ with specific relative phases that can be implemented perfectly by quantum control.

However, the above phase gate control could only generate particular phases ϕ_s on the qubit state $|1\rangle$, which is completely determined by the coupling factor g and the controller field frequency ω. In this sense we cannot achieve a quantum control of implementing universal phase gates for a given total system with fixed g and the controller field frequency ω. To overcome this problem the local parameters of the initial states of the controller should be used in the quantum control rather than the fixed global parameters of the total system. We will explore this possibility in Sec. V where the quantum decoherence will be considered based on the uncertainty relation that relates to a multimode coherent field.

III. QUANTUM CONTROL BY A GENERAL CONTROLLER

Staring with an idealized model, the above investigations provide us some insights into the quantum control problem. In order to consider more practical cases, we will analyze quantum controllability in this section. To focus on the central idea we do not consider the influence of the environment yet. The entire system that we consider is an isolated system including the controller C with Hamiltonian H_c and the controlled system Q with Hamiltonian H_q. To bring out more clearly the physical picture of such a quantum control, the minimal assumption is that the Hamiltonian includes only two items: H_{qc} and H_c. Matching this assumption, there exists a practical case that the nondemolition control satisfies $[H_q,H_{qc}]=0$ and then the free evolution of the controlled system is eliminated.

Conveniently we work in the interaction picture with the Schrödinger equation

$$i\hbar\frac{d}{dt}|\Psi^I(t)\rangle = H^I_{qc}(t)|\Psi^I(t)\rangle. \tag{23}$$

Formally, quantum control requires that the interaction Hamiltonian

$$H^I_{qc}(t) = e^{iH_c t/\hbar}H_{qc}e^{-iH_c t/\hbar} \tag{24}$$

be automatically turned on and off at certain instants $t=0$ and $t=T$ during the evolution of the controller system. Under quantum control a quantum gate operation is accomplished by the controlled system. Besides, it is also required that the controlling parameters depend on the initial state of the controller system. By applying them to quantum computing, the quantum computer implements the operations programmed by the controller.

Without loss of the generality, we still take the controlled system as a qubit with two basis states $|0\rangle$ and $|1\rangle$. An ideal quantum control with $U_q(T)$ exerting on the qubit can be described as a factorized evolution

$$U_I(T) = e^{-(i/\hbar)(H_{qc}+H_c)T} = U_q(T) \otimes U_c(T) \tag{25}$$

of the total system, so that a controlled evolution of the qubit system is implemented as $|\psi_q(T)\rangle=U_q(T)|\psi_q(0)\rangle$, while $|\psi_c(T)\rangle=U_c(T)|\psi_c(0)\rangle$ defines the final state of the controller. Here, $|\psi_q(0)\rangle=c_0|0\rangle+c_1|1\rangle$ and $|\psi_c(0)\rangle$ are the initial states of the qubit and controller, respectively. We note that, because

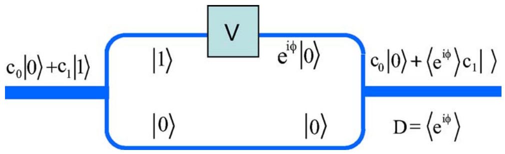

FIG. 2. (Color online) Schematic illustration of the nondemolition interaction for phase quantum control: the effective potential V only act on the component $|1\rangle$, but not on the component $|0\rangle$ in the superposition. Besides the wanted phase to be generated, such an interaction also induced a fluctuation of phase reflected by the factor $|D|\approx\exp\left[-\frac{1}{2}(\Delta\Phi)^2\right]$ with module less than 1.

the Hermitian operators H_{qc} and H_c do not commute with each other, thus there is not simply $\exp(-iH_{qc}T)=U_q(T)$ in practice. We emphasize that, due to the limitation resulting from the Heisenberg uncertainty principle, realistic control cannot be carried out in such a perfect way as a completely factorized evolution.

Generally, the Hermitian operators H_{qc} and H_c do not commute with each other and there exists an uncertainty relation:

$$\Delta H_{qc}^I \Delta H_c \geq \frac{1}{2}|\langle[H_{qc}^I,H_c]\rangle|, \tag{26}$$

where the variations $\Delta A=\sqrt{\langle A^2-\langle A\rangle^2\rangle}$, $A=H_{qc}$ and H_c. In a consistent approach for quantum measurement [12], this uncertainty relation is also responsible for the decoherences induced by the detector as well as those induced by quantum control. Roughly speaking, the variation $\Delta H_{qc}^I(t)$ is relavent to the induced decoherence in the qubit system, while the term $|\langle[H_{qc}^I(t),H_c]\rangle|$ indicates the influence of quantum control and ΔH_c is associated with the power or the average energy of the controller. The conservation laws throw some limits on such an implementation of quantum gates [5]. For example, a quantum control to complete a controlled-NOT (CNOT) gate usually concerns the transfer of some conservation quantities between qubits. To focus on the problems in the following, we will only consider the quantum control itself, which does not involve the transfer of any known conservation quantities.

Now we assume a nondemolition controlling interaction $H_{qc}=|1\rangle\langle 1|\otimes V$ with a potential V that acts only on the qubit state $|1\rangle$ (see Fig. 2). It does not play any role at the beginning and end of the gate operation, but we require that it be generated by the controller and a nontrivial phase be left on the qubit state $|1\rangle$. Actually, as for the quantum controls in quantum information processing, it is expected that a quantum computer could work like electronic computers: when the programs are designed and then stored in it initially, the quantum computer should be able to carry out computations without any other assistance. The basic requirement for quantum control is that the interaction can be switched on and off automatically at certain instants—e.g., at $t=0$ and $t=T$,

$$\langle V_I(0)\rangle = \langle\psi_c(0)|V_I(0)|\psi_c(0)\rangle = 0,$$

$$\langle V_I(T)\rangle = \langle\psi_c(T)|V_I(T)|\psi_c(T)\rangle = 0, \tag{27}$$

where $V_I(t)\equiv\exp(iH_ct/\hbar)V(-iH_ct/\hbar)$. The above equations (27) are the general controllable condition. The sandwich $\langle V\rangle$ is defined as the average of the operator V over the controller state. This means that the effective interaction is obtained by taking the average of $V_I(t)$ over the instantaneous controller states $|\psi_c(t)\rangle$.

Generally, the controller in physical implementations of the quantum control are various fields that are supposed to be classical. For example, microwave electromagnetic fields are used to manipulate the nuclear spin qubits in NMR, the laser fields are applied to control the atomic qubits, and classical magnetic flux and voltage are utilized to adjust the Josephson-junction-based qubits. However, the controlling fields are essentially of quantization and are usually described by coherent states or some quantum-mechanical mixture.

Starting from the initial state where the qubit is in $|\psi_q\rangle=c_0|0\rangle+c_1|1\rangle$, the total system evolves according to the entangled state

$$|\Psi(t)\ = c_0|0\rangle\otimes|\psi_c(0)\rangle + c_1|1\rangle\otimes e^{i\mathbf{\Phi}(t)}|\psi_c(0)\rangle, \tag{28}$$

where we have defined the time-order integral

$$e^{i\mathbf{\Phi}} = \hat{T}\exp\left(-\frac{i}{\hbar}\int_0^t V_I(\tau)d\tau\right). \tag{29}$$

The decoherence factor [13] is an expectation of the unitary operator

$$D(T) = \langle\psi_c(0)|e^{i\mathbf{\Phi}}|\psi_c(0)\rangle, \tag{30}$$

which can be used to characterize quantum controllability.

Now we need to consider that in what cases the above entangled state $|\Psi(t)\rangle$ can become a factorized state, Eq. (5), at a certain instant $t=T$ so that the ideal quantum control is realized by choosing the initial state $|\psi_c(0)\rangle$ of the controlling system. The simplest illustration is that $V_I(t)=V$ is a static potential and thus

$$e^{i\mathbf{\Phi}} = \exp(-iTV/\hbar). \tag{31}$$

If we choose $|\psi_c(0)\rangle=|\phi\rangle$ with the eigenvalue ϕ, then $\exp(i\mathbf{\Phi})$ becomes a c-number phase factor φ and the time evolution automatically generates a phase gate operation with the c-number phase:

$$|\Psi(T)\ = (c_0|0\rangle + c_1e^{i\varphi}|1\rangle)\otimes|\psi_c(0)\rangle. \tag{32}$$

Indeed, the phase ϕ multiplied to the qubit state $|1\rangle$ is well defined and can be generated with arbitrary precision at a suitable instant T by choosing the initial state $|\psi_c(0)\rangle=|\phi\rangle$ of the controller. This is what we want: the qubit system to be controlled by the parameters of the initial state as well as the evolution time. It seems that no fundamental restrictions exist for $|\psi_c(0)\rangle$ and T.

However, the above idealized situation is far from realistic cases in practical quantum controls. First, the precision of quantum control is guaranteed by the stability of the potential—i.e., $[V,H_c]=0=[H_{qc},H_c]$. However, this means that the free Hamiltonian evolution of the controller has no influence on the effective interaction by H_{qc} and thus the CABC cannot be satisfied automatically. Therefore, we infer that, in order to realize a quantum control with the "switched on and off," the potential $V_I(t)$ could not be a static one. In this case the c-number phase is not well defined by the initial state of the controller and thus there exists a phase fluctuation $\Delta\Phi$ in the implementation of quantum control.

To explore the possibility of assorting with the CABC and the precision of quantum control, we distinguish two cases by whether the potential $V_I(t)$ generated by the controller is commutative or not at different instants, i.e.,

$$\text{case 1:}\quad [V_I(t),V_I(t')]=0, \tag{33}$$

$$\text{case 2:}\quad [V_I(t),V_I(t')]\neq 0. \tag{34}$$

In the first case a phase factor operator can be simply defined as

$$\mathbf{\Phi}=-\frac{1}{\hbar}\int_0^T V_I(\tau)d\tau. \tag{35}$$

Under the small variation $\Delta\mathbf{\Phi}\ll 1$, the decoherence factor can be calculated as

$$D(T)\approx e^{i\langle\mathbf{\Phi}\rangle-(\Delta\mathbf{\Phi})^2/2}\equiv e^{i\langle\mathbf{\Phi}\rangle}d(T). \tag{36}$$

Similar to the arguments about the exactly solvable model in Sec. II, an observation is that the ideal quantum control can be characterized by whether or not the decoherence factor $|D(T)|=|\langle\exp(i\mathbf{\Phi})\rangle|$ can reach unity. Actually the phase multiplied by the qubit state $|1\rangle$ is the real part of the expectation value of the phase factor operator $\langle\mathbf{\Phi}\rangle$ plus a decay factor from its quantum fluctuation $(\Delta\mathbf{\Phi})^2$ [11]. Thus the quantum controllability is destroyed by the phase fluctuation $(\Delta\mathbf{\Phi})^2$ in general.

In the following, we will show that the phase fluctuation $(\Delta\mathbf{\Phi})^2$ will result in a loss of quantum coherence or quantum dephasing. To this end we calculate

$$(\Delta\mathbf{\Phi})^2\sim\frac{1}{\hbar^2}\int_0^T dt\int_0^t \Delta V_I(t)\Delta V_I(\tau)d\tau, \tag{37}$$

which shows that the phase fluctuation $(\Delta\mathbf{\Phi})^2$ is just the correlated fluctuation of the Heisenberg interaction. Thus $d(T)=\exp[-(\Delta\mathbf{\Phi})^2/2]$ is a decaying factor in $D(T)$ accompanying the off-diagonal terms of the reduced density matrix of the qubit system. To quantitatively describe that to what extent the target sate

$$|\psi_t\rangle=c_0|0\rangle+c_1e^{i\langle\mathbf{\Phi}\rangle}|1\rangle$$

can be reached by the controlled time evolution $|\Psi(t)\rangle$, the control fidelity

$$\begin{aligned}F(t)&=\mathrm{Tr}[|\Psi(t)\rangle\langle\Psi(t)|(1\otimes|\psi_t\rangle\langle\psi_t|)]\\&=\mathrm{Tr}_c[\langle\psi_t|\Psi(t)\rangle\langle\Psi(t)|\psi_t\rangle]\\&=\mathrm{Tr}(\rho_t\rho_r)\end{aligned} \tag{38}$$

is defined in terms of the reduced density matrix ρ_t and the reduced density matrix $\rho_r=\mathrm{Tr}_c[|\Psi(t)\rangle\langle\Psi(t)|]$, where Tr_c indicates tracing over the variables of the controller. In this case the result is obtained as

$$F(t)=1-2|c_0|^2|c_1|^2(1-e^{-(\Delta\mathbf{\Phi})^2}/2).$$

Thus the corresponding error measure

$$\varepsilon=1-F(t)=2|c_0|^2|c_1|^2[1-d(t)] \tag{39}$$

describes the failure probability of the quantum control.

For the second case, due to the nonvanishing commutator between $V_I(t)$ at different instants, we cannot generally define a phase factor operator $\mathbf{\Phi}$, but we can still formally write $D(T)=\langle\exp(i\mathbf{\Phi})\rangle$ or

$$D(T)=e^{i\mathbf{\Phi}-\xi}\simeq\exp\left(i\langle\mathbf{\Phi}\rangle-\frac{1}{2}(\Delta\mathbf{\Phi})^2\right). \tag{40}$$

This can give all similar results as case 1. The exactly solvable model in Sec. II elongs to the second case. This result is exact for the above example presented in the last section where

$$\frac{1}{2}(\Delta\Phi)^2=\xi(t),\quad\langle\mathbf{\Phi}\rangle=\phi(t). \tag{41}$$

As discussed above, the decoherence-induced limit to the quantum control has been explained based on the phase uncertainty. In fact, this understanding reveals once again the inherence of quantum decoherence in the generalized two-slit experiment about $|0\rangle$ and $|1\rangle$, whose interference fringe vanishes when one determines which slit the particle comes from. According to Heisenberg, this is due to the randomness of relative phases [10] from quantum control. Furthermore, we can conclude from the above exact solution that the large random phase change just originates from Heisenberg's position-momentum uncertainty relation $\Delta x_k\Delta p_k=1/2$. This observation will help us to discover a bound on quantum control.

IV. PHASE UNCERTAINTY DUE TO THE STANDARD QUANTUM LIMIT

Based on our previous explorations of the relation between the two explanations for quantum decoherence [12], using the position-momentum uncertainty relation, we now can associate the physical limit of quantum control with the standard quantum limit in a quantum measurement context [3] through a concrete example as follows.

This is a more practical example, where the qubit is controlled by a multimode electromagnetic field

$$E=\sum_k[u_k(x)a_ke^{-i\omega_kt}+\mathrm{H.c.}], \tag{42}$$

with the mode functions $u_k(x)$. The controlling Hamiltonian $H_{qc}(t)=|1\rangle\langle 1|\otimes V_I(t)$ in the interaction picture reads as

$$H_{qc}(t) = |1\rangle\langle 1| \otimes \sum_k H_k \equiv |1\rangle\langle 1| \otimes \sum_k \hbar(g_k a_k e^{-i\omega_k t} + \text{H.c.}), \tag{43}$$

where ω_k are the mode frequencies, a_k and $a_k^\dagger$ the creation and annihilation operators, respectively, and g_k the mode coupling constants between the qubit and field modes. We suppose that the electromagnetic field is initially prepared in a multimode coherent state

$$|\psi_c(0)\rangle = |\alpha\rangle \equiv \prod_k |\alpha_k\rangle \tag{44}$$

as a direct product of the coherent state $|\alpha_k\rangle$ of the kth mode. In such an initial state, the observable is the average of the field operator,

$$\langle \alpha|E|\alpha\rangle = \sum_k [u_k(x)\alpha_k e^{-i\omega_k t} + \text{H.c.}], \tag{45}$$

which is a wave packet, the superposition of many plane waves. This means that, to realize a more realistic quantum control, we need a wave packet rather than a single mode or a plane wave.

The free Hamiltonian of the qubit system has been omitted without loss of generality. The potential $V_I(t)$ exerts on the qubit state $|1\rangle$, but not on the qubit state $|0\rangle$. Then the evolution can be obtained as

$$U(t) = |0\rangle\langle 0| \otimes \mathbf{1} + |1\rangle\langle 1| \otimes e^{i\mathbf{\Phi}},$$

where

$$e^{i\mathbf{\Phi}} = \prod_k e^{i\Phi_k} = \prod_k U_k \equiv \prod_k \hat{T} \exp\left(-\frac{i}{\hbar}\int_0^t H_k(\tau)d\tau\right). \tag{46}$$

We can explicitly calculate the phase operator $\Phi = \Sigma_k \Phi_k$ defined above by a method similarly to that used for the example about the single-mode field in Sec. II. It is obtained by

$$\Phi_k = \eta_k(t)a_k + \text{H.c.} + \varphi_k(t) + i\xi_k(t),$$

with three time-dependent parameters

$$\eta_k(t) = i\frac{g_k}{\omega_k}(1 - e^{-i\omega_k t}),$$

$$\varphi_k(t) = \frac{|g_k|^2}{\omega_k^2}(\omega_k t - \sin \omega_k t),$$

$$\xi_k(t) = \frac{|g_k|^2}{\omega_k^2}(1 - \cos \omega_k t). \tag{47}$$

The phase operator can be rewritten as $\mathbf{\Phi} = \Omega(t) + \mathbf{\Phi}_a$ in terms of the constant phase $\Omega(t) = \Sigma_k \varphi_k(t)$ plus the operator

$$\mathbf{\Phi}_a = \sum_k \mathbf{\Phi}_{ak}(t) = \sum_k [\eta_k(t)a_k + \text{H.c.}]. \tag{48}$$

The decoherence factor can be calculated similarly as

$$D(t) = \langle \alpha|e^{i\mathbf{\Phi}}|\alpha\rangle = e^{i\phi(t) - \xi(t)}, \tag{49}$$

where $\xi(t) = \Sigma_k \xi_k(t)$ and

$$\phi(t) = 2\sum_k \left\{ \text{Im}\left[\frac{g_k\alpha_k}{\omega_k}(1 - e^{-i\omega_k t})\right]\right\} + \sum_k \frac{|g_k|^2}{\omega_k^2}(\omega_k t - \sin \omega_k t). \tag{50}$$

It is easy to check that the phase generated by the quantum control is just the average value of the phase operator

$$\langle \alpha|\mathbf{\Phi}|\alpha\rangle = \langle \alpha|\mathbf{\Phi}_a|\alpha\rangle + \Omega(t) = \sum_k [\eta_k(t)\alpha_k + \text{H.c.}] + \sum_k \varphi(t). \tag{51}$$

The analytical expression of the phase fluctuation is

$$(\Delta\Phi)^2 = (\Delta\Phi_a)^2 = \sum_{k=1}^N (\Delta\Phi_{ak})^2 = \sum_{k=1}^N |\eta_k|^2 = 2\sum_{k=1}^N \xi_k(t) = 2\xi(t), \tag{52}$$

where we have considered each uncertain phase change as an independent stochastic variable. Namely, the relation $\xi(t) = (\Delta\Phi)^2/2$ or the exact expression $D(T) = \exp(i\langle\mathbf{\Phi}\rangle - (\Delta\mathbf{\Phi})^2/2)$ still holds for the multimode case with the specialized initial state. Correspondingly, the error measure is estimated as

$$\varepsilon = 1 - F(t) = \lambda(\Delta\mathbf{\Phi}_a)^2 = 2\lambda\xi(t), \tag{53}$$

where $\lambda = |c_0|^2|c_1|^2$. Different from the single-mode case, it is hard to find a proper instant T such that $\varepsilon = \xi(T) = 0$ in general. Namely, it is hard to achieve an ideal quantum control without any error.

In the above discussions, the realization of quantum control boils down to the appearance of the c-number phase $\phi(t)$ that contains the controllable part depending on the initial state of the controller. An ideal quantum control means a vanishing error $\lambda(\Delta\mathbf{\Phi}_a)^2$. But it is almost impossible because of the intrinsic decoherence due to quantum control itself. In fact, if the electromagnetic field could carry out a completely efficient control of the controlled system, then the interaction Hamiltonian should not commute with that of the controller. These facts are responsible for the inaccuracy of the phase gate or decoherence in the controlled system under the quantum control. We have to point out that the conclusion drawn above seems to depend on the choice of initial state, but now we can argue that this is not the case with the above consideration. So we need to consider the universality of the conclusions.

Physically, every variable of the controller can independently exert a different impact on the different components of controller state. Since every uncertain phase is an independent stochastic variable, we have

$$(\Delta\mathbf{\Phi}_a)^2 = \sum_{k=1}^N (\Delta\mathbf{\Phi}_{ak})^2 \geqslant N \min\{(\Delta\mathbf{\Phi}_{ak})^2 | k = 1,2,\ldots,N\}$$

for a general initial state of the controller. We note that the phase uncertainty $(\Delta\mathbf{\Phi}_a)^2$ caused by the controller variables can be amplified to a number much larger than unity when

$N \to \infty$; i.e., the system states acquire a very large random phase factor. The decay factor

$$|D(t)| = e^{-(\Delta\mathbf{\Phi})^2/2} \leq \exp\left(-\frac{N}{2}\min\{(\Delta\mathbf{\Phi}_{ak})^2|k=1,2\ldots N\}\right).$$

So $|D(t)| \to 0$ when $N \to \infty$; i.e., the macroscopic controller can wash out the quantum coherence of the controlled system.

To be more concrete we assume that, in the initial state $|0\rangle = \otimes_{k=1}^{N}|\psi_k\rangle$ of the controller, each component $|\psi_k\rangle$ is a wave packet, symmetric with respect to both the "canonical coordinate" $x_k=(a_k+a_k^\dagger)/\sqrt{2}$ and the corresponding "canonical momentum" $p_k=-i(a_k-a_k^\dagger)/\sqrt{2}$. So $\langle x_k\rangle \equiv \langle\psi_k|x_k|\psi_k\rangle=0$ and $\langle p_k\rangle=0$. We do not need the concrete form of the initial state. For convience we assume it to be of Gaussian type with variance $\sigma_k=\Delta x_k$ in x_k space. Physically, once Δx_k is given, the variance of p_k cannot be arbitrary since there is a Heisenberg's position-momentum uncertainty relation $\Delta x_k \Delta p_k \geq 1/2$. In the following we will show that the uncertainty relation will give a low bound to the variance of $\Delta\mathbf{\Phi}_a$. In the above reasoning about $|D(t)| \to 0$ when $N \to \infty$, we have considered that there exists a finite minimum value of $(\Delta\mathbf{\Phi}_{ak})^2$. In the quantum measurement theory, the finite minimum value of $\{(\Delta\mathbf{\Phi}_{ak})^2|k=1,2,\ldots,N\}$ is implied by the so-called SQL on the continuous measurement of phase operator.

To see this we rewrite the phase operator

$$\mathbf{\Phi}_a = \sum_k \mathbf{\Phi}_{ak} = \sum_k [\varkappa_k(t)x_k + \mu_k(t)p_k], \tag{54}$$

in terms of the "canonical coordinate" and the corresponding "canonical momentum," and the coefficients are

$$\varkappa_k(t) = \frac{1}{\sqrt{2}}[\eta_k(t) + \eta_k^*(t)],$$

$$\mu_k(t) = \frac{i}{\sqrt{2}}[\eta_k(t) - \eta_k^*(t)].$$

The existence of the SQL is guaranteed by Heisenberg's position-momentum uncertainty relation, because each $\mathbf{\Phi}_{ak}=\varkappa_k(t)x_k+\mu_k(t)p_k$ is a linear combination of x_k and p_k with a property $\langle x_k p_k\rangle+\langle p_k x_k\rangle=0$ for the average over the real initial state. The phase fluctuation $\Delta\mathbf{\Phi}_{ak}$ can be derived as

$$\Delta\mathbf{\Phi}_{ak} = \sqrt{|\varkappa_k(t)|^2(\Delta x_k)^2 + |\mu_k(t)|^2(\Delta p_k)^2} \geq \sqrt{|\varkappa_k(t)\mu_k(t)|}$$

or

$$(\Delta\mathbf{\Phi}_{ak})^2 \geq 8\frac{g_k^2}{\omega_k^2}\left|\sin^3\frac{\omega_k t}{2}\cos\frac{\omega_k t}{2}\right|. \tag{55}$$

Here, we considere the variance $\Delta(\xi x)=|\xi|(\Delta x)$ for a stochastic variable x and a real number ξ, and suppose g_k/ω_k being a real number.

In the above arguments, x_k and p_k are not only regarded as a pair of uncorrelated stochastic variables in the terminology of a classical stochastic process, the uncertainty relation $\Delta x_k \Delta p_k \sim 1/2$ of them is also taken into account. This constraint just reflects the uncertainty of the phase change in the quantum control process. Therefore, we have a time-dependent minimum value of phase uncertainty with a low bound

$$(\Delta\mathbf{\Phi}_a)^2 \geq N\min\{|\varkappa_k(t)\mu_k(t)||k=1,2,\ldots,N\}.$$

This result qualitatively illustrates the many-particle amplification effect of uncertain phase change due to quantum control itself. The large random phase variance $(\Delta\mathbf{\Phi}_a)^2$ implies that it is hard to satisfy the exact condition $(\Delta\mathbf{\Phi}_a)^2=0$ in principle, and thus one can only optimize both the system parameters and the initial state of the controller to approach what we want.

To see the above observation analytically, we calculate $\langle\mathbf{\Phi}\rangle$ in comparison with $\Delta\mathbf{\Phi}$ in the decoherence factor $D(T)=\exp[i\langle\mathbf{\Phi}\rangle-(\Delta\mathbf{\Phi})^2/2]$. The most simple, but somewhat trivial case is that all modes are degenerate—i.e., $g_k=g$ and $\omega_k=\omega$—then,

$$\Delta\mathbf{\Phi} = \sqrt{8N}\frac{|g|}{\omega}\left|\sin\frac{\omega t}{2}\right|, \tag{56}$$

while the phase we wanted is

$$\phi(t) = 2N\left\{\mathrm{Im}\left[\frac{g\alpha}{\omega}(1-e^{-i\omega t})\right]\right\} + 2N\frac{|g|^2}{\omega^2}(\omega t - \sin\omega t). \tag{57}$$

Obviously, for very large N, the phase fluctuation $\Delta\mathbf{\Phi}$ can be neglected since $\Delta\mathbf{\Phi}/|\phi(t)| \sim 1/\sqrt{N} \to 0$. In general, we need to consider divergence of the phase fluctuation

$$(\Delta\mathbf{\Phi})^2 = \sum_{k=1}^{N} 16\frac{g_k^2}{\omega_k^2}\sin^2\frac{\omega_k t}{2} = \int_{-\infty}^{\infty} 16\frac{g_k^2}{\omega_k^2}\rho(\omega_k)\sin^2\frac{\omega_k t}{2}d\omega_k \tag{58}$$

for various spectrum distributions of the controller, where an unspecific spectrum distribution $\rho(\omega_k)$ is used to discuss the case with a continuous spectrum. For example, when $\rho(\omega_k)=\gamma/g_k^2$, the decoherence factor is exponentially decaying since the above integral converges to a number $8\pi\gamma t/9$ proportional to time t. Another example is the Ohmic distribution $\rho(\omega_k)=2\eta\omega_k^2/(\pi g_k^2)$, which results in a diverging phase fluctuation for $t \neq 0$.

V. LOW BOUND OF CONTROL-INDUCED DECOHERENCE AND QUANTUM COMPUTATION

In this section we will show that it is the back action of the controller on the controlled system, implied by Heisenberg's position-momentum uncertainty relation, that disturbs the phases of states of the controlled system and then induces a quantum decoherence, which is relevant to the SQL. In order to quantitatively characterize such a limit to quantum controllability, we now return to the discussion about quantum control with a multimode field initially prepared in a coherent state.

The commutation relation of the number operator $\mathbf{N}$ and the phase operator $\mathbf{\Phi}_a$ defines an operator

$$\mathbf{\Theta} = i\sum_k [-\eta_k(t)a_k + \eta_k^*(t)a_k^\dagger] \tag{59}$$

dual to the phase operator $\mathbf{\Phi}_a$—that is,

$$\mathbf{\Theta} = i[\mathbf{N},\mathbf{\Phi}_a]. \tag{60}$$

To see the meaning of the defined $\mathbf{\Theta}$, we calculate the commutation relation of $\mathbf{N}$ and $\mathbf{\Phi}_a$ to find a close algebra by

$$[\mathbf{N},\mathbf{\Theta}] = i\mathbf{\Phi}_a,$$

$$[\mathbf{\Phi}_a,\mathbf{\Theta}] = iF(t), \tag{61}$$

where $F(t)=2\Sigma_k|\eta_k(t)|^2$ is a time-dependent constant. This means that $\mathbf{P}=\mathbf{\Theta}/F(t)$ is a conjugate variable with respect to $\mathbf{\Phi}_a$ since we have the canonical commutation relation $[\mathbf{\Phi}_a,\mathbf{P}]=1$. In this sense we call $\mathbf{\Theta}$ a dual-phase operator (DPO). A constant uncertainty relation can be found for $\mathbf{\Phi}_a$ and $\mathbf{P}$, which can be minimized by the corresponding coherent state $|\alpha\rangle=\Pi_k|\alpha_k\rangle$.

The above arguments about minimization of the uncertainty by $[\mathbf{\Phi}_a,\mathbf{\Theta}]$ can enlighten us to find a low bound for the control-induced decoherence. To this end we consider the uncertainty relation

$$\langle\mathbf{N}\rangle\Delta\mathbf{\Phi}_a = \Delta\mathbf{N}\Delta\mathbf{\Theta} \geqslant \frac{1}{2}|\langle[\mathbf{\Theta},\mathbf{N}]\rangle| = \frac{1}{2}|\langle\mathbf{\Phi}_a\rangle| \tag{62}$$

about the DPO and photon number operator $\mathbf{N}=\Sigma_k a_k^\dagger a_k$.

To derive the above uncertainty relation (62), we have considered

$$\Delta\mathbf{N} = \langle\mathbf{N}\rangle,$$

$$(\Delta\mathbf{\Phi}_a)^2 = (\Delta\mathbf{\Theta})^2 \tag{63}$$

for the average $\langle\cdots\rangle$ over the coherent state $|\alpha\rangle$. We check the above results (63) by the straightforward calculations

$$(\Delta\Phi_a)^2 = \langle\alpha|\Phi_a^2 - \langle\Phi_a\rangle^2|\alpha\rangle = \sum_k |\eta_k(t)|^2,$$

$$(\Delta\mathbf{\Theta})^2 = \langle\alpha|\mathbf{\Theta}^2 - \langle\mathbf{\Theta}\rangle^2|\alpha\rangle = \sum_k |\eta_k(t)|^2.$$

The novel uncertainty relation (62) defines a low bound for the phase variation $\Delta\Phi_a$ for a given phase $\langle\mathbf{\Phi}_a\rangle$ to be achieved by quantum control—i.e.,

$$\Delta\mathbf{\Phi}_a \geqslant \frac{|\langle\mathbf{\Phi}_a\rangle|}{2\langle\mathbf{N}\rangle}. \tag{64}$$

Equation (64) clearly implies that we need much larger energy to reduce the low bound of the phase fluctuation. Actually, we can formally write down the expectation of the photon energy of the controller,

$$E = \left\langle \hbar\sum_k \omega_k a_k^\dagger a_k \right\rangle \equiv \hbar\langle\mathbf{N}\rangle\langle\omega\rangle, \tag{65}$$

in terms of the average photon number $\langle\mathbf{N}\rangle=\Sigma_k|\alpha_k|^2$ and the average frequency of photons

$$\langle\omega\rangle = \frac{\sum_k \omega_k|\alpha_k|^2}{\sum_k |\alpha_k|^2}. \tag{66}$$

Then Eq. (64) becomes

$$\Delta\Phi_a \geqslant \frac{\hbar\langle\omega\rangle}{2E}|\langle\mathbf{\Phi}_a\rangle|. \tag{67}$$

The small low bound requires that a large quantum controller (implied by large $\langle\mathbf{N}\rangle$ or large energy E) possess a very small average frequency. In this sense Eq. (62) defines a necessary condition for the quantum control that can manipulate the qubit system reaching the target state accurately. This requirement is very similar to that where the apparatus should be sufficiently so "large" as to be "classical" in quantum measurement in the so-called "Copenhagen interpretation." Since quantum control relies on the ability to preserve the quantum coherence of the qubit system during controlling it, the controller should be much "larger" than the controlled system. In this sense, the back action of the qubit system on the controller can be neglected.

Next we consider the controllable condition (27) that the controller field is switched on and off over a time duration T, which can be roughly realized as a periodic phenomenon with the average period $T\sim 2\pi/\langle\omega\rangle$. Since the average frequency of the field can be approximated by $\langle\omega\rangle\sim 2\pi/T$, there is a low bound

$$\varepsilon \geqslant \frac{\lambda h^2}{4E^2T^2}|\langle\mathbf{\Phi}_a\rangle|^2 \equiv \frac{\lambda h^2}{4S^2}|\langle\mathbf{\Phi}_a\rangle|^2 \tag{68}$$

for the error measure estimation of $\varepsilon\sim\lambda(\Delta\mathbf{\Phi}_a)^2$ of the quantum control. So the larger action $S=ET$ from the controller is brought on the qubit system, the less quantum decoherence characterized by the control induced error ε becomes; the more one wants to change by the phase $\langle\mathbf{\Phi}_a\rangle$ of the qubit system, the larger quantum decoherence is induced by the quantum control. Therefore Eq. (68) imposes a fundamental limit on the accuracy of quantum control. In the following we can consider this physical limit for quantum computing.

It is well known that the controllability of qubits is a basic requirement for universal quantum computations, but according to the above arguments a well-armed control in quantum computing would cause extra decoherence in the qubit system. Thus, in competition with the induced decoherence, the controllability for quantum computation is limited.

In the last section a low bound of decoherence from quantum control is obtained. It throws an accuracy limit on the quantum controls in quantum computation. According to Eq. (68) this limit is about 10^{-20} for the typical setting $\langle\Phi_a\rangle=\pi$, $E=10^{-9}$ J and $T=1$ μs in an ion trap scheme. This is such a small limit that it is negligible in comparison with other errors, such as environment-induced errors in the current experiments of implementing quantum computation. However, in principle, Eq. (68) does throw a fundamental limit on the accuracy of the quantum control and thus on quantum computations. There are some numerical estimates in Fig. 3, which demonstrate a similar limit to the power of quantum computers. It is known that for an algorithm consisting of L

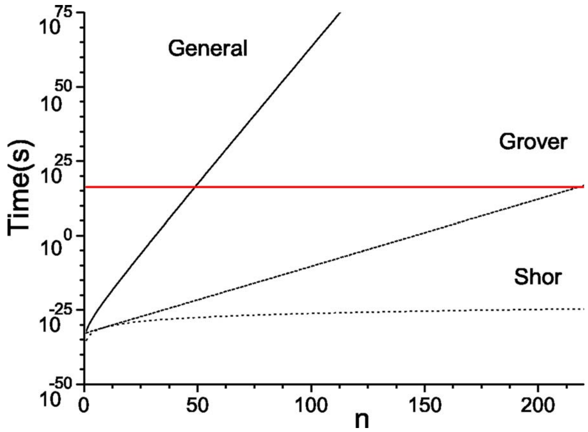

FIG. 3. (Color online) The minimum amount of time needed for some algorithms, in logarithmic scale. n is the number of qubits existing in the quantum algorithm. The horizontal line at about 10^{16} indicates the age of the Universe.

operations on the qubits system, an upper bound of error $\epsilon \sim 1/L$ is required in each operation for a faithful result of the entire computation. Inequality (68) tells us that the minimum amount of time needed for a single gate is

$$t_{\min} = \frac{h\sqrt{\lambda L}}{2E} |\langle \Phi_a \rangle|, \tag{69}$$

and so the total time needed to carry out a particular algorithm consisting of L elementary gates is about $Lt_{\min}$.

For a general algorithm such as an arbitrary unitary operation on n qubits, the amount of elementary gates L needed is about $O(n^2 4^n)$ [14]; for the Grover algorithm on n qubits, the amount is about $O(\sqrt{2^n})$; for the Shor large number factorization, the amount is about $O(n^3 \ln^3 2)$. The time duration needed for a general algorithm, the Grover algorithm, and the Shor's algorithm are estimated with optimistic assumptions, in which the only restriction is from quantum control. Thus, in Fig. 3 it could be found that the practice of quantum computation heavily depends on sophisticated quantum algorithms and arbitrary quantum operations on about several tens of qubits is already inaccessible even in principle. This handicap in quantum computation stands when quantum computation is carried out by tandem elementary gates under quantum control.

VI. CONCLUSION

In this paper we present a universal description for quantum control based on the quantized controller. We discovered the complementarity about the competition between the controllability and control-induced quantum decoherence in the view of quantum measurement. Starting with an exactly soluble example, a general model of quantum control is proposed to describe this novel complementarity or a new type of uncertainty relation. Our investigations show that it is possible to realize decoherence-free quantum controls only with some special phases at the finite-energy scale and in finite time. If the parameters of the phase are to be determined by the initial state of the controller, then there exists a low bound for the systematic errors resulting from the decoherence causedf by quantum control itself.

The above arguments also show that the decoherences from quantum control are different from those induced by the environment through unwanted interactions. This is because the negative influence of the controller happens in the quantum control process itself. If one eliminates this influence out and out, the positive role of quantum control would perish together. Therefore, for quantum computing, these kinds of errors induced by the control itself cannot be overcome totally by conventional error management protocols [15]. At least it has not been proved that control-induced decoherence can also be conquered efficiently by well-estabilished error management protocols. A better method to solve this problem is to optimize the control operations when the target of control is given. Without a doubt, this is an open question which is a challenge for the physical implementation of quantum computing as well as other protocols of quantum information processing.

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