

# Operator quantum error-correcting subsystems for self-correcting quantum memories

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The most general method for encoding quantum information is not to encode the information into a subspace of a Hilbert space, but to encode information into a subsystem of a Hilbert space. Recently this notion has led to a more general notion of quantum error correction known as operator quantum error correction. In standard quantum error-correcting codes, one requires the ability to apply a procedure which exactly reverses on the error-correcting subspace any correctable error. In contrast, for operator error-correcting subsystems, the correction procedure need not undo the error which has occurred, but instead one must perform corrections only modulo the subsystem structure. This does not lead to codes which differ from subspace codes, but does lead to recovery routines which explicitly make use of the subsystem structure. Here we present two examples of such operator error-correcting subsystems. These examples are motivated by simple spatially local Hamiltonians on square and cubic lattices. In three dimensions we provide evidence, in the form a simple mean field theory, that our Hamiltonian gives rise to a system which is self-correcting. Such a system will be a natural high-temperature quantum memory, robust to noise without external intervening quantum error-correction procedures.

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## I. INTRODUCTION

In the early days of quantum computation, the analog nature of quantum information and quantum transforms, as well as the effect of noise processes on quantum systems, were thought to pose severe obstacles [1,2] towards the experimental realization of the exponential speedups promised by quantum computers over classical computers [3–6]. Soon, however, a remarkable theory of fault-tolerant quantum computation [7–17] emerged which dealt with these problems and showed that quantum computers are indeed more similar to probabilistic classical computers than to analog devices. Analog computers have a computational power which is dependent on a lack of noise and on exponential precision, whereas probabilistic classical computers can be error-corrected and made effectively digital even in the presence of noise and nonexponential precision. The theory of fault-tolerant quantum computation establishes that quantum computers are truly digital devices deserving of the moniker computer. An essential idea in the development of the theory of fault-tolerant quantum computation was the notion that quantum information could be encoded into subspaces [7–10] (quantum error-correcting codes) and thereafter protected from degradation via active procedures of detection and correction of errors. Encoding quantum information into subspaces, however, is not the most general method of encoding quantum information into a quantum system. The most general notion for encoding quantum information is to encode the information into a subsystem of the quantum system [9,18,19]. This has been perhaps best exploited in the theory of noiseless subsystems [18,20–24] and dynamic recoupling schemes [22,25,26]. Recently a very general notion of quantum error correction has appeared under the moniker of “operator quantum error correction.” [27,28] In this work the possibility of encoding into subsystems for active error correction is explicitly examined. While it was found [27,28] that the notion of an encoding into a subsystem does not lead

to new codes (all subsystem codes could be thought of as arising from subspace codes), encoding into a subsystem *does* lead to different recovery procedures for quantum information which has been encoded into a subsystem. Hence, operator quantum error-correcting codes, while not offering the hope of more general codes, do offer the possibility of new quantum error-correcting routines, and, in particular, to the possibility of codes which might help improve the threshold for fault-tolerant quantum computation due to the lessened complexity of the error-correcting routine.

In this paper we present two examples of operator quantum error-correcting codes which use subsystem encodings. The codes we present have the interesting property that the recovery routine does not restore information encoded into a subspace, but recovers the information encoded into a subsystem. Using the  $[n, d, k]$  labeling a quantum error-correcting code, where  $n$  is the number of qubits used in the code,  $d$  is the distance of the code, and  $k$  is the number of encoded qubits for the code, our codes are  $[n^2, n, 1]$  and  $[n^3, n, 1]$  quantum error-correcting codes. The subsystem structure of our codes is explicitly exploited in the recovery routine for the code, and because of this they are substantially simpler than any subspace code derived from these codes.

While the two codes we present are interesting in their own right, there is a further motivation for these codes above and beyond their exploitation of the subsystem structure in the recovery routine. The two operator quantum error-correcting subsystems we present are motivated by two interesting Hamiltonians defined on two- and three-dimensional square and cubic lattices of qubits with certain anisotropic spin-spin interactions [21]. The three-dimensional version of this system is particularly intriguing since it offers the possibility of being a self-correcting quantum memory. In a self-correcting quantum memory, the quantum error correction is enacted not by the external con-

trol of a complicated quantum error-correction scheme, but instead by the natural physics of the device. Such a quantum memory offers the possibility of removing the need for a quantum microarchitecture to perform quantum error correction and could, therefore, profoundly speed up the process of building a quantum computer. In this paper we present evidence, in the form of a simple mean field argument, that the three-dimensional system we consider is a self-correcting quantum memory. We also show that the operator error-correcting subsystem structure of this code is an important component to not only the self-correcting properties of this system, but also to encoding and decoding information in this system.

The organization of the paper is as follows. In Sec. II we review the notion of encoding information into a subsystem and discuss the various ways in which this has been applied to noiseless subsystems and dynamic recoupling methods for protecting quantum information. Next, in Sec. III, we discuss how operator error-correcting subsystems work and how they differ from standard quantum error-correcting codes. Our first example of a operator quantum error-correcting subsystem is presented in Sec. IV where we introduce an example on a square lattice. The second, and more interesting, example of an operator quantum error-correcting subsystem is given in Sec. V where we discuss an example on a cubic lattice. In Sec. VI we introduce the notion of a self-correcting quantum memory and present arguments that a particular Hamiltonian on a cubic lattice related to our cubic lattice subsystem is self-correcting. We conclude in Sec. VII with a discussion of open problems and the prospects for operator quantum error-correcting subsystems and self-correcting quantum memories.

## II. SUBSYSTEM ENCODING

Consider two qubits. The Hilbert space of these qubits is given by  $\mathbb{C}^2 \otimes \mathbb{C}^2$ . Pick some fiducial basis for each qubit labeled by  $|0\rangle$  and  $|1\rangle$ . One way to encode a single qubit of information into these two qubits is to encode the information into a subspace of the joint system. For example, we can define the logical basis states  $|0_L\rangle = 1/\sqrt{2}(|01\rangle - |10\rangle)$  and  $|1_L\rangle = |11\rangle$  such that a single qubit of information can be encoded as  $\alpha|0_L\rangle + \beta|1_L\rangle = \alpha/\sqrt{2}(|01\rangle - |10\rangle) + \beta|11\rangle$ . This is an example of an idea of encoding quantum information into a subspace, in this case the subspace spanned by  $|0_L\rangle$  and  $|1_L\rangle$ . But another way to encode a single qubit of information is to encode this information into one of the two qubits. In particular, if we prepare the state  $|\psi\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)$  for an arbitrary single qubit state  $|\psi\rangle$ , then we have also encoded a single qubit of information in our system. This time, however, we have encoded the information into a subsystem of the system. It is important to note that the subsystem encoding works for an arbitrary state  $|\psi\rangle$ . If we fix  $|\psi\rangle$  to some known state, then we are again encoding into a subspace. We reserve the nomenclature of “encoding into a subsystem” to times in which  $|\psi\rangle$  is arbitrary.

More generally, if we have some Hilbert space  $\mathcal{H}$ , then a subsystem  $\mathcal{C}$  is a Hilbert space arising from  $\mathcal{H}$  as [9,18]

$$\mathcal{H} = (\mathcal{C} \otimes \mathcal{D}) \oplus \mathcal{E}. \quad (1)$$

Here we have taken our Hilbert space and partitioned it into two subspaces,  $\mathcal{E}$  and a subspace perpendicular to  $\mathcal{E}$ . On these perpendicular subspaces, we have introduced a tensor product structure,  $\mathcal{C} \otimes \mathcal{D}$ . We can then encode information into the first subsystem  $\mathcal{C}$ . This can be achieved by preparing the quantum information we wish to encode  $\rho_C$  into the first subsystem,  $\mathcal{C}$  along with any arbitrary state  $\rho_D$  into the second subsystem  $\mathcal{D}$ :

$$\rho = (\rho_C \otimes \rho_D) \oplus 0. \quad (2)$$

The fact that quantum information can most generally be encoded into a subsystem was an essential insight used in the construction of noiseless (decoherence-free) subsystems [18,20–24]. Suppose we have a system with Hilbert space  $\mathcal{H}_S$  and an environment with Hilbert space  $\mathcal{H}_E$ . The coupling between these two systems will be described by an interaction Hamiltonian  $H_{int}$  which acts on the tensor product of these two spaces  $\mathcal{H}_S \otimes \mathcal{H}_E$ . The idea of a noiseless subsystem is that it is often the case that there is a symmetry of the system-environment interaction such that the action of the interaction Hamiltonian factors with respect to some subsystem structure on the system’s Hilbert space,

$$H_{int} = \sum_{\alpha} [(I_d \otimes D_{\alpha}) \oplus E_{\alpha}] \otimes B_{\alpha}, \quad (3)$$

where  $I_d$  is the  $d$ -dimensional identity operator acting on the subsystem code space  $\mathcal{C}$ ,  $D_{\alpha}$  acts on the subsystem  $\mathcal{D}$ ,  $E_{\alpha}$  acts on the orthogonal subspace  $\mathcal{E}$ , and  $B_{\alpha}$  operates on the environment Hilbert space  $\mathcal{H}_E$ . When our interaction Hamiltonian possesses a symmetry leading to such a structure, then, if we encode quantum information into  $\mathcal{C}$ , this information will not be affected by the system-environment coupling. Thus information encoded in such a subsystem will be protected from the effect of decoherence and hence exists in a noiseless subsystem. Noiseless subsystems were a generalization of decoherence-free subspaces [29,30], this latter idea occurring when the subsystem structure is not exploited,  $\mathcal{D}=\mathcal{C}$ , and then encoding quantum information is simply encoding quantum information into a subspace. Subsystems have also been used in dynamic recoupling techniques [22,25,26] where symmetries are produced by an active symmetrization of the system’s component of the system-environment evolution.

## III. OPERATOR QUANTUM ERROR-CORRECTING SUBSYSTEMS

Here we examine the implications of encoding information into a subsystem for quantum error-correcting protocols [27,28]. Suppose that we encode quantum information into a subsystem  $\mathcal{C}$  of some quantum system with full Hilbert space  $\mathcal{H} = (\mathcal{C} \otimes \mathcal{D}) \oplus \mathcal{E}$ . Now suppose some quantum operation (corresponding to an error) occurs on our system. Following the standard quantum error-correcting paradigm, we then apply a recovery procedure to the system. When  $\mathcal{D}=\mathcal{C}$ , i.e., when we are encoding into a quantum error-correcting subspace, then the quantum error-correcting criteria are simply that the ef-

fect of the error process followed by the recovery operation should act as identity on this subspace. If we encode information into a subsystem, however, these criteria are changed to *only* requiring that the error process followed by the recovery operation should act as identity on the subsystem  $\mathcal{C}$ . In particular, we do not care if the effect of an error followed by our recovery procedure enacts some nontrivial procedure on the  $\mathcal{D}$  subsystem. In fact our error correcting procedure may induce some nontrivial action on the  $\mathcal{D}$  subsystem in the process of restoring information encoded in the  $\mathcal{C}$  subsystem.

How does the above observation modify the basic theory of quantum error-correcting codes? In a standard quantum-error correction, we encode into some error-correcting subspace with basis  $|i\rangle$ . The necessary and sufficient condition for there to be a procedure under which quantum information can be restored under a given set of errors  $E_a$  is given by

$$\langle i|E_a^\dagger E_b|j\rangle = \delta_{i,j}c_{a,b}. \quad (4)$$

For the case of encoding into a subsystem this necessary and sufficient condition is modified as follows: Let  $|i\rangle \otimes |k\rangle$  denote a basis for the subspace  $\mathcal{C} \otimes \mathcal{D}$ . Then Kribs and co-workers [27,28] showed a necessary condition [9,10] for the quantum error correcting given by

$$(\langle i| \otimes \langle k|)E_a^\dagger E_b(|j\rangle \otimes |l\rangle) = \delta_{i,j}m_{a,b,k,l}. \quad (5)$$

That this condition is also sufficient has recently also been shown by Nielsen and Poulin [31]. As noted in [27,28], a code constructed from the subsystem operator quantum error-correcting criteria can always be used to construct a subspace code which satisfies the subspace criteria Eq. (4). We note, however, that while this implies that the notion of using subsystems for quantum error correction does not lead to new quantum error-correcting codes above and beyond subspace encodings, the codes constructed which exploit the subsystem structure have error recovery routines which are distinct from those which arise when encoding into a subspace. In particular, when one encodes into a subsystem, the recovery routine does not need to fix errors which occur on other subsystems. Below we will present examples of subsystem encodings in which the subsystem structure of the encoding is essential not for the existence of the quantum error-correcting properties, but it is essential for the simple recovery routine we present.

#### IV. TWO-DIMENSIONAL OPERATOR QUANTUM ERROR-CORRECTING SUBSYSTEM

Here we construct an operator quantum error-correcting subsystem for a code which lives on a two-dimensional square lattice. This code makes explicit use of the subsystem structure in its error recovery procedure. A familiarity with the stabilizer formalism for quantum error-correcting codes is assumed (see [32,33] for overviews.)

##### A. Preliminary definitions

Consider a square lattice of size  $n \times n$  with qubits located at the vertices of this lattice. Let  $O_{i,j}$  denote the operator  $O$  acting on the qubit located at the  $i$ th row and  $j$ th column of

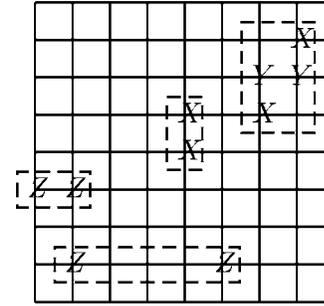


FIG. 1. Above we have represented elements of the  $\mathcal{T}$ . Each set of operators enclosed in a rectangle represents a Pauli operator acting on the particular qubits tensored with the identity on all other qubits. Each of the operators enclosed in the dotted rectangles are elements of  $\mathcal{T}$ .

this lattice tensored with identity on all other qubits. Recall that the Pauli operators on a single qubit are

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad iY = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \text{and } Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (6)$$

It is convenient to use a compact notation to denote Pauli operators on our  $n^2$  qubits by using two  $n^2$  bit strings,

$$P(a,b) = \prod_{i,j=1}^n X_{i,j}^{a_{i,j}} Z_{i,j}^{b_{i,j}} = \prod_{i,j=1}^n \begin{cases} X_{i,j} & \text{if } a_{i,j} = 1 \text{ and } b_{i,j} = 0 \\ Z_{i,j} & \text{if } a_{i,j} = 0 \text{ and } b_{i,j} = 1 \\ -iY_{i,j} & \text{if } a_{i,j} = 1 \text{ and } b_{i,j} = 1 \end{cases}, \quad (7)$$

where  $a, b \in \mathbb{Z}_2^{n^2}$  are  $n$  by  $n$  matrices of bits. Together with a phase,  $i^\phi, \phi \in \mathbb{Z}_4$ , a generic element of the Pauli group on our  $n^2$  qubits is given by  $i^\phi P(a,b)$ . We will often refer to a Pauli operator as being made up of  $X$  and  $Z$  operators, noting that when both appear, the actual Pauli operator is the  $iY$  operator.

We begin by defining three sets of operators which are essential to understanding the subsystem structure of our qubits. Each of these sets will be made up of Pauli operators.

The first set of Pauli operators which will concern us,  $\mathcal{T}$ , is made up of Pauli operators which have an even number of  $X_{i,j}$  operators in each column and an even number of  $Z_{i,j}$  operators in each row:

$$\mathcal{T} = \left\{ (-1)^\phi P(a,b) \mid \phi \in \mathbb{Z}_2, \bigoplus_{i=1}^n a_{i,j} = 0, \text{ and } \bigoplus_{j=1}^n b_{i,j} = 0 \right\}, \quad (8)$$

where  $\oplus$  denotes the binary exclusive-or operation (we use it interchangeably with the direct sum operation with context distinguishing the two uses.) Note that these operators form a group under multiplication. This group can be generated by nearest neighbor operators on our cubic lattice

$$\mathcal{T} = \langle X_{i,j} X_{i+1,j}, Z_{j,i} Z_{j,i+1}, \forall i \in \mathbb{Z}_{n-1}, j \in \mathbb{Z}_n \rangle. \quad (9)$$

Examples of elements of the group  $\mathcal{T}$  are diagrammed in Fig. 1.

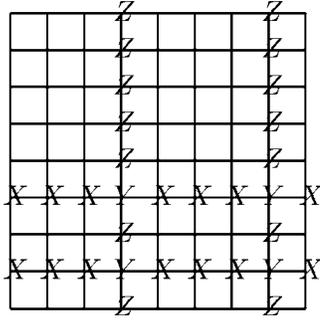


FIG. 2. A nontrivial element of the group  $\mathcal{S}$ . This element has an even number of columns which are entirely  $X$  operators multiplied by an even number of rows which are entirely  $Y$  operators. Notice how the  $Y$  elements appear where both of these conditions are met.

The second set of Pauli operators we will be interested in is a subset of  $\mathcal{T}$ , which we denote  $\mathcal{S}$ .  $\mathcal{S}$  consists of Pauli operators which are made up of an even number of rows consisting entirely of  $X$  operators and an even number of columns consisting entirely of  $Z$  operators,

$$\mathcal{S} = \left\{ P(a,b) \left| \bigoplus_{i=1}^n \left( \bigwedge_{j=1}^n a_{i,j} \right) = 0, \bigoplus_{j=1}^n \left( \bigwedge_{i=1}^n b_{i,j} \right) = 0 \right. \right\} \quad (10)$$

where  $\wedge$  is the binary and operation.  $\mathcal{S}$  is also a group. In fact it is an Abelian subgroup of  $\mathcal{T}$ . Further, all of the elements of  $\mathcal{S}$  commute not just with each other, but with all of the elements of  $\mathcal{T}$ . It can be generated by nearest row and column operators,

$$\mathcal{S} = \left\langle \prod_{i=1}^n X_{j,i} X_{j+1,i}, \prod_{i=1}^n Z_{i,j} Z_{i,j+1}, \forall j \in \mathbb{Z}_{n-1} \right\rangle. \quad (11)$$

These generators will be particularly important to us, so we will denote them by

$$S_i^X = \prod_{j=1}^n X_{i,j} X_{i+1,j}, \quad \text{and} \quad S_j^Z = \prod_{i=1}^n Z_{i,j} Z_{i,j+1}. \quad (12)$$

$\mathcal{S}$  is a stabilizer group familiar from the standard theory of quantum error-correcting codes. An example of an element in  $\mathcal{S}$  is given in Fig. 2

The final set of operators which we will consider,  $\mathcal{L}$ , is similar to  $\mathcal{S}$  except that the evenness condition becomes an oddness condition,

$$\mathcal{L} = \left\{ (-1)^\phi P(a,b) \left| \phi \in \mathbb{Z}_2, \bigoplus_{i=1}^n \left( \bigwedge_{j=1}^n a_{i,j} \right) = 1, \bigoplus_{j=1}^n \left( \bigwedge_{i=1}^n b_{i,j} \right) = 1 \right. \right\}. \quad (13)$$

This set does not by itself form a group, but together with  $\mathcal{S}$  it does form a group. This combined group is not Abelian.  $\mathcal{L}$  has the property that all of its elements commute with those of  $\mathcal{T}$  and  $\mathcal{S}$ . A nontrivial element of  $\mathcal{L}$  is given in Fig. 3.

### B. Subsystem structure

We will now elucidate how  $\mathcal{T}$ ,  $\mathcal{S}$ , and  $\mathcal{L}$  are related to a subsystem structure on our  $n^2$  qubits. Let  $\mathcal{H} = (\mathbb{C}^2)^{n^2}$  denote the Hilbert space of our  $n^2$  qubits.

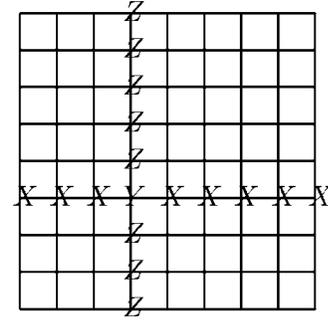


FIG. 3. A nontrivial element of the set  $\mathcal{L}$ . This element has an odd number of columns which are entirely  $X$  operators multiplied by an odd number of rows which are entirely  $Y$  operators. It represents an encoded  $Y$  operator on the encoded qubit as described in Sec. IV D.

We first note that since  $\mathcal{S}$  consists of a set of mutually commuting observables, we can use these observables to label subspaces of  $\mathcal{H}$ . In particular, we can label these subspaces by the  $2(n-1) \pm 1$ -valued eigenvalues of the  $S_i^X$  and  $S_i^Z$  operators. Let us denote these eigenvalues by  $s_i^X$  and  $s_i^Z$ , respectively, and the length  $n-1$  string of these  $\pm 1$  eigenvalues by  $s^X$  and  $s^Z$ . We can thus decompose  $\mathcal{H}$  into subspaces as

$$\mathcal{H} = \bigoplus_{s_1^X, \dots, s_{n-1}^X, s_1^Z, \dots, s_{n-1}^Z = \pm 1} \mathcal{H}_{s^X, s^Z} = \bigoplus_{s^X, s^Z} \mathcal{H}_{s^X, s^Z}. \quad (14)$$

By standard arguments in the stabilizer formalism, each of the  $\mathcal{H}_{s^X, s^Z}$  subspaces is of the dimension  $d = 2^{n^2 - 2(n-1)}$ . Just for completeness, we note that the operators  $S_i^X$  and  $S_i^Z$  act under this decomposition as

$$S_i^X = \bigoplus_{s^X, s^Z} s_i^X I_{2^{n^2 - 2(n-1)}},$$

$$S_i^Z = \bigoplus_{s^X, s^Z} s_i^Z I_{2^{n^2 - 2(n-1)}}. \quad (15)$$

Now examine the two groups  $\mathcal{T}$  and the group generated by elements of  $\mathcal{L}$  and  $\mathcal{S}$ . Both of these groups are non-Abelian. All of the elements of  $\mathcal{T}$  and  $\mathcal{L}$  commute with elements of  $\mathcal{S}$ . Further, all of the elements of  $\mathcal{T}$  and  $\mathcal{L}$  commute with each other. This implies, via Schur's lemma [34,35], that  $\mathcal{L}$  and  $\mathcal{T}$  must be represented on  $\mathcal{H}_{s^X, s^Z}$  by a subsystem action. In particular, the full Hilbert space splits as

$$\mathcal{H} = \bigoplus_{s^X, s^Z} \mathcal{H}_{s^X, s^Z}^{\mathcal{T}} \otimes \mathcal{H}_{s^X, s^Z}^{\mathcal{L}}, \quad (16)$$

such that operators from  $T \in \mathcal{T}$  act on the first tensor product

$$T = \bigoplus_{s^X, s^Z} T_{s^X, s^Z} \otimes I_2, \quad \forall T \in \mathcal{T}, \quad (17)$$

and the operators from  $L \in \mathcal{L}$  act on the second tensor product

$$L = \bigoplus_{s^X, s^Z} I_{2^{(n-1)^2}} \otimes L_{s^X, s^Z}, \quad \forall L \in \mathcal{L}. \quad (18)$$

Here we have assigned dimensions  $2^{(n-1)^2}$  and 2 to these tensor product spaces. To see why these dimensionalities arise we will appeal to the stabilizer formalism. We note that

modulo the stabilizer structure of  $\mathcal{S}$ ,  $\mathcal{L}$  is a single encoded qubit. Similarly if one examines the following set of  $(n-1)^2$  operators from  $\mathcal{T}$ :

$$\bar{Z}_{i,j} = Z_{i,j}Z_{i,j+1}, \quad \bar{X}_{i,j} = \prod_{k=1}^j X_{i,k}X_{n-1,k}, \quad (19)$$

where  $i \in \mathbb{Z}_{n-1}, j \in \mathbb{Z}_{n-1}$ , one finds that modulo the stabilizer they are equivalent to  $(n-1)^2$  encoded Pauli operators.

The subsystem code we now propose encodes a single qubit into the Hilbert space  $\mathcal{H}_{s^X, s^Z}^{\mathcal{L}}$  with  $s_i^X = s_i^Z = +1, \forall i \in \mathbb{Z}_{n-1}$  (choices with other  $\pm 1$  choices form an equivalent code in the same way that stabilizer codes can be chosen for different stabilizer generator eigenvalues.) The code we propose thus encodes one qubit of quantum information into a subsystem of the  $n^2$  ‘‘bare’’ qubits. We stress that the encoding we perform is truly a subsystem encoding: we do not care what the state of the  $\mathcal{H}_{s^X, s^Z}^{\mathcal{T}}$  subsystem is. For simplicity it may be possible to begin by encoding into a subspace which includes our particular subsystem (i.e., by fixing the state on  $\mathcal{H}_{s^X, s^Z}^{\mathcal{T}}$ ) but this encoding is not necessary and indeed, after our recover routine for the information encoded into  $\mathcal{H}_{s^X, s^Z}^{\mathcal{L}}$ , we will not know the state of the  $\mathcal{H}_{s^X, s^Z}^{\mathcal{T}}$  subsystem. We will denote the Hilbert space  $\mathcal{H}_{s^X, s^Z}^{\mathcal{L}}$  with  $s_i^X = s_i^Z = +1, \forall i \in \mathbb{Z}_{n-1}$  by  $\mathcal{H}_{s^X, s^Z = \{+1\}^{n-1}}^{\mathcal{L}}$ .

### C. Subsystem error-correcting procedure

If we encode quantum information into the subsystem  $\mathcal{H}_{s^X, s^Z = \{+1\}^{n-1}}^{\mathcal{L}}$ , then what sort of error-correcting properties does this encoding result in? We will see that the  $\mathcal{S}$  operators can be used to perform an error correcting procedure which restores the information on  $\mathcal{H}_{s^X, s^Z = \{+1\}^{n-1}}^{\mathcal{L}}$ , but which often acts nontrivially on the subsystem  $\mathcal{H}_{s^X, s^Z = \{+1\}^{n-1}}^{\mathcal{T}}$ . This exploitation of the subsystem structure in the correction procedure is what distinguishes our subsystem operator quantum error-correcting code from standard subspace quantum error-correcting codes.

Suppose that a Pauli error  $P(a, b)$  occurs on our system. For a Pauli operator  $P(a, b)$ , define the following error strings:

$$e_j(a) = \bigoplus_{i=1}^n a_{i,j}, \quad f_i(b) = \bigoplus_{j=1}^n b_{i,j}. \quad (20)$$

Notice that if  $e_j = f_i = 0, \forall i, j$ , then this implies that  $P(a, b)$  is in the set  $\mathcal{T}$ . Further note that in this case, the effect of  $P(a, b)$  is to only act on the  $\mathcal{H}_{s^X, s^Z}^{\mathcal{T}}$  subsystems, i.e.,  $P(a, b)$  is a block diagonal under our subsystem decomposition, Eq. (16), acting as

$$P(a, b) = \bigoplus_{s^X, s^Z} E_{s^X, s^Z}(a, b) \otimes I_2, \quad (21)$$

where  $E_{s^X, s^Z}(a, b)$  is a nontrivial operator depending on the subspace labels  $s^X, s^Z$ , and the type of Pauli error  $(a, b)$ . Therefore errors of this form ( $e_j = f_i = 0$ ) do not cause errors on our information encoded in  $\mathcal{H}_{s^X, s^Z = \{+1\}^{n-1}}^{\mathcal{L}}$ . With respect to

the errors of this form, the information is encoded into a noiseless subsystem [18].

Returning now to the case of a general  $P(a, b)$ , from the above argument we see that if we can apply a Pauli operator  $Q(c, d)$  such that  $Q(c, d)P(a, b)$  is a new error, call it  $R(a', b')$ , which has error strings  $e'_j(a') = f'_i(b') = 0, \forall i, j$ , then we will have a procedure for fixing the error  $P(a, b)$ , modulo the subsystem structure of our encoded quantum information. In other words, our error-correcting procedure need not result in producing the identity action on the subspace labeled by  $s_i^X = s_i^Z = +1, \forall i \in \mathbb{Z}_{n-1}$ , but need only produce the identity action on the subsystem  $\mathcal{H}_{s^X, s^Z = \{+1\}^{n-1}}^{\mathcal{L}}$ . We can perform just such a procedure by using the elements of  $\mathcal{S}$  as a syndrome for which errors of small enough size can be corrected.

To see how this works, suppose  $P(a, b)$  occurs on our system. Then note that measuring  $S_i^X$  is equivalent to determining

$$\bigoplus_{j=1}^n (b_{i,j} \oplus b_{i+1,j}) = f_i(b) \oplus f_{i+1}(b), \quad (22)$$

and similarly measuring  $S_j^Z$  is equivalent to determining

$$\bigoplus_{i=1}^n (a_{i,j} \oplus a_{i,j+1}) = e_j(a) \oplus e_{j+1}(a). \quad (23)$$

Note that all  $2(n-1)$  of these measurements can be performed simultaneously since the elements of  $\mathcal{S}$  all commute with each other. We wish to use these measurement outcomes to restore the system to  $e_j(a) = f_i(b) = 0$  (if possible).

To see how to do this, treat the  $f_i(b)$  as a  $n$  bit codeword for a simple redundancy code [i.e., the two codewords are  $f_i(b) = 0, \forall i$  and  $f_i(b) = 1, \forall i$ ]. A similar procedure will hold for the  $e_i(b)$ . Measuring the  $n-1$  operators  $S_i^X$  is equivalent to measuring the syndrome of our redundancy code. In particular, we can use this syndrome to apply an error-correcting procedure for the  $f_i(b)$  bit strings. The result of this correction procedure is to restore the system to either the codeword  $f_i(b) = 0, \forall i$  or the codeword  $f_i(b) = 1, \forall i$ . The former corresponds to an error correction procedure which can succeed [given that an equivalent procedure for the  $e_j(a)$  bit strings also succeeds], whereas the latter procedure is one where the error-correction procedure will fail. Notice that our error-correcting procedure, when it succeeds, is only guaranteed to restore the system to  $f_i(b) = 0$  and  $e_j(b) = 0$ , and thus the full effect of the procedure may be to apply some nontrivial operator to the  $\mathcal{H}_{s^X, s^Z}^{\mathcal{T}}$  subsystems.

Let us be more detailed in describing the error-correcting procedure for the  $f_i(b)$  code words. Let  $s_i^X$  be the result of our measurements of the  $S_i^X$  operators. Given the  $s_i^X$  we can construct two possible bit strings  $f'_i$  and  $\neg f'_i$  ( $\neg$  denotes the negation operation) consistent with these measurements. Let  $H(f')$  and  $H(\neg f')$  denote the Hamming weight of these bit strings [i.e.,  $H(f')$  is the number of 1s, in the  $n$  bits  $f'_i$ ] and define  $f''$  to be the bit string  $f'$  or  $\neg f'$  with the smallest Hamming weight. We now apply an operation consisting only of  $Z_{i,j}$  operators. In particular, we apply the operator

$$Q_1(f^n) = \prod_{i=1}^n Z_{i,j_0}^{\delta_{f_i,1}}, \quad (24)$$

for any fixed column index  $j_0$ . The operator  $Q_1(f^n)P(a,b)$  is then seen to be of one of two forms: either this new operator has the  $Z$  error string equal to all zeros or all ones. In the first case we have successfully restored the system to the all  $f_i(b)=0$  codeword, whereas for the second case, we have failed. How many  $Z$  errors can be corrected in this fashion? If  $P(a,b)$  consisted of  $Z$  errors  $b$  with an error string  $f_i(b)$  with a Hamming weight of this string  $H(f)$  which is less than or equal to  $\lfloor(n-1)/2\rfloor$ , then the correction procedure will succeed. Thus the code we have constructed is a  $[[n^2, n, 1]]$  code: it encodes a single qubit into  $n^2$  qubits and has a distance  $n$ .

Above we have focused on the case of Pauli  $Z$  errors. Clearly an analogous argument holds for Pauli  $X$  errors (with the role of the rows and columns reversed.) Further, Pauli  $Y$  errors are taken care of by the combined action of these two procedures. By the standard arguments of digitizing errors in quantum error-correcting codes, we have thus shown how our operator quantum error-correcting subsystem code can correct up to  $\lfloor(n-1)/2\rfloor$  arbitrary single qubit errors.

#### D. Logical operators

We comment here on the logical operators (operators which act on the encoded subsystem) for this code. From our analysis of the subsystem structure, it is clear that elements of  $\mathcal{L}$  act on the subsystem. Thus, for instance, the effect of a row of Pauli  $X$  operators is to enact an encoded Pauli  $X$  operation on the coded subsystem. We can choose a labeling of the subsystem such that

$$\bar{X} = \prod_{j=1}^n X_{1,j} = \bigoplus_{s^X, s^Z} I_{2^{(n-1)^2}} \otimes X, \quad (25)$$

while in the same basis the effect of a column of Pauli  $Z$  operators is to enact an encoded Pauli  $Z$  on the coded subsystem,

$$\bar{Z} = \prod_{i=1}^n Z_{i,1} = \bigoplus_{s^X, s^Z} I_{2^{(n-1)^2}} \otimes Z. \quad (26)$$

These two operators can then be used to enact any Pauli operator on the encoded quantum information. Notice that other elements of  $\mathcal{L}$  also act as encoded Pauli operators on  $\mathcal{H}_{s^X=s^Z=\{+1\}^{n-1}}^{\mathcal{L}}$  (but act with differing signs on the other  $s^X, s^Z$  labeled subspaces.) An important property of these logical operators is that they can be enacted by performing single qubit operators and no coupling between the different qubits is needed. This is important because it will allow us to assume an independent error model for errors which occur when we imprecisely implement these gates on our encoded quantum information. Not only can the above construction be used to implement the Pauli operators on our subsystem code, it can also be used to measure the Pauli operators on our subsystem code.

Another easily implementable operation on our code is a logical controlled NOT operation. Suppose we take two iden-

tically sized two-dimensional codes and stack them on top of each other. Then the application of a transverse controlled NOT operator between all  $n^2$  of these two systems will enact a logical controlled NOT operator between the two encoded qubits. To see this note that if we treat the elements of the set  $\mathcal{S}$  as a stabilizer code, then these transverse operators preserve the combined stabilizer  $\mathcal{S} \times \mathcal{S}$  and that the action of the  $n^2$  controlled NOT gates do not mix the  $\mathcal{L} \times \mathcal{L}$  and  $\mathcal{T} \times \mathcal{T}$  operators.

Gottesman [15,32] has shown that given the ability to measure and apply the encoded Pauli operators along with the ability to perform a controlled NOT operation on a stabilizer code, one can perform any encoded operation which is in the normalizer of the Pauli group (i.e., the gate set relevant to the Gottesman-Knill theorem [32]).

We have seen how to implement encoded Pauli operators and the controlled NOT operation on the information encoded into our subsystem. These operations do not allow for universal quantum computation, so an important open question for our subsystem code is to find an easily implementable method for completing this gateset to a universal set of gates.

#### E. Hamiltonian model of the two-dimensional subsystem code

An interesting offshoot of the above two-dimensional operator quantum error-correcting subsystem is the analysis of a particularly simple Hamiltonian whose ground state has a degeneracy which corresponds to the subsystem code. We introduce this Hamiltonian here in order to make our analysis of a similar Hamiltonian for our three-dimensional subsystem code more transparent. The Hamiltonian is given by nearest neighbor interactions constructed entirely from operators in the set  $\mathcal{T}$ ,

$$H = -\lambda \sum_{i=1}^n \sum_{j=1}^{n-1} (Z_{i,j} Z_{i,j+1} + X_{j,i} X_{j+1,i}). \quad (27)$$

Since  $H$  is constructed entirely from elements of  $\mathcal{T}$ , this Hamiltonian can be decomposed as

$$H = \bigoplus_{s^X, s^Z} H_{s^X, s^Z} \otimes I_2. \quad (28)$$

To understand the exact nature of this Hamiltonian, we would need to diagonalize each of the  $H_{s^X, s^Z}$ . What can be said, however, is that the ground state of the system will arise as the ground state of one or more of the  $H_{s^X, s^Z}$  (numerical diagonalization of systems with a few qubits show that the ground state comes from only the  $s_i^X = s_i^Z = +1$  subsystem and we conjecture that this subspace always contains the ground state.) If  $k$  of the  $H_{s^X, s^Z}$  contribute to the ground state, the degeneracy of the ground state will be  $2k$  due to the subsystem corresponding to the  $\mathcal{L}$  operators. Thus we see that we can encode quantum information into the subsystem degeneracy of this Hamiltonian, in the similar manner that information is encoded into the ground state of a Hamiltonian related to the toric code [16,36,37].

Indeed, Douçot and collaborators [38] have studied the Hamiltonian, Eq. (27), and suggested that it might possibly

allow for the robust storage of quantum information in its ground state. These authors discuss the symmetries we have described above for the model, but they do not make the connection of these symmetries to a subsystem error correcting code. Recently, however, Dorier and collaborators [39] have given strong numerical evidence that this system has a vanishing gap between the ground state and excited states of this system. This suggests that only for small size systems will there be any benefit in encoding quantum information into the ground state of this system *à la* toric code [16,36,37].

In Sec. VI we will return to introduce a similar Hamiltonian for our three-dimensional operator quantum error-correcting subsystem.

### V. THREE-DIMENSIONAL OPERATOR QUANTUM-ERROR CORRECTING SUBSYSTEM

We now turn to a three-dimensional operator quantum error-correcting subsystem which is a generalization of the two-dimensional subsystem code we presented above. In particular, whereas the construction for the two-dimensional model relied on the structure of  $\mathcal{T}$  containing Pauli operators with an even number of Pauli  $Z$ 's in a row and an even number of Pauli  $X$ 's in a column, in the three-dimensional case we rely on a new set of operators with an even number of Pauli  $Z$ 's in the  $yz$  plane and an even number of Pauli  $X$ 's in the  $xy$  plane.

Consider a cubic lattice of size  $n \times n \times n$  with qubits located at the vertices of this lattice and let  $n$  be odd. Let  $O_{i,j,k}$  denote the operator  $O$  acting on the qubit located at the  $(i,j,k)$ th lattice site tensored with identity on all other qubits. We again use a compact notation to denote the Pauli operator on our  $n^3$  qubits by using two  $n^3$  bit strings,

$$\begin{aligned}
 P(a,b) &= \prod_{i,j,k=1}^n X_{i,j,k}^{a_{i,j,k}} Z_{i,j,k}^{b_{i,j,k}} \\
 &= \prod_{i,j,k=1}^n \begin{cases} X_{i,j,k} & \text{if } a_{i,j,k} = 1 \text{ and } b_{i,j,k} = 0 \\ Z_{i,j,k} & \text{if } a_{i,j,k} = 0 \text{ and } b_{i,j,k} = 1, \\ -iY_{i,j,k} & \text{if } a_{i,j,k} = 1 \text{ and } b_{i,j,k} = 1 \end{cases}
 \end{aligned} \tag{29}$$

where  $a, b \in \mathbb{Z}_2^{n^3}$  are  $n$  by  $n$  by  $n$  arrays of bits.

As in the two-dimensional case, we will define three sets of operators,  $\mathcal{T}_3$ ,  $\mathcal{S}_3$ , and  $\mathcal{L}_3$  which are essential to understanding the subsystem structure of our qubits.

The first set of Pauli operators which will concern us,  $\mathcal{T}_3$ , is made up of Pauli operators which have an even number of  $X_{i,j,k}$  operators in each  $xy$  plane and an even number of  $Z_{i,j,k}$  operators in each  $yz$  plane:

$$\begin{aligned}
 \mathcal{T}_3 &= \left\{ (-1)^\phi P(a,b) \mid \phi \in \mathbb{Z}_2, \bigoplus_{i,j=1}^n a_{i,j,k} = 0, \right. \\
 &\quad \left. \text{and } \bigoplus_{j,k=1}^n b_{i,j,k} = 0 \right\}.
 \end{aligned} \tag{30}$$

These operators, like the analogous two-dimensional  $\mathcal{T}$  form

a group under multiplication. This group can be generated by nearest neighbor operators on our cubic lattice

$$\begin{aligned}
 \mathcal{T}_3 &= \langle X_{k,i,j} X_{k+1,i,j}, X_{i,k,j} X_{i,k+1,j}, Z_{i,j,k} Z_{i,j,k+1}, Z_{i,k,j} Z_{i,k+1,j}, \\
 &\quad \forall i,j \in \mathbb{Z}_n, k \in \mathbb{Z}_{n-1} \rangle
 \end{aligned} \tag{31}$$

The second set of the Pauli operators that we will be interested in is a subset of  $\mathcal{T}_3$ , which we denote  $\mathcal{S}_3$ .  $\mathcal{S}_3$  consists of Pauli operators which are made up of an even number of  $xy$  planes made entirely of Pauli  $Z$  operators and an even number of  $yz$  planes made entirely of Pauli  $X$  operators:

$$\mathcal{S}_3 = \left\{ P(a,b) \Big| \bigoplus_{i=1}^n \bigwedge_{j,k=1}^n a_{i,j,k} = 0, \bigoplus_{k=1}^n \bigwedge_{i,j=1}^n b_{i,j,k} = 0 \right\}. \tag{32}$$

$\mathcal{S}_3$  is an Abelian subgroup of  $\mathcal{T}_3$  and all of the elements of  $\mathcal{S}_3$  commute with all of the elements of  $\mathcal{T}_3$ . It can be generated by nearest  $xy$ -plane and  $yz$ -plane operators:

$$\mathcal{S}_3 = \left\langle \prod_{i,j=1}^n X_{i,j,k} X_{i,i,k+1}, \prod_{i,j=1}^n Z_{k,i,j} Z_{k+1,i,j}, \forall k \in \mathbb{Z}_{n-1} \right\rangle. \tag{33}$$

We label these generators, as before:

$$S_k^X = \prod_{i,j=1}^n X_{i,j,k} X_{i,i,k}, \quad \text{and} \quad S_i^Z = \prod_{j,k=1}^n Z_{i,j,k} Z_{i+1,j,k}. \tag{34}$$

$\mathcal{S}_3$  is again a stabilizer group.

The final set of operators which we will consider,  $\mathcal{L}_3$ , is similar to  $\mathcal{S}_3$  except that the evenness condition becomes an oddness condition

$$\begin{aligned}
 \mathcal{L}_3 &= \left\{ (-1)^\phi P(a,b) \mid \phi \in \mathbb{Z}_2, \bigoplus_{i=1}^n \bigwedge_{j,k=1}^n a_{i,j,k} = 1, \right. \\
 &\quad \left. \bigoplus_{k=1}^n \bigwedge_{i,j=1}^n b_{i,j,k} = 1 \right\}.
 \end{aligned} \tag{35}$$

$\mathcal{L}_3$  together with  $\mathcal{S}_3$  forms a group and all of the elements of  $\mathcal{L}_3$  commute with those of  $\mathcal{T}_3$ .

#### A. Subsystem structure

All three of the sets,  $\mathcal{T}_3$ ,  $\mathcal{S}_3$ , and  $\mathcal{L}_3$  will play a directly analogous role to the sets  $\mathcal{T}$ ,  $\mathcal{S}$ , and  $\mathcal{L}$  in our two-dimensional model. In particular if we let  $\mathcal{H}$  denote the Hilbert space of our  $n^3$  qubits, then we can partition this space into subspaces labeled by the  $2(n-1)$  different  $\pm 1$  eigenvalues of the operators  $S_k^X$  and  $S_i^Z$  of Eq. (34). Again we will label these eigenvalues by  $s_k^X$  and  $s_i^Z$ , with  $s^X$  and  $s^Z$  labeling these strings. The Hilbert space of the system then decomposes as

$$\mathcal{H} = \bigoplus_{s_1^X, \dots, s_{n-1}^X, s_1^Z, \dots, s_{n-1}^Z = \pm 1} \mathcal{H}_{s^X, s^Z} = \bigoplus_{s^X, s^Z} \mathcal{H}_{s^X, s^Z}. \tag{36}$$

Again, the  $\mathcal{H}_{s^X, s^Z}$  subspaces have a tensor product structure,  $\mathcal{H}_{s^X, s^Z} = \mathcal{H}_{s^X}^T \otimes \mathcal{H}_{s^Z}^T$ , such that elements of  $\mathcal{L}_3$  act as

$$L = \bigoplus_{s^X, s^Z} I_{2^{n^3-2n+1}} \otimes L_{s^X, s^Z}, \quad \forall L \in \mathcal{L}_3, \quad (37)$$

and those of  $\mathcal{T}_3$  act as

$$T = \bigoplus_{s^X, s^Z} T_{s^X, s^Z} \otimes I_2, \quad \forall T \in \mathcal{T}_3. \quad (38)$$

### B. Subsystem quantum error-correcting procedure

The subsystem error-correcting procedure for the three-dimensional code nearly directly mimics that of the two-dimensional code. Here we discuss how the subsystem error-correcting procedure works without going into the details as we did in the two-dimensional case. The three-dimensional procedure is nearly identical to that of the two-dimensional procedure with the sets  $\mathcal{T}$ ,  $\mathcal{S}$ , and  $\mathcal{L}$  interchanged with the sets  $\mathcal{T}_3$ ,  $\mathcal{S}_3$ , and  $\mathcal{L}_3$ , respectively.

We will again encode our quantum information into the  $\mathcal{H}_{s^X=s^Z=\{+1\}^{n-1}}^{\mathcal{L}}$  subsystem. Whereas for the two-dimensional code we defined error strings for the rows and column conditions of the set  $\mathcal{T}$ , now we define error strings for the  $xy$  and  $yz$ -plane conditions of the set  $\mathcal{T}_3$ . If the Pauli error  $P(a, b)$  occurs on our system, then we can define the two error strings

$$e_k(a) = \bigoplus_{i,j=1}^n a_{i,j,k} \quad \text{and} \quad f_i(b) = \bigoplus_{j,k=1}^n b_{i,j,k}. \quad (39)$$

Pauli errors with  $e_k(a)=f_i(b)=0$  are errors from  $\mathcal{T}_3$  and act trivially on the information encoded into the  $\mathcal{H}_{s_k^X=s_i^Z=+1}^{\mathcal{L}}$  subsystem.

The quantum error-correction procedure is then directly analogous to the one for the two-dimensional code. We measure the  $s_k^X$  and  $s_i^Z$  operators and treat these as nearest neighbor parity checks for a redundancy code on the  $f_i(b)$  and  $e_k(a)$ , respectively. Then in direct analogy with the two-dimensional code, we can apply a subsystem error-correcting procedure that restores the system modulo the subsystem structure. The three-dimensional code is a  $[[n^3, n, 1]]$  code. We have thus gained nothing in terms of the distance of the code, but, as we will argue in the next section, the three-dimensional code when converted to a Hamiltonian whose ground state is the subsystem code has intriguing features not found in the two-dimensional code.

### C. Logical operators

The logical operators for the three-dimensional code are directly analogous to those in the two-dimensional code. As in the two-dimensional code, operators from  $\mathcal{L}_3$  can be used to enact Pauli operators on the information encoded into the  $\mathcal{H}_{s_k^X=s_i^Z=+1}^{\mathcal{L}}$  subsystem. Similarly, we can enact a controlled NOT operation between two encoded qubits by performing  $n^3$  controlled NOT gates between two identical copies of the code. This allows us to again perform any operation in the normalizer of the Pauli group on our encoded qubits.

## VI. SELF-CORRECTION IN THE THREE-DIMENSIONAL EXAMPLE

In the 1930s, when Alan Turing wrote his papers [40,41] laying out the foundations of computer science, there was absolutely no reason to believe that any computing device such as the one described by Turing could actually be built. One of the foremost problems, immediately apparent to the engineers of the day, was the lack of reliable components out of which a computer could be built. Von Neumann solved this problem, in theory, by showing that robust encoding of the classical information could be used to overcome errors and faulty components in a computer [42]. Despite Von Neumann's theoretical ideas, it took the invention of the transistor and the integrated circuit, to mention only the broadest innovations, in order to bring forth the technological movement now known as the computer revolution. The overarching result of the technological innovations responsible for the computer revolution was the development of techniques which exhibited Von Neumann's theoretical ideas in a *natural setting*. Modern computers naturally correct errors in both the storage and manipulation of classical information. The task of robust storage and manipulation of the data is essentially guaranteed by the physics of these devices. *There are distinct physical reasons why robust storage and manipulation of classical information is possible.*

If there are distinct physical reasons why robust storage and manipulation of classical information is possible, an obvious question to ask in the quantum information sciences is whether we can mimic these effects in the quantum domain. Do there exist, or can we engineer, physical systems whose physics ensure that the robust storage and manipulation of quantum information is possible? In this section, we will present evidence, in the form of a mean field argument, that a Hamiltonian related to the three-dimensional subsystem code might exactly be this type of system.

### A. Self-correcting quantum memories

The traditional approach to building a robust fault-tolerant quantum computer imagines building the computer using a complex microarchitecture of quantum error-correcting fault-tolerant procedures. This poses a severe technological overhead of controlling thousands of qubits in a complex manner, simply to get a single robust qubit. Kitaev [16,43] suggested that an alternative, less complex, method to constructing a fault-tolerant quantum computer might be possible. Kitaev showed that there exists a quantum error-correcting code, the toric code, which is the degenerate ground state of a certain four-body spatially local Hamiltonian on a two-dimensional lattice of qubits. Kitaev imagined encoding quantum information into the ground state of this system and then, because there is an energy gap between the ground state of this system and the first excited state and because the errors which will destroy quantum information consist of errors which scale like the size of the lattice, this quantum information would be protected from decoherence due to the environment as long as the temperature of the environment was sufficiently low.

It is important to note that the Hamiltonian implementation of Kitaev's toric code (by which we mean encoding information into a physical system governed by the four-body Hamiltonian associated with the toric code), while providing a mechanism for the robust storage of quantum information, does not provide a full fault-tolerant method for quantum computation. The reason for this is that during the implementation of the physical processes which manipulate the information encoded into the ground state of the system, real excitations will be created which will disorder the system. This distinction has been confused in a manner because the toric codes *can* be used to construct a fault-tolerant quantum computer, but only with the aid of the external quantum control which serves to identify and correct errors which occur during the manipulation of the information encoded into the system (see, for example, [36]).

The idea of a self-correcting quantum memory is to overcome the limitations of Kitaev's original model by constructing a physical system whose energy levels not only correspond to a quantum error-correcting code (in our case a subsystem code), but which also uses the energetics of this system to actively correct real errors created when the quantum information is being manipulated. Thus, while in the toric codes, a single real error on the system can create excitations which can disorder the system, in a self-correcting system, a single real error on the system cannot disorder the system.

In order to explain the distinction of a self-correcting memory from the original toric code we will compare the situation to that of the one-dimensional and two-dimensional classical ferromagnetic Ising model. These models will be analogous to the toric code Hamiltonian model and a self-correcting Hamiltonian model, respectively.

Recall that in a ferromagnetic Ising model one takes a lattice of classical spins and these are coupled by Ising interactions between the neighboring spins via a Hamiltonian

$$H = -\frac{J}{2} \sum_{\langle i,j \rangle} s_i s_j, \quad (40)$$

where  $s_i \in \{\pm 1\}$  are the spin variables, the sum  $\langle i,j \rangle$  is over neighbors in the lattice, and  $J > 0$ . Notice that the ground state of this Hamiltonian corresponds to the uniform states  $s_i = +1, \forall i$  or  $s_i = -1, \forall i$ . These, of course, are also the two codewords for a classical redundancy code. Thus we can imagine that we encode classical information into the ground state of this Hamiltonian in direct analogy to the way in which information (but quantum this time) is encoded into the ground state of the toric code. Errors on the Ising codes are just bit flips. From hereon in when we refer to the Ising model we will implicitly be discussing the ferromagnetic Ising model.

Recall some basic properties of the one- and two-dimensional Ising models (see, for example [44]). We begin by discussing the thermal equilibrium values of the total magnetization,

$$M = \sum_i s_i, \quad (41)$$

of these models. In one dimension, for any  $T > 0$ , the total magnetization of the Ising model vanishes in thermal equi-

librium, whereas for the two-dimensional Ising model, the magnetization is zero above some critical temperature  $T_c$  and below this temperature, two magnetizations of equal magnitude and opposite signs are maintained. Since the magnetization is a measure of the information recorded in the redundancy code, we see that if we encode information into the ground state of the one-dimensional Ising model and this system is allowed to reach thermal equilibrium, then this information will be destroyed. On the other hand, for the two-dimensional Ising model, if we encode information into the ground state and the system is below the critical temperature  $T_c$ , then this information will be maintained. Above  $T_c$ , like the one-dimensional Ising model, the information will be destroyed. From the point of view of storing the information in the thermal states of these models, the two-dimensional Ising model is a robust medium, but the one-dimensional Ising model is not.

But what about the properties of the Ising models on the way to reaching equilibrium (i.e., during the time evolution with the environment)? In the one-dimensional case we find that the system will generically (depending on the exact method of relaxation) take a time which is suppressed like a Boltzmann factor  $e^{-J/T}$ . Thus at low enough temperature, we can encode information into the ground state of the one-dimensional Ising model and it will be protected for a long time. While the scaling of this decay rate is favorable in the temperature  $T$ , this type of approach is different from what is done in standard error corrections where a larger redundancy can be used to overcome errors without changing the error rate (as long as that error rate is below the threshold for the error-correcting code.)

What happens for the time evolution of a two-dimensional Ising model? If we start the system in one of the redundancy code states, then far below  $T_c$  the system will relax quickly to the closest of the two equilibrium states with a large total magnetization. As we raise the temperature closer to  $T_c$ , this relaxation will slow down. Above  $T_c$  the picture is similar to that of the one-dimensional Ising model that if we are close to  $T_c$ , then the relaxation to vanishing magnetization is suppressed like  $e^{-J/(T-T_c)}$ .

What are the main reasons for the differences in the ability to store information in the one- and two-dimensional Ising models? A rough heuristic of what is happening is that in the two-dimensional Ising model, the errors self-correct [45]. Consider starting the one-dimensional Ising model in the all  $s_i = +1$  state. Now flip one of the spins at the end of the chain. This will cost an energy  $J$ . Flipping the neighbor of this spin will then cost no energy. Proceeding along the chain in this manner one sees that one can expend energy  $J$  to turn the system from the codeword all in  $s_i = +1$  states to the all  $s_i = -1$  state. Thus the environment need only supply this energy to disorder the system. However, in the two-dimensional Ising model, something different happens. Suppose again that we start in the all  $s_i = +1$  state. Here if we flip a single spin (say on the boundary of the lattice) then the energy required to flip this spin is  $J$  times the number of bonds this spin has with its neighbors. Now flipping a neighbor will cost energy: in the two-dimensional model the energy cost of flipping a connected domain of spins is proportional to the perimeter of this domain. Since to get from the

all  $s_i = +1$  codeword to the all  $s_i = -1$  codeword we need to build a domain of the size of the entire lattice, we see that we will require at least an energy times the size of the lattice to disorder the system. Now suppose that errors are happening at some rate to all of the spins in the Ising models. In the one-dimensional model, once one creates a single error, there is no energy barrier to disordering the system. In the two-dimensional model, however, there is now an energy barrier. In particular, the system coupled to its environment will not only cause errors, but will also cause errors to be corrected by shrinking the domains of flipped errors. As long as the error rate is not too strong (which corresponds loosely to being below the critical temperature  $T_c$ ) the pathways that fix the error will dominate the actual creation of errors.

Thus we see that a two-dimensional Ising model operating below the critical temperature is performing a classical error correction on information stored in a redundancy code. In the one-dimensional Ising model and in the two-dimensional Ising model above the critical temperature, there is suppression due to a Boltzmann factor, but there is no self-correction (or the self-correction is not fast enough) and the information stored in the redundancy code is destroyed.

The two-dimensional anyon models of topological quantum computing and variations [16,36–39,43,46–57], including Kitaev's toric model, all share the property with the one-dimensional Ising model that the system can disorder using only an energy proportional to the gap in the Hamiltonian. (The models in [53] contain errors similar to those in the two-dimensional Ising model for certain types of quantum errors, but not for both phase and bit flip errors.) This can provide protection via an exponential suppression due to a Boltzmann factor, but this does not provide indefinite correction. The idea of a self-correcting quantum memory, however is to mimic the two-dimensional Ising model. In particular, below a critical temperature, quantum information stored in the system should persist even when the system is in thermal equilibrium and further the system should have a mechanism whereby real errors are corrected by the energetics of the system faster than when the real errors occur when operating below the critical temperature.

Finally, we note that there is one system which is widely suspected to be a self-correcting quantum memory. This is a version of the toric code on a four-dimensional lattice [36]. The problem with this model is that it exists in four dimensions and that it requires greater than two-qubit interactions in order to implement and is therefore not realistic for practical implementations. Our original motivation for considering the subsystem codes presented in this paper was to obtain a self-correcting system in the realistic setting of three or lower dimensions and using two-qubit interactions.

### B. The three-dimensional Hamiltonian

Next we turn our attention to a system which may be a self-correcting quantum memory. We provide evidence for this by showing that in a mean field approximation this Hamiltonian has properties for the expectation value of its energy which is similar to the energetics of the two-dimensional Ising model.

Consider the following Hamiltonian on a cubic lattice of qubits constructed exclusively from elements of  $\mathcal{T}_3$ ,

$$H = -\lambda \sum_{i,j=1}^n \sum_{k=1}^{n-1} (X_{k,i,j} X_{k+1,i,j} + X_{i,k,j} X_{i,k+1,j} + Z_{i,k,j} Z_{i,k+1,j} + Z_{i,j,k} Z_{i,j,k+1}), \quad (42)$$

with  $\lambda > 0$ . This Hamiltonian consists of Ising couplings along the direction  $x$  in the  $xy$  plane and along the direction  $z$  in the  $yz$  plane. As in the two-dimensional Hamiltonian, Eq. (27), we can use the fact that  $H$  is a sum of operators from  $\mathcal{T}_3$  to block diagonalize  $H$  with respect to the subsystem structure of our three-dimensional operator quantum error correcting subsystem:

$$H = \bigoplus_{s^x, s^z} H_{s^x, s^z} \otimes I_2. \quad (43)$$

Information can then be encoded into the  $I_2$  subsystem. We would like to show that if we perform such an encoding, then the information stored in this subsystem will be protected from the effect of general quantum errors in a manner similar to that of the two-dimensional Ising model.

### C. The mean field argument

Ignore for the moment the boundary conditions for the Hamiltonian and suppose that the ground state of Eq. (42) has the following properties for the expectation values of the Ising bonds in the system:

$$\begin{aligned} \langle X_{k,i,j} X_{k+1,i,j} \rangle_G &= c_{xx}, \\ \langle X_{i,k,j} X_{i,k+1,j} \rangle_G &= c_{xy}, \\ \langle Z_{i,k,j} Z_{i,k+1,j} \rangle_G &= c_{zy}, \\ \langle Z_{i,j,k} Z_{i,j,k+1} \rangle_G &= c_{zz}, \end{aligned} \quad (44)$$

where  $c_{\alpha\beta} > 0$ . In such a phase, the ground state energy is given by

$$E_G = \langle H \rangle_G = -\lambda n^2 (n-1) (c_{xx} + c_{xy} + c_{zy} + c_{zz}). \quad (45)$$

Now consider the effect of a single Pauli error on a single qubit in the lattice (assume this is away from the boundary). For example, consider a Pauli  $X$  error. Using the fact that  $X$  commutes with Ising bonds oriented along the  $x$  direction but anticommutes with Ising bonds oriented along the  $z$  direction, it then follows that expectation value of the energy of the system changes to

$$E_1 = \langle H \rangle_1 = E_G + 2\lambda (c_{zy} + c_{zy} + c_{zz} + c_{zz}), \quad (46)$$

where we have separated out each term arising from each of the four  $z$  direction Ising bonds that connect to the lattice site where the  $X$  error has occurred. Thus we see that a single bit flip error on the ground state will cause the expectation value of the energy to increase. Now consider the effect of applying a second Pauli  $X$  error which is a nearest neighbor to the original spin in the same  $yz$  plane. Now the Ising bond between these two errors does not contribute to the change in

the expectation value of the ground state, but all of the  $z$  direction Ising bonds around the perimeters of the two flipped spins do contribute. Thus, for example, if the spins are neighbors along the  $y$  direction, the expectation value of the energy of this state is

$$E_2 = \langle H \rangle_2 = E_G + 2\lambda(2c_{zy} + 4c_{zz}). \quad (47)$$

Generalizing the above argument we see that a connected domain of Pauli  $X$  errors in an  $yz$  plane will result in an energy increase proportional to the perimeter of the domain. A similar argument will hold for Pauli  $Z$  errors in an  $xy$  plane, but now the Ising bonds along the  $x$  direction will contribute to the change in expectation value, and those along the  $z$  will not contribute.

Suppose that a general error  $P(a, b)$  occurs on the ground state of  $H$ . Then in each plane  $yz$  plane the part of the error coming from  $a$  will produce excitations whose expectation of the energy scales like the perimeter of domains of errors in  $a$  and similarly for each  $xy$  plane, but now for the  $b$  component of the errors. From this argument we see that, at least for the expectation value of the energy, the system looks very similar to the two-dimensional Ising model. But now instead of only bit flip errors, more general quantum errors produce changes in the energy of the system that are proportional to the perimeter of the erred domain. Thus we argue that this provides evidence that the model we have presented will be self-correcting. For the same reasons that the two-dimensional Ising model will not disorder the classical information stored in a redundancy code [i.e., since our errors require (expected) energy proportional to the perimeter of the erred domain] we expect that the quantum information stored in our system will not disorder up to some critical temperature.

Of course, the evidence we have provided is based on numerous assumptions arising from our mean field model. First of all we have ignored boundary conditions. It is possible that certain edge states could disorder the system. Secondly we have only made arguments about the expectation value of the energy after errors have occurred to the ground state. This does not give us concrete information about the energy level structure of our Hamiltonian. It could be that while the expectation value scales like the perimeter, there are actually error pathways whose energetics are much less favorable. Third we have assumed the existence of a phase with the desired expectation values of the bond energies. This phase may not exist, i.e., it may be that in the thermodynamic limit, the expectation values all vanish. This would totally invalidate the mean field argument we have given above. Given these caveats our mean field argument only suggests that the system will be self-correcting. Clearly rigorously establishing whether our memory is self-correcting is a challenging open problem.

#### D. The quantum error-correcting order parameter

In the Ising model, an indication that the information stored in a redundancy code is still there after thermalization is the persistence of the total magnetization of the system. In particular, for the two-dimensional Ising model at a tempera-

ture between zero and the critical temperature, the magnetization in equilibrium is never exactly equal to its maximal value,  $\pm n^2$ . This is because there are always small domains of flipped spins due to the interaction of the system with its environment. However, a magnetization which is different from zero may be interpreted as a measurement of the majority vote for the redundancy code in this system. If we are going to establish that our quantum system is actually self-correcting, it is important to identify an order parameter for our system which can be used to reveal the presence of quantum information in our system. This can be done using the operator quantum error-correcting subsystem properties of the three-dimensional code.

Suppose we encode quantum information into a quantum error-correcting code and apply a number of quantum errors less than the number which the code has been designed to correct. We know from the theory of quantum error correction that the encoded information in this system can be recovered by the measurement of an appropriate error syndrome and the application of the appropriate recovery procedure. We can use this to construct an order parameter for any quantum error-correcting code.

A note about the nature of order parameters for quantum information before we describe this parameter for our three-dimensional model. In the classical Ising model, we found that below the critical temperature there was a bifurcation of the system into two magnetizations of equal and opposite magnitude. Since the classical information is based upon a bit, we are not surprised to find that such a bifurcation into two states happens. Actually, depending on the initial state of the system before it is thermalized, the thermal state of the system can be any value between these two extremes. But if we start our system by encoding into one of the two codestates (the all  $\pm 1$  state) then only the two bifurcated values will result after thermalization. What is the analogous situation for quantum information? For quantum information, we must show not that the bifurcation happens for a bit of encoded information but instead for a qubit of encoded information. Since a qubit is parametrized by the Bloch-sphere, one might expect that one needs an order parameter with similar properties. Such an order parameter can be constructed, but we can get away with examining fewer parameters in order to show the robustness of the quantum information. In particular if we make measurements along the  $x$ ,  $y$ , and  $z$  directions of the quantum information, then because of the linearity of the density operator we can use this to show that the information has been preserved. In particular, we can imagine encoding into one of the eigenstates of Pauli operators along these directions and looking at this system after it has thermalized. Notice now that instead of a single order parameter, we will have three order parameters. In order to demonstrate the self-correcting nature of our three-dimensional Hamiltonian, we will need to show that the expectation value of these three order parameters each bifurcate below some critical temperature.

Consider, now, the active recovery procedure for our three-dimensional subsystem code. We begin by measuring the  $S_k^x$  and  $S_i^z$  operators. Given these measurements, we can, as in the active recovery procedure, deduce an appropriate recovery operator to restore the information originally en-

coded into a subsystem, modulo the subsystem structure of the system. If we were to apply this syndrome and measure the encoded Pauli logical operators for the code (which were given in Sec. V C) then this would serve as an order parameter for our system. We note, however that if we are simply interested in measuring the Pauli logical operators and not in fully restoring the information into the original subsystem encoding, we do not need to actually apply the syndrome. This is because the syndrome we diagnose will either commute or anticommute with the encoded Pauli operator we wish to measure. Thus given the syndrome we can, instead of applying the appropriate recovery operation, simply flip or not flip the answer we get from measuring the encoded Pauli operator depending on the exact syndrome measured.

Now we can explicitly describe our order parameters. Suppose we measure the  $S_k^X$  and  $S_i^Z$  operators and obtain the values  $s_k^X$  and  $s_i^Z$  and we measure one of the encoded Pauli operators. Suppose, for example, that this encoded Pauli operator is an encoded  $X$  operator which we measure by measuring all of the Pauli  $X$  operators in a fixed  $xy$  plane. Given the  $s_k^X$  we can deduce whether an even or an odd number of Pauli  $Z$  errors will need to apply to the fixed  $xy$  plane to restore the quantum information in the subsystem (if possible). If this number is odd, then we simply flip the value of what we measure for the encoded Pauli  $X$  operator and if this number is even, we do not flip the value for the encoded  $X$  operator. Thus we see that we measure the  $x$  directional order parameter from our system by measuring the  $S_k^X$  and the encoded  $X$  operator and, as a function of these values, produce a single number representing the  $x$  directional order parameter. Similar comments hold for the order parameters along the other cardinal directions.

## VII. CONCLUSION

In this paper we have constructed a class of quantum error-correcting procedures based upon the notion of encoding quantum information into a subsystem. By encoding into a subsystem, we were able to demonstrate a recovery routine that explicitly used the subsystem structure. The three-dimensional code we constructed was shown to be related to a three-dimensional spin lattice system which we gave evidence for being a self-correcting quantum memory. We close by remarking on some open problems for this three-dimensional system and some thoughts about future directions for constructing self-correcting quantum systems.

The first open question concerns the implementation of our model in a physical system. A particularly promising system for such an implementation is with ultracold atoms trapped in an optical lattice [58]. Duan, Demler, and Lukin

[59] showed how to simulate a large class of spin-spin interactions for these systems. An open question is whether their techniques allow one to implement our three-dimensional anisotropic spin-spin Hamiltonian. Of particular concern is the magnitude of the spin-spin coupling which one can achieve in these models. This will directly effect the critical temperature for any self-correction that occurs in the system. A further concern for the physical implementation in an optical lattice is the ability to measure the syndrome operators  $S_k^X$  and  $S_i^Z$  along with appropriate logical Pauli operators. Finally one would like to understand how to produce an effective controlled NOT coupling between two such encoded lattices. Solutions to all of these problems would allow one to propose an experiment in which a self-correcting quantum memory could be demonstrated.

A second open question is, of course, whether our three-dimensional system is indeed self-correcting?. Noting that the Hamiltonian for this system does not possess a sign problem, one approach to verifying this question would proceed by performing quantum Monte Carlo simulations of this system. The order parameters we have described could then be simulated at finite temperature and evidence for self-correction could then be examined. Another promising approach is to use recent ideas in the density matrix renormalization group [60,61] to simulate the thermal properties of this system.

Another important question is whether one can design a self-correcting quantum system in two dimensions? This would be particularly desirable if one wishes to physically implement the self-correction in a solid state system.

Finally a large open question is what role operator quantum error-correcting subsystem codes can play in quantum information science? What do other such subsystem codes look like? Since these codes have important properties due to the fact that they exploit degenerate quantum codes, can subsystem codes be used to beat the quantum Hamming bound? [32] Further, results which relied on showing that there were no subspace codes with certain properties (for example, as in [62]) need to be reexamined in light of the existence of operator quantum error-correcting subsystem codes.

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