Quantum entanglement in the two-impurity Kondo model

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In order to quantify quantum entanglement in two-impurity Kondo systems, we calculate the concurrence, negativity, and von Neumann entropy. The entanglement of the two Kondo impurities is shown to be determined by two competing many-body effects, namely the Kondo effect and the Ruderman-Kittel-Kasuya-Yosida (RKKY) interaction, I. Due to the spin-rotational invariance of the ground state, the concurrence and negativity are uniquely determined by the spin-spin correlation between the impurities. It is found that there exists a critical minimum value of the antiferromagnetic correlation between the impurity spins which is necessary for entanglement of the two impurity spins. The critical value is discussed in relation with the unstable fixed point in the two-impurity Kondo problem. Specifically, at the fixed point there is no entanglement between the impurity spins. Entanglement will only be created [and quantum information processing (QIP) will only be possible] if the RKKY interaction exchange energy, I, is at least several times larger than the Kondo temperature, T_K. Quantitative criteria for QIP are given in terms of the impurity spin-spin correlation.

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I. INTRODUCTION

The potential of quantum information processing and quantum communication has led to numerous proposals of specific material systems for the creation and manipulation of entanglement in solid-state qubits [1–6]. Condensed-matter systems have several appealing features: (i) natural qubits such as single spin-$\frac{1}{2}$, (ii) the dream of scalability found in the solid-state technology that is the basis of classical computers, and (iii) the presence of strong interactions between qubits, such as spin exchange, which can create entanglement. Furthermore, even when there is no direct interaction between qubits, the interaction of the individual qubits with their environment can lead to an indirect interaction between qubits [7]. A concrete example of such an indirect interaction is the Ruderman-Kittel-Kasuya-Yosida (RKKY) interaction [8] between two localized spins interacting with the itinerant spins in a metal. This has led to several recent proposals to use the RKKY interaction to produce and manipulate entanglement in solid-state qubits [9–13].

In considering these proposals for solid-state quantum information processing, it is important to bear in mind some results from quantum information theory concerning the entanglement in mixed states. Even though entangled states result from interactions and exhibit certain correlations (e.g., antiferromagnetic interactions can produce singlet states which exhibit antiferromagnetic correlations), such interactions and correlations are necessary but not sufficient for the presence of entanglement. In particular, Werner [14] defined a subclass of mixed states of pairs of qudits that had two particularly interesting subfamilies. One family of states had "classical" correlations in the sense that the two-qudit density matrix could be written as a convex combination of product (i.e., unentangled) states. Such states can be modeled by a hidden-variable theory and satisfy Bell's inequalities. A second distinct subfamily was entangled but could be modeled by a hidden-variable theory. In this paper, we consider the implications of this for the specific case of the two-impurity Kondo model, which describes the interaction of two localized spin-$\frac{1}{2}$'s (qubits) via the Heisenberg exchange interaction with the itinerant electrons in a metal. We investigate how the competition between the Kondo effect [15] and the RKKY interaction determines the parameter regime for which entanglement of the two qubits can occur. Although we focus on this specific system, many of the results and concepts considered can be readily adapted to other solid-state qubit systems. For example, this is another example of how the entanglement in the whole system is "shared" [16]: the extent to which two qubits can be entangled with each other is limited by how entangled the individual qubits are with the environment.

Manipulation of many-body quantum states in solid-state physics has come to reality. For example, the Kondo effect [17] and superconducting qubits [18] have been realized experimentally in a controllable manner. For a quantum dot (QD) fabricated in a semiconductor two-dimensional electron gas (2DEG) system, system parameters can be varied in a tunable manner [19] to explore various many-body effects in previously inaccessible regimes. Electron transport through QD's in the unitary limit has manifested that the ground state is a many-body Kondo singlet [20] as a result of the Kondo resonance [21,22]. This means that the localized magnetic moment is entangled with the itinerant electrons. Further, it has recently been proposed that a tunable RKKY interaction could be used to entangle two spatially separated spins and perform quantum information processing (QIP) electrically [9,10] or optically [11] in coupled QDs or with endohedral fullerenes inside carbon nanotubes [12]. Varying the RKKY interaction, to induce the effect of transitions between different ground states has been theoretically investigated [23,24]. Furthermore, quantifying entanglement in quantum many-body systems has recently been investigated

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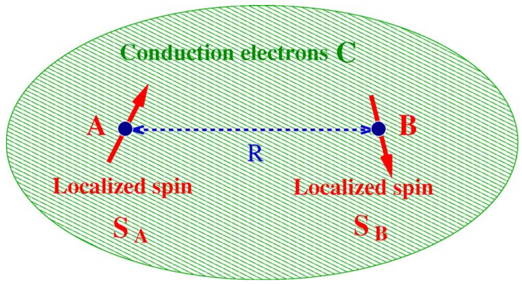

FIG. 1. (Color online) Two-impurity Kondo system. The total system may be regarded as a combined system containing three subsystems: A, B, and C. Two localized spins A and B are separated by a distance R, and C is the conduction electrons. These three subsystems interact with one another both directly and indirectly. For example, the localized spins interact directly with the conduction electrons by the spin-exchange interaction J and indirectly with each other by the RKKY interaction I, which is mediated by the conduction electrons.

[25–38]. Motivated by a recent experiment of nonlocal spin control in a coupled-QD system [10], it is important to understand how two spatially separated spins are entangled by tunable quantum many-body effects. We quantify quantum entanglement in two-impurity Kondo systems. A general expression for the reduced density matrix for the two-impurity spins is given in terms of the spin-spin correlation. It is found that to be entangled, the two-impurity spins need a minimum nonzero antiferromagnetic (AFM) correlation determined by the competition between the Kondo effect and the RKKY interaction. We point out that at the unstable fixed point [39] in the two-impurity Kondo problem, the AFM correlation has this critical value.

II. THE TWO-IMPURITY KONDO MODEL

The two-impurity Kondo model describes two localized spins interacting with itinerant conduction electrons. One may then suppose that the total system has three subsystems, consisting of the two localized spins (A and B) and the conduction electrons (C) as the environment (see Fig. 1). The Hamiltonian describing the two-impurity Kondo model is

$$H = H_C - J[\mathbf{S}_A \cdot \mathbf{s}_c(A) + \mathbf{S}_B \cdot \mathbf{s}_c(B)], \qquad (1)$$

where $H_C=\Sigma_{k\sigma}\varepsilon_k c^\dagger_{k\sigma}c_{k\sigma}$ denotes the Hamiltonian for the conduction electrons. J is the spin-exchange coupling between the impurity spins $\mathbf{S}_{A(B)}$ and the conduction-electron spin densities, $\mathbf{s}_c(\mathbf{R})=(1/2N_e)\Sigma_k c^\dagger_{k\sigma}\boldsymbol{\sigma}_{\sigma\sigma'}c_{k'\sigma'}\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{R}]$, at the impurity sites A and B, where N_e is the number of different $\mathbf{k}$ vectors. The relevant energy scale governing a single spin impurity model is the Kondo temperature [15], $T_K \approx D\sqrt{J\varrho_F}\exp[-1/J\varrho_F]$, with the conduction-band width, D, and the single-particle density of states at the Fermi energy ϱ_F. The conduction electron spins mediate a spin-exchange interaction between the two spatially separated impurity spins, the RKKY interaction, even though it does not explicitly appear in Eq. (1) of the Hamiltonian. To second order in J, the RKKY interaction between two impurity spins can be described by the Hamiltonian

$$H_{\mathrm{RKKY}} = I(R)\mathbf{S}_A \cdot \mathbf{S}_B, \qquad (2)$$

where $I(R)$ characterizes the effective spin-exchange interaction between the two-impurity spins depending on the distance R. The exchange interaction varies as $I(R) = 4\pi J^2 \varepsilon_F F_{1,3}(2k_F R)$ with $F_3(x)=(\sin x - x\cos x)/x^4$ in three dimensions [41] and $F_1(x)=-(1/4)\int_x^\infty dy \sin y/y$ in one dimension [42]. ε_F and k_F are the Fermi energy and the Fermi wave vector, respectively. Recently the RKKY interaction in single-walled nanotubes has been studied [43] theoretically. Note that the sign of $I(R)$ depends on the distance R between the two-impurity spins. AFM coupling occurs for $I(R)>0$ and ferromagnetic (FM) coupling occurs for $I(R)<0$. The competition between the RKKY interaction and the Kondo effect determines the characteristics of the system by the ratio of the relevant energy scales, $I(R)/T_K$. For instance, for a strong ferromagnetic RKKY interaction, $|I| \gg T_K$, the two-stage Kondo effect [44] is seen in the temperature dependence of susceptibility; there are three distinct temperatures at which the susceptibility decreases. The ratio $I(R)/T_K$ can be varied by changing J or R.

III. REDUCED DENSITY MATRIX FOR THE TWO-IMPURITY SPINS

At zero temperature ($T=0$), the total system should be in a ground state, $|\Psi_G\rangle$, which is pure. The ground state should be a spin singlet [39]. This means that it is invariant under joint rotation of all the spins. To quantify entanglement between the two localized spins, let $\rho=|\Psi_G\rangle\langle\Psi_G|$ be the density matrix for the ground state of the total system. Although the total system is in a pure state, the two localized spins are in a mixed state. For any two qubits, here two Kondo impurity spins $\mathbf{S}_A$ and $\mathbf{S}_B$, the density matrix can be written in the form [40]

$$\rho_{AB} = \mathrm{Tr}_C(\rho) = \frac{1}{4}\sum_{\alpha,\beta=0,x,y,z} r_{\alpha\beta}\sigma_A^\alpha \otimes \sigma_B^\beta, \qquad (3)$$

where the coefficients in this operator expansion are determined by the relation

$$r_{\alpha\beta} = \mathrm{Tr}(\sigma_A^\alpha \sigma_B^\beta \rho_{AB}) = \langle \sigma_A^\alpha \sigma_B^\beta \rangle. \qquad (4)$$

σ_j^0 and σ_j^μ ($\mu=x,y,z$) are the identity matrix and the Pauli matrices, respectively, and $j=A$ and B. The reduced (four by four) density matrix ρ_{AB} is obtained from ρ by taking the partial trace over the states of subsystem C (conduction electrons). All influences of the direct and indirect interactions between the two Kondo spins are contained in the correlation functions defined by the coefficients $r_{\alpha\beta}$.

For the two Kondo spins, we derive a general expression for the reduced density matrix which is valid when any total system considered satisfies the following symmetries. The reflection symmetry of the system implies that $\rho_{AB}=\rho_{BA}$. Since the Hamiltonian describing the system is real, $\rho^*_{AB} = \rho_{BA}$. Furthermore, if the ground state is a total spin singlet, i.e., spin rotationally invariant, then $r_{\alpha\beta}=0$ if $\alpha \neq \beta$ and $r_{xx} = r_{yy} = r_{zz} = r$. The symmetries require that the only nonzero coefficients in the operator expansion are r_{00}, r_{xx}, r_{yy}, and r_{zz}.

In addition, $r_{00}=1=\mathrm{Tr}(\rho_{AB})$ because the density matrix must have trace unity. The reduced density matrix may depend only on the distance R between the two Kondo impurity spins $\mathbf{S}_A$ and $\mathbf{S}_B$ because in our study the indirect RKKY interaction between the Kondo impurity spins is mediated by the conduction electrons. In the basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$, the reduced density matrix ρ_{AB} can be rewritten entirely in terms of the spin-spin correlation function, $f_s \equiv \langle \mathbf{S}_A \cdot \mathbf{S}_B \rangle = 3r/4$, as follows:

$$\rho_{AB} = \frac{1}{4}\left(\mathbf{I} + r \sum_{\alpha=x,y,z} \sigma_A^\alpha \otimes \sigma_B^\alpha\right), \tag{5}$$

where $\mathbf{I}$ denotes the four by four identity matrix.

To get more insight into the entanglement for the two impurity Kondo system, we rewrite the reduced density matrix in the Bell basis of maximally entangled states $\{|\Psi^\pm\rangle, |\Phi^\pm\rangle\}$, where $|\Psi^\pm\rangle = (1/\sqrt{2})(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle)$ and $|\Phi^\pm\rangle = (1\sqrt{2})(|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle)$. Note that $|\Psi^-\rangle$ is the spin singlet state. The result is

$$\rho_{AB} = p_s|\Psi^-\rangle\langle\Psi^-| + p_t\left(|\Psi^+\rangle\langle\Psi^+| + \sum_{i=\pm}|\Phi^i\rangle\langle\Phi^i|\right), \tag{6}$$

where $p_s=(1/4)-f_s$ and $p_t=(1/4)+f_s/3$. This state is a singlet-triplet mixture. The spin-spin correlation function of any two spins is bounded: $-3/4 \leqslant f_s \leqslant 1/4$. Thus, the probabilities for the singlet and three triplet states are $0 \leqslant p_s \leqslant 1$ and $0 \leqslant p_t \leqslant 1/3$. The spin-spin correlation determines the properties of the state for the two spins. The probability of the two spins being in a singlet state is $P(S)=p_s=(1/4)-f_s$ and the total probability of triplet states is $P(T)=3p_t=(3/4)+f_s$. In the limit of a pure AFM singlet of the two spins, i.e., $f_s=-3/4$, the two spins are in a maximally entangled state; $\rho_S=|\Psi^-\rangle\langle\Psi^-|$, with $P(S)=1$ and $P(T)=0$. While in the limit of no singlets, i.e., $f_s=1/4$, the two spins are in an equal mixture of three triplet states, $\rho_T=\frac{1}{3}(|\Psi^+\rangle \times\langle\Psi^+| + \Sigma_{i=\pm}|\Phi^\pm\rangle\langle\Phi^\pm|)$, with $P(S)=0$ and $P(T)=1$. However, the concurrence/negativity for this particular mixture of triplet states is zero, as will be discussed below. When the spin-spin correlation vanishes, i.e., $f_s=0$, the reduced density matrix becomes $\rho_{AB}=\frac{1}{4}\mathbf{I}$, the totally mixed density matrix which is "garbage" for QIP. Then there is no entanglement between the two spins. In this case, the probabilities for the singlet and triplet states are $P(S)=1/4$ and $P(T)=3/4$. When the probabilities for the singlet and triplet states are equal, i.e., $P(S)=P(T)=1/2$, the spin-spin correlation is $f_s^c=-1/4$. The state for the two localized spins can be regarded as an equal mixture of the total spin of impurities $S_{\mathrm{imp}}=0$ and $S_{\mathrm{imp}}=1$.

IV. CONCURRENCE/NEGATIVITY AND A CRITICAL VALUE OF CORRELATION

ρ_{AB} is actually a Werner state [14] and can be written as

$$\rho_{AB} = \frac{4p_s-1}{3}|\Psi^-\rangle\langle\Psi^-| + \frac{1-p_s}{3}\mathbf{I}. \tag{7}$$

This state is characterized by a single parameter p_s called the fidelity because $p_s=\langle\Psi^-|\rho_{AB}|\Psi^-\rangle$ measures the overlap of the Werner state with the spin singlet Bell state. One measure of entanglement is the concurrence [45]. For the Werner state ρ_{AB}, the concurrence is given by [45]

$$C(\rho_{AB}) = \max\{2p_s - 1, 0\}. \tag{8}$$

For $0 \leqslant p_s \leqslant 1/2$ (i.e., $-1/4 \leqslant f_s \leqslant 1/4$), the concurrence is zero and the reduced density matrix can be written as a convex combination of (disentangled) product states. For $1/2 \leqslant p_s \leqslant 1$ (i.e., $-3/4 \leqslant f_s \leqslant -1/4$), the concurrence ranges from zero to one (a maximally entangled state), and it is related to the spin-spin correlation function monotonically. Therefore, at $p_s^c=1/2$, there exists a critical value of the spin-spin correlation, $f_s^c=-1/4$, separating entangled states from unentangled states. In a quantum spin system, a critical value of spin-spin correlation has been discussed for a system consisting of a spin S and a spin $\frac{1}{2}$ [46]. Another important measure of entanglement is the negativity $N(\rho_{AB})$ [48,49]. Similarly to the concurrence, the negativity ranges from zero to one. The negativity of the Werner state [50] is equal to the concurrence,

$$N(\rho_{AB}) = C(\rho_{AB}). \tag{9}$$

Hence, the negativity gives exactly the same critical value of the spin-spin correlation for the absence of entanglement. In fact, any measure of the entanglement shows that the critical value of the spin-spin correlation, $f_s^c=-1/4$, is a unique point for the two-impurity Kondo problem. We will see below that the critical correlation can be related to the unstable fixed point in the two-impurity Kondo model [39].

V. QUANTUM TELEPORTATION, BELL INEQUALITIES, AND CORRELATION

There are rigid constraints on the value of the spin-spin correlation required to use the two Kondo impurities for QIP. The state of Eq. (7) for two Kondo impurities is a Werner state that is highly symmetric and $\mathrm{SU}(2)\otimes\mathrm{SU}(2)$-invariant [14,51]. The Werner state can be entangled but not violate any Bell inequality (i.e., be described by a hidden variable theory) for some values of the fidelity p_s. In fact, a Werner state with $p_s \leqslant (1+3/\sqrt{2})/4 \approx 0.78$ satisfies the Clauser-Horne-Shimony-Holt (CHSH) inequality [47,52], i.e., it does not have the nonlocal correlations characteristic of maximally entangled states. This criterion corresponds to $f_s \geqslant -(3/4)\sqrt{2} \approx -0.53$ in the two-impurity Kondo system. The values of the spin-spin correlation, for an entangled state without the violation of the Bell-CHSH inequality, is determined by the concurrence/negativity. The entangled state, for $f_s < -1/4$, can then be used for QIP including teleportation [53,54]. To provide a clear comparison of criteria in terms of the spin-spin correlation for the two-impurity Kondo problem, Table I shows values required for Bell inequalities, quantum teleportation, and entanglement.

We summarize pictorially the main results of this study; the relationship between the concurrence/negativity and the probabilities of the states for two-impurity spins as a function of the spin-spin correlation is shown in the top panel of Fig. 2.

TABLE I. Comparison of criteria for entanglement, quantum teleportation, and violation of a Bell inequality in terms of the fidelity, $p_s=(1/4)-f_s$, and the spin-spin correlation, $f_s \equiv \langle \mathbf{S}_A \cdot \mathbf{S}_B \rangle$, for which the two spin-$\frac{1}{2}$ is in a spin-rotationally invariant mixed state. Note that the requirement for violation of the Bell inequality is a more stringent condition than the presence of entanglement.

	Fidelity p_s	Correlation f_s
Concurrence	$p_s \geqslant 1/2$	$f_s \leqslant -1/4$
Quantum teleportation	$p_s \geqslant 1/2$ [53]	$f_s \leqslant -1/4$
Violation of Bell-CHSH inequality	$p_s > (1+3/\sqrt{2})/4$ [47]	$f_s < -3/(4\sqrt{2})$

VI. RELATIONSHIP BETWEEN CONCURRENCE/NEGATIVITY AND QUANTUM PHASE TRANSITIONS

A connection between entanglement and quantum phase transition (QPT) has been proposed for a particular class of Hamiltonians [34]. For first- and second-order QPTs, there occurs a discontinuity in the ground-state concurrence/negativity and its first derivative, respectively, due to nonanalyticities in the ground-state energy. In the case of the two-impurity Kondo problem, the concurrence/negativity is a continuous function and its first derivative has a discontinuity at the critical value of spin-spin correlation. However, the discontinuity does not come from nonanalyticity in the ground-state energy but from the requirement of nonnegative concurrence [37] or nonpositive negativity. Consequently, in general, the critical point of the spin-spin correlation in the concurrence/negativity is not necessarily related to a QPT [34]. Thus, to use the concurrence/negativity as a signature of QPTs, the nonanalyticities in the ground-state energy should occur at values of spin-spin correlation $-3/4 \leqslant f_s \leqslant -1/4$, even if the two-impurity Kondo system has a definite QPT. Otherwise, since the concurrence/negativity is zero for $-1/4 \leqslant f_s \leqslant 1/4$, we see that nonanalytic behavior of the concurrence/negativity is not a definitive signature of a QPT.

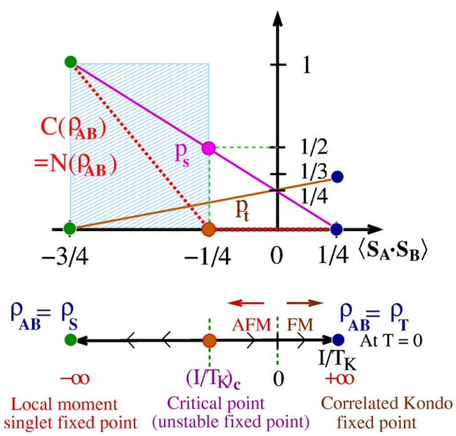

FIG. 2. (Color online) Top: Relationship between the probabilities for spin singlet (p_s) and triplet (p_t) states between the two Kondo impurity spins $\mathbf{S}_A$ and $\mathbf{S}_B$, and the entanglement measures concurrence (C)/negativity (N) [$C(\rho_{AB})=N(\rho_{AB})=2p_s-1 \geqslant 0$, otherwise $C=N=0$] and the spin-spin correlation function $f_s \equiv \langle \mathbf{S}_A \cdot \mathbf{S}_B \rangle$. Here, p_s corresponds to the singlet fidelity. Only for $-3/4 \leqslant f_s \leqslant -1/4$ the concurrence/negativity has a nonzero value. This implies that the critical value of $f_s^c=-1/4$ separates entangled states (hatched) from disentangled states (nonhatched). The entangled state is useful for quantum teleportation for which the criterion is given by $p_s \geqslant 1/2$ ($f_s \leqslant -1/4$) [53]. Then, for $-3/4\sqrt{2} \leqslant f_s \leqslant -1/4$, the entangled states which do not violate the Bell-CHSH inequality can be used for quantum teleportation. Bottom: Schematic renormalization-group flow on the axis of the ratio of RKKY interaction to Kondo temperature, I/T_K, at zero temperature taken from Ref. [39]. Note that there is a one-to-one correspondence between the fixed point and the critical value of the spin-spin correlations.

VII. VANISHING ENTANGLEMENT AT A QUANTUM CRITICAL POINT

We now consider how the vanishing entanglement at $f_s^c = -1/4$ may relate to the unstable fixed point (QPT) of the two-impurity Kondo model found by Jones, Varma, and Wilkins [39]. Wilson's numerical renormalization-group technique [39] and conformal field theory approaches [55] have shown that at the unstable fixed point, the staggered susceptibility and the specific-heat coefficient, γ, diverge. The critical value of $(I/T_K)_c$ separates the regimes of renormalization-group flows to the stable Kondo effect fixed point for $I/T_K > (I/T_K)_c$ and the locked-impurity singlet fixed point for $I/T_K < (I/T_K)_c$ [44]. It should be stressed that this critical point only exists when there is a symmetry between even- and odd-parity channels [39,56–58]. The divergence of thermodynamic properties implies that, at and around the unstable fixed point, a local description of the impurity and conduction electron degrees of freedom in terms of a local Fermi liquid is not possible. Interestingly, in addition, the spin-spin correlation of the ground state varies continuously as a function of I/T_K and, at the unstable fixed point $(I/T_K)_c$, approaches the critical value of $f_s^c=-1/4$ within numerical accuracy in the wide range of values of T_K [39]. This value at the critical point was also found analytically [57]. The schematics in the top and bottom panels of Fig. 2 show a correspondence between the entanglement and renormalization-group flow for the two-impurity Kondo system. When the symmetry of even-odd parity is broken, the critical point is replaced by a crossover [56,57]. This might suggest that the quantum entanglement for this two-impurity Kondo problem plays an important role in this quantum phase transition. However, if the even-odd symmetry is not present, then the entanglement still vanishes for a critical value of $(I/T_K)_c$ but there is no quantum phase transition [56,57].

VIII. ENTANGLEMENT BETWEEN THE CONDUCTION ELECTRONS AND THE KONDO IMPURITIES

The von Neumann entropy [40] is a good measure of entanglement between two subsystems of a pure state $|\Psi_G\rangle$ [59]. Although above we considered the total system in terms of the three subsystems (two impurity spins A and B, and the conduction electrons C), it can also be regarded as a bipartite system having two subsystems $\mathcal{A}$ and $\mathcal{B}$. There are two options: (i) one-impurity spin ($\mathcal{A}=j$) and the remainder of the total system [$\mathcal{B}=j'(\neq j)\cup C$] or (ii) two-impurity spins ($\mathcal{A}=A\cup B$) and the conduction electrons ($\mathcal{B}=C$). For the pure state $|\Psi_G\rangle$ of the bipartite systems, the von Neumann entropy $E(\rho)=-\mathrm{Tr}\rho\log\rho$ is given by the density matrix associated with either of the two subsystems, i.e., $E(\rho_{\mathcal{A}})=E(\rho_{\mathcal{B}})$. The logarithm is taken in the base 2.

To quantify the entanglement of one-impurity spin (j) with the remainder ($j'C$) of the total system, the reduced (two by two) density matrix of the one-impurity spin, $\rho_j=\mathrm{Tr}_{j'C}(\rho)=\mathrm{Tr}_{j'}(\rho_{AB})$, needs to be evaluated by taking trace over the state of the remainder of the total system. In terms of the Pauli matrices, it has the form $\rho_j=(\sigma_j^0+\Sigma_\alpha r_\alpha\sigma_j^\alpha)/2$ with $r_\alpha=\langle\sigma_j^\alpha\rangle$. As expected, $\rho_A=\rho_B=\sigma_j^0/2$ because the expectation value of each impurity spin is zero, $\langle\sigma_j^\alpha\rangle=0$, due to the spin-rotational invariance of the system. Then we have

$$E(\rho_j) = -\mathrm{Tr}\rho_j \log_2 \rho_j = 1. \tag{10}$$

Note that the von Neumann entropy of each impurity spin is not dependent on the spin-spin correlation f_s of the two-impurity spins. Hence, each Kondo spin is always maximally entangled with the remainder of the total system [30]. The entanglement of two-impurity spins (A and B) with the conduction electrons (C) is quantified by the von Neumann entropy of the reduced density matrix ρ_{AB},

$$E(\rho_{AB}) = -p_s \log_2 p_s - (1-p_s)\log_2 \frac{1-p_s}{3}. \tag{11}$$

Figure 3 shows the von Neumann entropy, $E(\rho_{AB})$, and the singlet fidelity, p_s, as a function of the spin-spin correlation, f_s. When $f_s=-3/4$ ($p_s=1$), $E(\rho_{AB})=0$ and the two Kondo spins are completely disentangled from the conduction electrons. The maximum degree of the entanglement of one Kondo spin with the remainder of the total system is then attributed to the other Kondo spin rather than the conduction electron spins. The concurrence, $C(\rho_{AB})=1$, as a measure of the entanglement between the two Kondo spins shows that they form the AFM spin-singlet state in the limit of $I/T_K \ll 0$, as shown in Ref. [44]. In the language of Kondo screening, one Kondo spin perfectly screens the other Kondo spin and the conduction electrons do not participate in screening any Kondo spin. As the RKKY interaction increases up to $I/T_K=0$, i.e., $p_s=1/4$, the entropy of Eq. (11) increases monotonically and reaches its maximum value of 2. As discussed, each Kondo spin is always maximally entangled with the conduction electrons but the entanglement of two Kondo spins disappears for $f_s \geq -1/4$. Thus, partial screenings by one Kondo spin and the conduction electrons accomplish a complete screening of the other Kondo spin. In fact, the

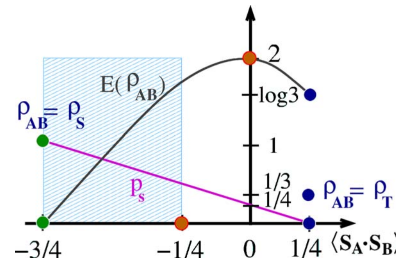

FIG. 3. (Color online) The von Neumann entropy $E(\rho_{AB})$ for entanglement between the two Kondo spins and the conduction electrons is shown as a function of the spin-spin correlation function $f_s \equiv \langle \mathbf{S}_A\cdot\mathbf{S}_B\rangle$. The von Neumann entropy is zero only at $f_s=-3/4$. Thus the two Kondo spins are entangled with the conduction electrons except for the extreme limit of a pure singlet of two Kondo spins, i.e., $I/T_K \ll 0$. The hatched region for $-3/4 \leq f_s \leq -1/4$ is compared to show the entanglement between two Kondo spins (compare Fig. 2). No unique behavior is seen in the von Neumann entropy at $f_s^c=-1/4$. When the indirect RKKY interaction disappears at $f_s=0$ ($I=0$), the two Kondo spins are maximally entangled with the conduction electrons. At $f_s=1/4$, the von Neumann entropy for the triplet states of the two Kondo spins is log 3.

competition between the Kondo effect, T_K, and the RKKY interaction, I, determines the extent of the partial screening of one Kondo spin by the other Kondo spin and the conduction electron spins. At $f_s=0$ ($p_s=1/4$), i.e., $I/T_K=0$, the two Kondo spins are maximally entangled with the conduction electrons but no entanglement between them exists. As the spin-spin correlation (RKKY interaction) increases to $I/T_K\to\infty$, i.e., $p_s=0$, the entropy of Eq. (11) decreases gradually to $E(\rho_{AB})=\log_2 3$ in the limit of the FM spin-triplet state. As a result, the entanglements of (i) one Kondo spin and the remainder of the total system and (ii) two Kondo spins and the conduction electrons exist in the whole range of the spin-spin correlation, and do not show any signatures of the unique behavior of the two-impurity Kondo system at the unstable fixed point $(I/T_K)_c$, i.e., $f_s^c=-1/4$.

IX. CONCLUSIONS

Any system of two spins which are a subsystem of a spin-rotationally invariant state will have similar entanglement properties. To our knowledge, this is the first discussion of an experimental solid-state realization of a Werner state. This work has significant implications for proposals using the RKKY interaction for QIP. We have shown that it is not sufficient to just use the RKKY interaction to produce antiferromagnetic correlations between spins. Entanglement will only be created when the AFM correlations are larger than a critical nonzero value. Hence, it is important that realistic estimates be made for the ratio I/T_K for candidate systems [10–12]. Similar physics will be relevant to proposals to couple Josephson-junction qubits via one-dimensional wires [60].

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