General setting for a geometric phase of mixed states under an arbitrary nonunitary evolution

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(Received 29 July 2005; published 18 January 2006)

The problem of a geometric phase for an open quantum system is reinvestigated in a unifying approach. Two of the existing methods to define geometric phase, one by Uhlmann's approach and the other by a kinematic approach, which have been considered to be distinct, are shown to be related in this framework. The method is based upon purification of a density matrix by its uniform decomposition and a generalization of the parallel transport condition obtained from this decomposition. It is shown that the generalized parallel transport condition can be satisfied when Uhlmann's condition holds. However, it does not mean that all solutions of the generalized parallel transport condition are compatible with those of Uhlmann's. It is also shown how to recover the earlier known definitions of geometric phase as well as how to generalize them when degeneracy exists and varies in time.

DOI: 10.1103/PhysRevA.73.012107 PACS number(s): 03.65.Vf, 42.50.Dv

I. INTRODUCTION

The concept of a geometric phase was originally introduced by Pancharatnam in the classical context of comparing two polarized light beams through their interference [1]. Later, Berry pointed out its importance even in quantum systems undergoing a cyclic adiabatic evolution [2]. After that, this important notion was a subject of interest in many different aspects, which has led to many different generalizations and applications [3,4]. Of course in general cases to retain a purely geometrical nature of the phase one has to put some constraints, namely parallel transport (PT) conditions. In this manner, the geometric phase is a feature that only depends on the geometry of the path traversed by the system in its motion during evolution.

It is also worth noting that an important source of the renewed interest in geometric phases is their relevance to geometric quantum computation and holonomic quantum computation [5]. Indeed, it is known that quantum logic gates can be implemented only by using the concept of geometric phases. It is believed that the purely geometric nature of this phase makes such computations intrinsically fault-tolerant and robust against noise [6].

A pure state is merely an idealization, and in real experiments a description of the system in terms of mixed states is usually required. This point accounts for attempts toward extending the concept of the geometric phase to mixed states. In fact, Uhlmann was the first to tackle the problem through the mathematical approach of purification of mixed states [7]. This method is rather general in that it is independent of the type of evolution of the system. Next, Sjöqvist *et al.* put forward a quantum interferometric based definition for the geometric phase of nondegenerate density matrices undergoing a *unitary* evolution [8]. Later, Singh *et al.* proposed a kinematic description and extended the results to the case of degenerate mixed states [9]. It must be mentioned that there also exists another, differential geometric, approach to define the geometric phase for mixed states undergoing a unitary evolution [10]. In this approach, the mixed-state geometric phase appears as an immediate and direct generalization of the pure-state case.

Indeed, in the case of environmental effects such as decoherence, one has to consider nonunitary evolutions of mixed states. Some generalizations in this direction have been addressed in Refs. [7,11–18]. The proposition in Ref. [12] for completely positive maps in spite of being operationally well-defined depends on the specific Kraus representation for the map. In Refs. [15,16], the problem of a geometric phase of an initially pure open quantum system, based on the standard definition of a pure-state geometric phase, has been addressed through the quantum jump method. A more recent effort is based on a kinematic approach, with no *a priori* assumption about the dynamics of the system [17]. However, most of these different definitions do not agree with each other. In fact, Uhlmann's method, even in the case of unitary evolution, does not agree with the interferometric definition [19,20]. The source of such disagreement is known to be the use of different types of PT conditions. Hence, it has been argued [19–21] that these approaches are not generally equivalent and one cannot obtain one from the other. Therefore, it could be desirable to find a more unified approach that can bring together the previous general ideas. Recently, in the unitary evolution case it was argued that using (nonorthogonal) decompositions different from spectral decomposition can make it possible to unify the kinematic and Uhlmann approaches [22]. In this framework, a suitable notion of PT condition of the mixed state is based on the PT condition of the vectors constituting this decomposition.

In this paper, we shall use a rather similar mechanism plus uniform decomposition of density matrices, and propose a generalized kinematic approach for the geometric phase of mixed states under an arbitrary nonunitary evolution. This approach vividly shows how it is possible to merge Uhlmann's approach and the kinematic approach. It is also shown how to recover the earlier definitions of the geometric phase from this more general approach. In addition, it is shown that the approach can be easily modified to include

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the more general case of degenerate mixed states. This investigation may also be useful in the study of the robustness of geometric phases against decoherence [23]. Also, another possible application of the method can be putting forward a framework in which a rather deeper understanding of the notion of a geometric phase can be obtained.

The structure of the paper is as follows. In Sec. II, after a short review of Uhlmann's and the kinematic approaches, the structure of the generalized approach, which is based on uniform decomposition, is established. Next, some of the conditions of the approach are relaxed. The paper is concluded in Sec. III.

II. GENERALIZED APPROACH

Let us suppose that the density matrix of our system of interest (with the Hilbert space $\mathcal{H}_s$) is $\varrho(t) = \sum_{k=1}^{N} p_k(t)|w_k(t)\rangle\langle w_k(t)|$, in which $p_k(t)$'s [$|w_k(t)\rangle$'s] are considered to be its eigenvalues (normalized eigenvectors). In a *general* evolution both p_k and $|w_k\rangle$ are subject to change in time. For simplicity of our discussion, in the sequel we assume that the rank of this matrix is constant at all instants, and even more that the matrix is nondegenerate. In the case of *unitary* evolution, we have $p_k(t)=p_k(0)$ and $|w_k(t)\rangle = U(t)|w_k(0)\rangle$, where $U(t)$ is the unitary evolution operator. However, when evolution is nonunitary, the eigenvalues p_k can also vary in time. Thus, generally $U(t) = \sum_k |w_k(t)\rangle\langle w_k(0)|$ does not encompass the whole dynamical information. In fact, in such cases, to obtain $\varrho(t)$ one often has to resort to some approximative methods in the theory of open quantum systems [24], such as the Lindblad equation [25].

Since in our construction we use Uhlmann's PT condition [7], we need to recall it briefly. Uhlmann's approach is based upon the standard purification $w(t)$, where $\varrho(t)=w(t)w^{\dagger}(t)$, for density matrices. In other words, w can be considered as a purification of ϱ in the larger Hilbert space of Hilbert-Schmidt operators with scalar product $\langle w(t), w(t')\rangle = \mathrm{tr}[w^{\dagger}(t)w(t')]$ such that $ww^{\dagger}=\varrho$. It is clear that $w(t) = \sqrt{\varrho(t)}V(t)$ is an acceptable purification of ϱ for *any* unitary $V(t)$. For a special purification where each $|\langle w(t), w(t')\rangle|$ is constrained to its maximum value, Uhlmann has defined the geometric phase associated to the evolution from $\varrho(0)$ to $\varrho(\tau)$ as $\gamma_g(\tau)=\arg(\langle w(0), w(\tau)\rangle)$, where the PT condition $w^{\dagger}(t)\dot{w}(t)=\dot{w}^{\dagger}(t)w(t)$ has to be satisfied.

Let us also briefly review the construction of the geometric phase in Ref. [17]. Consider a purification for the density matrix $\varrho(t)$ as

$$|\Psi(t)\rangle_{sa} = \sum_k \sqrt{p_k(t)}|w_k(t)\rangle_s \otimes |a_k\rangle_a, \quad t \in [0,\tau]. \tag{1}$$

Now after imposing the PT condition, $\langle w_k(t)|d/dt|w_k(t)\rangle=0$, the geometric phase, defined according to Pancharatnam [1], $\gamma(\tau)=\arg[\langle\Psi(0)|\Psi(\tau)\rangle]$, reads $\gamma_g(\tau)=\arg[\sum_{k=1}^{N}\sqrt{p_k(0)p_k(\tau)}\langle w_k(0)|w_k(\tau)\rangle e^{-\int_0^{\tau}\langle w_k(t)|\dot{w}_k(t)\rangle dt}]$. Indeed, by using the PT condition one fixes the general form of the unitary operators, which, like $U(t)$, can run the system's dynamics. As is clear, in this method purification of the mixed state of the system is done based on its spectral decomposition, and the PT condition is considered to be the PT condition of all the vectors constituting this (spectral) decomposition. We know that a purification as in Eq. (1) is only one of the possible purifications that can give rise to the correct mixed state of the system. So, one has the freedom to choose other decompositions and study the problem of the geometric phase with respect to them. In the sequel, we follow such a strategy and look for a specific purification in which all normalized terms can be treated in a naturally uniform manner, unlike Eq. (1), where the contribution of the kth normalized term is the time-dependent variable $\sqrt{p_k(t)}$. In other words, instead of starting from the spectral decomposition of a density matrix, which is the usual starting point of purification-based approaches, we start with another decomposition that can result in the mentioned uniformity. In order to do so, we need the next two important theorems on different decompositions of a density matrix ϱ.

Theorem 1 [26]: Let ϱ have the spectral ensemble $\{p_k, |w_k\rangle\}$. Then $\{q_l, |x_l\rangle\}$ is another ensemble for it iff there exists a *unitary* matrix $\mathcal{U}=(\mathcal{U}_{kl})$ such that

$$\sqrt{q_l}|x_l\rangle = \sum_k \sqrt{p_k}\mathcal{U}_{lk}|w_k\rangle. \tag{2}$$

Theorem 2 [27]: Let $\{q_l\}$ be a probability distribution. Then there exist normalized quantum states $\{|x_l\rangle\}$ such that $\varrho=\sum_l q_l|x_l\rangle\langle x_l|$, iff $\vec{q}$ is majorized by $\vec{p}$.

An immediate corollary of Theorem 2 is the existence of a *uniform* ensemble for *any* density matrix. Therefore, there exist normalized pure states $|x_1\rangle, \ldots, |x_{\mathcal{N}}\rangle$ such that ϱ is an equal mixture of these states with probability $1/\mathcal{N}$ ($\mathcal{N} \geqslant N$), i.e., $\varrho=(1/\mathcal{N})\sum_{l=1}^{\mathcal{N}}|x_l\rangle\langle x_l|$. For the rest of the discussion we assume that $\mathcal{N}=N$. Now, let us see how this uniform decomposition is related to the spectral decomposition. By using Theorem 1, we have $(1/\sqrt{N})|x_k\rangle=\sum_{l=1}^{N}\sqrt{p_l}\mathcal{U}_{kl}|w_l\rangle$. It is easy to see that if one chooses an $N\times N$ Fourier matrix (corresponding to discrete Fourier transformations [28]) $\mathcal{U}_{kl} = (1/\sqrt{N})e^{-2\pi i(kl/N)}$ ($k,l=0,\ldots,N-1$), and momentarily runs all indices from 0 to $N-1$ (rather than 1 to N), this equation is satisfied. Then, by using a Fourier matrix one can find a uniform ensemble for any density matrix. If we define $C(t) = \sum_k \sqrt{p_k(t)}|w_k(0)\rangle\langle w_k(0)|$ and use the definition of $U(t)$, we can rewrite $|x_k(t)\rangle$ in the following matrix form:

$$|x_k(t)\rangle = \sqrt{N}U(t)C(t)\mathcal{U}|w_k(0)\rangle. \tag{3}$$

Now we show that the above-mentioned uniform decomposition is useful in our discussion of geometric phase. Consider the following pure state of the combined system sa:

$$|\Phi(t)\rangle_{sa} = \frac{1}{\sqrt{N}}\sum_k |x_k(t)\rangle_s \otimes V(t)|a_k\rangle_a, \tag{4}$$

where $V(t)$ is the unitary evolution of the $|a_k\rangle$'s. This state is a legitimate purification of the density matrix $\varrho(t)$ of the system, $\varrho(t)=\mathrm{tr}_a[|\Phi(t)\rangle_{sa}\langle\Phi(t)|]$. If $V(t)=I$, since $\langle x_k|x_k\rangle=1$ and all $|x_k(t)\rangle$ vectors enter with equal and constant probability of $1/N$ in the decomposition of the density matrix, it seems natural to consider our (generalized) PT conditions in the form of

$$\langle x_k(t)|\frac{d}{dt}|x_k(t)\rangle = 0, \quad k = 1, \ldots, N, \tag{5}$$

that is, a density matrix undergoes a PT condition when all of the vectors in its uniform decomposition do so. Here a point is in order. It must be mentioned that, except for the pure state case, this PT condition is generally different from the one considered in the earlier literature [8,17].[1]

In general, in the purification (4) ancillary vectors could also vary in time, and we have to find a natural picture for the geometric phase in this case. Let us first recall a simple and useful property of Schmidt decomposition of bipartite pure states [22]. If $|\Phi\rangle_{ab}=\Sigma_k c_k|a_k\rangle_a|b_k\rangle_b$, then $(U\otimes V)|\Phi\rangle_{ab}=(UC\mathcal{V}^T\otimes I)\Sigma_k|a_k\rangle_a|b_k\rangle_b$, where C is a diagonal matrix in the $\{|a_k\rangle\}$ basis defined as $C=\Sigma_k c_k|a_k\rangle\langle a_k|$ and $\mathcal{V}=\Sigma_{kk'}\langle b_k|V|b_{k'}\rangle|a_k\rangle\langle a_{k'}|$. Here, for notational purposes, we omit the T sign of $\mathcal{V}^T$. Now, noting this property and assuming that the basis vectors of the ancillary Hilbert space are $\{|w_k(0)\rangle\}$, one can rewrite Eq. (4) as

$$|\Phi(t)\rangle_{sa} = \sum_k |\tilde{x}_k(t)\rangle_s \otimes |w_k(0)\rangle_a, \tag{6}$$

where $|\tilde{x}_k(t)\rangle = U(t)C(t)\mathcal{U}\mathcal{V}(t)|w_k(0)\rangle$. This purification now results into the nonorthogonal decomposition $\varrho(t)=\Sigma_k|\tilde{x}_k(t)\rangle\langle\tilde{x}_k(t)|$ for the density matrix. Unlike the $\{|x_k(t)\rangle\}$ decomposition, now for a general $\mathcal{V}$, $\langle\tilde{x}_k(t)|\tilde{x}_k(t)\rangle$ is not time-independent and, as well, is not equal for all k's. However, if we consider the normalized vectors $|\hat{\tilde{x}}(t)\rangle=|\tilde{x}_k(t)\rangle/\|\tilde{x}_k(t)\|$ it still looks natural to consider our generalized PT condition to be in the following form:

$$\langle\hat{\tilde{x}}_k(t)|\frac{d}{dt}|\hat{\tilde{x}}_k(t)\rangle = 0. \tag{7}$$

In terms of $|\tilde{x}_k(t)\rangle$ vectors this is equal to $\langle\tilde{x}_k(t)|d/dt|\tilde{x}_k(t)\rangle=\frac{1}{2}(d/dt)[\langle\tilde{x}_k(t)|\tilde{x}_k(t)\rangle]$, or equivalently in more detail it is

$$\langle w_k(0)|\mathcal{V}^\dagger\mathcal{U}^\dagger CU^\dagger\dot{U}C\mathcal{U}\mathcal{V} + \mathcal{V}^\dagger\mathcal{U}^\dagger C\dot{C}\mathcal{U}\mathcal{V} + \mathcal{V}^\dagger\mathcal{U}^\dagger C^2\mathcal{U}\dot{\mathcal{V}}|w_k(0)\rangle = \frac{1}{2}\frac{d}{dt}[\langle w_k(0)|\mathcal{V}^\dagger\mathcal{U}^\dagger C^2\mathcal{U}\mathcal{V}|w_k(0)\rangle]. \tag{8}$$

Now let us see what is the form of Uhlmann's PT condition. We note that the $w(t)$ operator reads $w(t)=U(t)C(t)\mathcal{U}\mathcal{V}(t)$. Hence, the explicit form of Uhlmann's PT condition is

$$\mathcal{V}^\dagger\mathcal{U}^\dagger CU^\dagger\dot{U}C\mathcal{U}\mathcal{V} + \mathcal{V}^\dagger\mathcal{U}^\dagger C\dot{C}\mathcal{U}\mathcal{V} + \mathcal{V}^\dagger\mathcal{U}^\dagger C^2\mathcal{U}\dot{\mathcal{V}} = \mathcal{V}^\dagger\mathcal{U}^\dagger C\dot{U}^\dagger UC\mathcal{U}\mathcal{V} + \mathcal{V}^\dagger\mathcal{U}^\dagger\dot{C}C\mathcal{U}\mathcal{V} + \dot{\mathcal{V}}^\dagger\mathcal{U}^\dagger C^2\mathcal{U}\mathcal{V}. \tag{9}$$

As is seen, the left-hand side (LHS) of this equation is exactly the expression within bra-ket of the PT condition (8). If sandwiched between $\langle w_k(0)|$ and $|w_k(0)\rangle$, Eq. (9) gives rise to

$$\text{LHS of Eq. (8)} = \frac{1}{2}\langle w_k(0)|\text{LHS}+\text{RHS of Eq. (9)}|w_k(0)\rangle = \frac{1}{2}\frac{d}{dt}[\langle w_k(0)|\mathcal{V}^\dagger\mathcal{U}^\dagger C^2\mathcal{U}\mathcal{V}|w_k(0)\rangle]. \tag{10}$$

This is what we wanted to show; by using Uhlmann's PT condition, the generalized PT conditions (8) are also satisfied. However, it must be noted that generally the number of equations of the two PT conditions is not equal. In other words, Eq. (9) is a matrix equation which constitutes N^2 different equations (for $\mathcal{V}$), while Eq. (7) is just a set of N equations. This simply means that the solutions of Eq. (8) are usually not solutions of Eq. (9). If it is assumed that $\mathcal{V}(t)=e^{-i\tilde{H}(t)}$, then the solution of Eq. (9) is as follows [7]:

$$-i\tilde{H}(t) = -2\sum_{kk'}\mathcal{U}^\dagger|w_{k'}(0)\rangle\langle w_k(0)|\mathcal{U} \times \int_0^t dt'\langle w_{k'}(t')|\dot{w}_k(t')\rangle\frac{\sqrt{p_{k'}(t')p_k(t')}}{p_{k'}(t')+p_k(t')}. \tag{11}$$

Now it is easy to show that Eq. (8) can have solutions other than Eq. (11). For example, if we suppose that $[\mathcal{U}\mathcal{V},C]=0$, and $\mathcal{U}\mathcal{V}=\Sigma_k e^{-il_k(t)}|w_k(0)\rangle\langle w_k(0)|$, then Eq. (8) gives

$$l_k(t) = -i\int_0^t dt'\langle w_k(t')|\dot{w}_k(t')\rangle, \tag{12}$$

which does not generate a $\mathcal{V}(t)$ compatible with Eq. (11). This comes from the fact that to satisfy Eq. (8) we only need to have the diagonal terms of Uhlmann's PT condition, whereas off-diagonal terms of this equation may put extra constraints that are redundant for the validity of Eq. (8).

Now the geometric phase can be simply defined according to Pancharatnam as

$$\gamma_g(t) = \arg[\langle\Phi(0)|\Phi(t)\rangle] = \arg\left[\sum_k \nu_k(t)e^{i\gamma_k(t)}\right], \tag{13}$$

where $\langle\tilde{x}_k(0)|\tilde{x}_k(t)\rangle=\nu_k(t)e^{i\gamma_k(t)}$, i.e., $\nu_k(t)$ $[\gamma_k(t)]$ is the visibility (geometric phase) of the kth component of $|\Phi(t)\rangle$. The explicit form is obtained by insertion of the definition of $|\tilde{x}_k(t)\rangle$ in this equation, which gives

$$\gamma_g(t) = \arg\left(\sum_{kk'}\sqrt{p_{k'}(0)p_k(t)}\langle w_{k'}(0)|w_k(t)\rangle \times \langle w_k(0)|\mathcal{U}\mathcal{V}(t)\mathcal{U}^\dagger|w_{k'}(0)\rangle\right). \tag{14}$$

This equation shows that the geometric phase, as described here to be combined with Uhlmann's definition, generally retains a memory of the evolution of both the system and the ancilla, that is, it is a general property of the whole system that depends on the history of the system as well as the history of the ancilla entangled with it [19].

In the remainder of the paper, we investigate how the earlier definitions of the geometric phase [8,17] can be obtained from the present framework as special cases. If we

[1] Tong *et al.*'s PT condition [17] imposes a constraint on the form of parallel transported evolution operator, $U^{\parallel}(t)$, as $\langle w_k(0)|U^{\parallel\dagger}\dot{U}^{\parallel}|w_k(0)\rangle=0$, $k=1,\ldots,N$, whereas Eq. (5) results into $\langle w_k(0)|\mathcal{U}^\dagger CU^{\parallel\dagger}\dot{U}^{\parallel}C\mathcal{U}|w_k(0)\rangle=0$, $k=1,\ldots,N$.

confine ourselves to a restriction of the solution of Eq. (11) for $\widetilde{\mathcal{V}}(t)$, such that $\widetilde{\mathcal{V}}_{kk'}(t)=\mathcal{V}_{kk'}(t)\delta_{kk'}$ and has the property

$$[\widetilde{\mathcal{V}},\mathcal{U}^{\dagger}C^2\mathcal{U}]=0, \qquad (15)$$

or equivalently $\widetilde{\mathcal{V}}(t)=\sum_k e^{-il_k(t)}\mathcal{U}^{\dagger}|w_k(0)\rangle\langle w_k(0)|\mathcal{U}$, where $l(t)$ is defined as in Eq. (12), then the explicit form of γ_g becomes

$$\gamma_g(t)=\arg\left[\sum_k \sqrt{p_k(0)p_k(t)}\langle w_k(0)|w_k(t)\rangle e^{-il_k(t)}\right], \qquad (16)$$

as in Ref. [17]. Thus, in the context of the discussion of Ref. [12], it can be said that the physical role of the commutation relation (15) is to remove memory effects of the ancilla's evolution from the geometric phase.

Let us end by mentioning some remarks on the initial assumptions of the approach, while our emphasis is still on the derivation of earlier results and their possible generalizations. Based upon Theorem 2, it is seen that one can always choose $\mathcal{N}$, the number of vectors in uniform decomposition, such that $\mathcal{N}\geqslant N$. For example, we can assume that $\mathcal{N}=\dim(\mathcal{H}_s)$. Now we show how the whole framework can be modified in the degenerate case. Consider the evolution for the density matrix of the system from $\varrho(0)$ to $\varrho(t)=\sum_{k=1}^{N}\sum_{\mu=1}^{n_k}p_k(t)|w_k^{\mu}(t)\rangle\langle w_k^{\mu}(t)|$, where $p_k(t)$, $k=1,\ldots,N$, are the n_k-fold degenerate eigenvalues of $\varrho(t)$, and $|w_k^{\mu}(t)\rangle$, $\mu=1,\ldots,n_k$, are considered the corresponding eigenvectors. In this case, the pure state of the total system is $|\Phi(t)\rangle_{sa}=\sum_{k=1}^{N}\sum_{\mu=1}^{n_k}|\tilde{x}_k^{\mu}(t)\rangle_s\otimes|w_k^{\mu}(0)\rangle_a$, where $|\tilde{x}_k^{\mu}(t)\rangle$ is defined as in Eq. (6) in which $|w_k(0)\rangle$ is replaced by $|w_k^{\mu}(0)\rangle$. Then, one notes that

$$\langle\Phi(0)|\Phi(t)\rangle=\sum_{kk'\mu\mu'}\sqrt{p_k(0)p_{k'}(t)}\langle w_k^{\mu}(0)|w_{k'}^{\mu'}(t)\rangle \times\langle w_k^{\mu}(0)|\mathcal{U}\mathcal{V}(t)\mathcal{U}^{\dagger}|w_{k'}^{\mu'}(0)\rangle, \qquad (17)$$

which is determined when all elements of $\mathcal{V}(t)$ are known. Now we choose our PT condition in this general case as

$$\langle\hat{\tilde{x}}_k^{\mu}(t)|\frac{d}{dt}|\hat{\tilde{x}}_k^{\mu'}(t)\rangle=0, \quad \mu,\mu'=1,\ldots,n_k. \qquad (18)$$

It can be checked that this PT condition can also be satisfied by assuming Uhlmann's PT condition, Eq. (9). In this case, it is easily seen that the most general form for $\widetilde{\mathcal{V}}$ that satisfies Eq. (15) is as follows:

$$\widetilde{\mathcal{V}}(t)=\sum_{k\mu\mu'}\alpha_k^{\mu\mu'}(t)\mathcal{U}^{\dagger}|w_k^{\mu}(0)\rangle\langle w_k^{\mu'}(0)|\mathcal{U}. \qquad (19)$$

After some algebra and using the commutation relation (15), it is obtained that $\alpha_k^{\mu\mu'}(t)=\langle w_k^{\mu}(0)|\mathbf{P}e^{-\int_0^t U^{\dagger}(t')\dot{U}(t')dt'}|w_k^{\mu'}(0)\rangle$, where $\mathbf{P}$ denotes path ordering. After inserting this relation back into Eq. (17), non-Abelian factors show up in the geometric phase.

In general, when degeneracies vary in time, a level-crossing-like behavior can occur. In this situation, in the discussion of differentiability of the eigenvalues (and eigenvectors), the notion of ordering of the eigenvalues becomes important. For example, it can happen that the natural ordering as $p_1(t)\geqslant\cdots\geqslant p_N(t)$ (for all t) destroys differentiability, thus one has to seek for some ordering which respects it [29]. If such an ordering can be found, then the operator $U(t)$, eigenvalues, and eigenvectors are still well-defined differentiable functions and our approach may be generalized as well.

III. CONCLUSION

In summary, the notion of geometric phase of a mixed state undergoing nonunitary evolution has been investigated in a unifying picture in which two of the previous general definitions, Uhlmann's definition and the kinematic approach, have been related to each other. In this formalism, we have used the idea of purification of state of a system by uniform decomposition of its density matrix rather than the spectral one, and by attaching a time varying ancilla to it. Then, as a natural choice for a parallel transport condition, we have considered that a mixed state is undergoing a parallel transport condition when all the (normalized) vectors of its corresponding purification are subject to this condition. This generalized parallel transport condition is different from the ones defined previously in the literature. It has been shown that the new conditions are satisfied when Uhlmann's condition holds. However, because of the different numbers of equations in the two parallel transport conditions, the generalized parallel conditions are only diagonal equations of Uhlmann's condition. Finally, it has been shown how to recover earlier definitions of the geometric phase of a mixed state. An extension of the method to the more general cases of degenerate density matrices with time-varying degeneracies has also been discussed. Since our approach comprises the previously existing definitions, it is clear that, like those models, it can be used in different physical applications. While the range of physical applicability of the presented approach is tantamount to the former ones, this latter appears now to be tied by an underlying conceptual structure. We hope that this new approach can shed light on the notion of a mixed-state geometric phase in physical applications and to remove some ambiguity in its definition.

ACKNOWLEDGMENTS

We thank N. Paunković for useful discussions. This work was supported by EU project TOPQIP under Contract No. IST-2001-39215.

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