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Vortex distribution in the lowest Landau level

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We study the vortex distribution of the wave functions minimizing the Gross-Pitaevskii energy for a fast rotating condensate in the lowest Landau level (LLL): we prove that the minimizer cannot have a finite number of zeroes, thus the lattice is infinite, but not uniform. This uses the explicit expression of the projector onto the LLL. We also show that any slow varying envelope function can be approximated in the LLL by distorting the lattice. This is used in particular to approximate the inverted parabola and understand the role of "invisible" vortices: the distortion of the lattice is very small in the Thomas-Fermi region but quite large outside, where the "invisible" vortices lie.

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The fast rotating regime for a Bose-Einstein condensate in a harmonic trap, observed experimentally in Refs. [1–3], displays analogies with type-II superconductor behaviors and quantum Hall physics. However, some different features have emerged and are of interest, in particular due to the existence of a potential trapping of the atoms.

A quantum fluid described by a macroscopic wave function rotates through the nucleation of quantized vortices [4,5]. For a condensate confined in a harmonic potential with cylindrical symmetry around the rotation axis, a limiting regime occurs when the rotational frequency Ω approaches the transverse trapping frequency: the centrifugal force nearly balances the trapping force so that the size of the condensate increases and the number of vortices diverges. The visible vortices arrange themselves in a triangular Abrikosov lattice. The system is strongly confined along the axis of rotation, and it is customary to restrict to a two-dimensional analysis in the x-y plane. We will call $z=x+iy$. The Hamiltonian is similar to that for a charged particle in a magnetic field: for rotational angular velocities just below the transverse trap frequency, the wave function of the condensate can be described using only components in the lowest Landau level (LLL): $\Psi(z)=\Phi_0\Pi_{i=1}^{N}(z-z_i)e^{-|z|^2/2}$, where Φ_0 is a normalization factor and the z_i are the location of the vortices. In rescaled units, the reduced energy in the LLL is [6–8]

$$\mathcal{E}_{LLL}(\Psi)=\int\left[(1-\Omega)|z|^2|\Psi|^2+\frac{G}{2}|\Psi|^4\right]d^2r \qquad (1)$$

under $\int d^2r|\Psi|^2=1$, where Ω is the rotational velocity, the transverse trap frequency is scaled to 1, and G models the interaction term: $G=Ng/(d\sqrt{2\pi})$, where g is the two-body interaction strength and d is the characteristic size of the harmonic oscillator in the direction of the rotation.

In the absence of a confining potential, the problem is reduced to the one studied by Abrikosov [9] for a type-II superconductor and the minimizer is a wave function with a uniform triangular lattice [10]; its modulus vanishes once in each cell and is periodic over the lattice. The presence of the confining potential is at the origin of a slow varying density profile, which can be described as the mean of the modulus of the wave function on many cells. Ho [6] predicted that for a uniform lattice, the smoothed density profile is a Gaussian. Various contributions [7,8,11] then pointed out that the energy can be lowered if this smoothed density distribution is an inverted parabola rather then a Gaussian. This type of density profile can be achieved either by taking wave functions with a uniform lattice but with components outside the LLL [7] or by remaining in the LLL and distorting the lattice. The study of the distortion has been the focus of recent papers [8,11,12] and raises the issue of the optimal vortex distribution. In the LLL description, there are two kinds of vortices: the "visible vortices," which lie in the region where the wave function is significant (for instance, inside the Thomas Fermi region in the case of the inverted parabola), and the "invisible vortices" which are in the region where the modulus of the wave function is small. The visible vortices form a regular triangular lattice, while the invisible ones seem to have a strong distorted shape, whose distribution is essential to recreate the inverted parabola profile inside the LLL approximation. These latter are not within reach of experimental evidence, but can be computed numerically [8,12]. An important theoretical question is the distribution of these invisible vortices, their number, or an estimate of how many of them are necessary to approximate the inverted parabola properly inside the LLL.

Our main result states that the minimizer of the energy in the LLL has an infinite number of vortices. It relies on an explicit expression of the projector onto the LLL. This projector also allows us to approximate any slow varying density profile by LLL wave functions.

We define a small parameter $\varepsilon=\sqrt{1-\Omega}$ and make the change of variables $\psi(z)=\sqrt{\varepsilon}\Psi(\sqrt{\varepsilon}z)$, so that the condensate is of size of order 1 and the lattice spacing is expected to be of order $\sqrt{\varepsilon}$. The energy gets rescaled as $\mathcal{E}_{LLL}(\Psi)=\varepsilon E_{LLL}(\psi)$, where

$$E_{LLL}(\psi) = \int \left[|z|^2|\psi|^2 + \frac{G}{2}|\psi|^4 \right] d^2r. \tag{2}$$

Moreover, ψ belongs to the LLL so that $f(z)=\psi(z)e^{|z|^2/2\varepsilon}$ is a holomorphic function and thus belongs to the so-called Fock-Bargmann space,

$$\mathcal{F} = \left\{ f \text{ is holomorphic}, \quad \int |f|^2 e^{-|z|^2/\varepsilon} d^2r < \infty \right\}. \tag{3}$$

Let us point out that such a function f is not only determined by its zeroes and normalization factor, but also by a globally defined phase, which is a holomorphic function. The space $\mathcal{F}$ is a Hilbert space endowed with the scalar product $\langle f,g\rangle = \int f(z)\bar{g}(z)e^{-|z|^2/\varepsilon}d^2r$. The point of considering this space is that the projection of a general function $\phi(z,\bar{z})$ onto $\mathcal{F}$ is explicit, and called the Szego projector [13,14],

$$\Pi(\phi) = \frac{1}{\pi\varepsilon}\int e^{z\overline{z'}/\varepsilon}e^{-|z'|^2/\varepsilon}\phi(z',\bar{z}')d^2r'. \tag{4}$$

If ϕ is a holomorphic function, then an integration by part yields $\Pi(\phi)=\phi$.

If one considers the minimization of $E_{LLL}(\psi)$ without the holomorphic constraint on f, then the minimization process yields that $|z|^2+G|\psi|^2-\mu=0$, where μ is the chemical potential due to the constraint $\int|\psi|^2=1$, so that $|\psi|$ is the inverted parabola

$$|\psi|^2(z) = \frac{2}{\pi R^2}\left(1-\frac{|z|^2}{R^2}\right)1_{\{|z|\leq R\}}, \quad R=\sqrt{\mu}=\left(\frac{2G}{\pi}\right)^{1/4}. \tag{5}$$

The restriction to the LLL prevents from achieving this specific inverted parabola since $\psi e^{|z|^2/2\varepsilon}$ cannot be a holomorphic function. The advantage of the explicit formulation of the projector Π is that it allows us to derive an equation satisfied by ψ or rather f when minimizing the energy in the LLL. A proper distribution of zeroes can approximate an inverted parabola profile but is going to modify the radius R as we will see below by a coefficient coming from the contribution of the vortex lattice to the energy.

If $f\in\mathcal{F}$ minimizes $E(f)=\int[|z|^2|f|^2e^{-|z|^2/\varepsilon} + (G/2)|f|^4e^{-2|z|^2/\varepsilon}]d^2r$ under $\int|f|^2e^{-|z|^2/\varepsilon}d^2r=1$, then for any g in $\mathcal{F}$ with $\langle f,g\rangle=0$, we have $\int[|z|^2\bar{g}fe^{-|z|^2/\varepsilon} + (G/2)|f|^2\bar{g}fe^{-2|z|^2/\varepsilon}]d^2r=0$. We use the scalar product in $\mathcal{F}$ and the definition of the projector to conclude that f is a solution of

$$\Pi\Big((|z|^2 + G|f|^2e^{-|z|^2/\varepsilon} - \mu)f\Big) = 0 \tag{6}$$

where μ is the chemical potential coming from the mass constraint. Note that given the relation between f and ψ, $E(f)$ and $E_{LLL}(\psi)$ are identical. Equation (6) was pointed out by [11] as potentially useful. Indeed, this equation allows us to derive that this minimizer cannot be a polynomial:

Theorem 1. If $f\in\mathcal{F}$ minimizes E, and ε is small enough, then f has an infinite number of zeroes.

(1) The proof first requires another formulation of (6). The projector Π has many properties [13,19]: in particular, one can check, using an integration by part in the expression of Π, that $\Pi(|z|^2f)=z\varepsilon\partial_z f+\varepsilon f$. As for the middle term in the equation, one can compute that $\Pi(e^{-|z|^2/\varepsilon}|f|^2f)=\Pi(e^{-|z|^2/\varepsilon}|f|^2)\Pi f=\Pi(\bar{f}(z))\Pi(e^{-|z|^2/\varepsilon}f^2) =\bar{f}(\varepsilon\partial_z)\Pi(e^{-|z|^2/\varepsilon}f^2)$. A simple change of variable yields $\Pi(e^{-|z|^2/\varepsilon}f^2)(z)=(\pi\varepsilon)^{-2}\int e^{-(z\overline{z'}-2|z'|^2)/\varepsilon}f^2(z')d^2r' =\frac{1}{2}\Pi(f^2(\cdot/\sqrt{2}))(z/\sqrt{2})=\frac{1}{2}f^2(z/2)$. Thus, we find the following simplification of (6):

$$z\varepsilon\partial_z f + \frac{G}{2}\bar{f}(\varepsilon\partial_z)[f^2(z/2)] - (\mu-\varepsilon)f = 0. \tag{7}$$

(2) Now we assume that f is a polynomial of degree n and a solution of (7). We are going to show that there is a contradiction due to the term of highest degree in the equation. Indeed, if f is a polynomial of degree n, then $(\varepsilon\partial_z)^k[f^2(z/2)]$ is of degree $2n-k$. But (7) implies that $\bar{f}(\varepsilon\partial_z)[f^2(z/2)]$ is of degree n, hence f must be equal to cz^n. This is indeed a solution of (7) if $n\varepsilon+G|c|^2\varepsilon^n(2n)!/(2^{2n+1}n!)-\mu+\varepsilon=0$. Using that $\int|f|^2e^{-|z|^2/\varepsilon}=1$, we find that $|c|^2\pi\varepsilon^{n+1}n!=1$. The Stirling formula provides the existence of a constant c_0 such that $n\epsilon+c_0G/(2\pi\epsilon\sqrt{n})\leq\mu$. For the minimizer, μ is of the same order as the energy, thus of order 1, so that if ε is too small, no n can satisfy this last identity hence the minimizer is not a polynomial.

(3) If f is a holomorphic function with a finite number of zeroes, then there exists a polynomial $P(z)$ and a holomorphic function $\phi(z)$ such that $f=Pe^{\phi}$. The fact that $f\in\mathcal{F}$ provides a decrease property on f [15] which implies that $\mathrm{Re}[\phi(z)]\leq|z|^2/(2\varepsilon)$. A classical result on holomorphic functions then yields that ϕ is a polynomial of degree at most two, and $f(z)=P(z)e^{\alpha+\beta z+\gamma z^2}$ with $|\gamma|\leq 1/(2\varepsilon)$. A similar argument to case 2, but with more involved computations, provides a contradiction with the degree of the polynomial P if ε is small enough. We conclude that f has an infinite number of zeroes. The detailed mathematical proof will be given in [19].

The Abrikosov problem [9] consists in minimizing the ratio $\langle|u|^4\rangle/\langle|u|^2\rangle^2$ over periodic functions, where $\langle\cdot\rangle$ denotes the average value over a cell, for functions u obtained as limits of LLL functions. The minimum is achieved for $u=u(z,e^{2i\pi/3})$ where [16]

$$u(z,\tau) = e^{-|z|^2/2\varepsilon}f(z,\tau), \quad f(z,\tau) = e^{z^2/2\varepsilon}\Theta\left(\sqrt{\frac{\tau_I}{\pi\varepsilon}}z,\tau\right), \tag{8}$$

and for any complex number $\tau=\tau_R+i\tau_I$,

$$\Theta(v,\tau) = \frac{1}{i}\sum_{n=-\infty}^{+\infty}(-1)^n e^{i\pi\tau(n+1/2)^2}e^{(2n+1)\pi iv}. \tag{9}$$

The Θ function has the following properties $\Theta(v+k+l\tau,\tau) =(-1)^{k+l}e^{-2i\pi lv}e^{-i\pi l\tau}\Theta(v,\tau)$ so that $|u(z,\tau)|$ is periodic over the lattice $\sqrt{\pi\varepsilon/\tau_I}\mathbf{Z}\oplus\sqrt{\pi\varepsilon/\tau_I}\mathbf{Z}\tau$, and vanishes at each point of the lattice. Without loss of generality, one can restrict τ to vary in $|\tau|\geqslant 1$, $-1/2\leqslant \tau_R < 1/2$: this is equivalent to requiring that the smallest period for Θ is 1 and along the x axis (see [17]). Any lattice in the plane can be obtained by similarity from one of these.

For any τ, f given by (8) is a solution of

$$\Pi(|f|^2 e^{-|z|^2/\varepsilon} f) = \lambda_\tau f \quad \text{with } \lambda_\tau = \langle |u|^2\rangle b(\tau), \tag{10}$$

and

$$b(\tau) = \frac{\langle |u|^4\rangle}{\langle |u|^2\rangle^2} = \sum_{k,l\in \mathbf{Z}} e^{-\pi|k\tau - l|^2/\tau_I}. \tag{11}$$

This expression can be obtained using arguments in [18]. The minimal value of $b(\tau)\sim 1.16$ is achieved for $\tau = e^{2i\pi/3}$, that is, for the triangular lattice [10]: in [10], it is argued that one can restrict to $\tau_R = -1/2$, and vary τ_I in $(1/2, \sqrt{3}/2)$. Accepting this restriction, they compute the variations of b which depends on a single parameter and is indeed minimal for the triangular lattice. In [19], we prove that this restriction is rigorous using the description of these lattices by varying τ for $|\tau|=1$ and $\tau_R\in(-1/2,0)$.

If one compares (6) and (10) and takes λ_τ to be the chemical potential in (6), one notices that they only differ by the term $\Pi(|z|^2 f) = \varepsilon z\partial_z f + \varepsilon f$, which is negligible on the lattice size, but plays a role on the shape of the density profile.

A natural candidate to approximate any slow varying profile $\alpha(z,\bar{z})$ is to take $\alpha(z,\bar{z})u(z,\tau)$, where u is the periodic function defined in Ref. [8]. Of course, such a function is not in the LLL, but can be well approximated in the LLL by $f^\alpha e^{-|z|^2/2\varepsilon}$, where $f^\alpha = \Pi(\alpha f)$, Π is the projector onto the LLL (4), and f comes from Eq (8). Estimating the energy of f^α yields $E(f^\alpha) - \int [|z|^2|\alpha|^2\langle |u|^2\rangle + [Gb(\tau)/2]|\alpha|^4\langle |u|^2\rangle^2] d^2r \sim C\varepsilon^{1/4}$. This computation uses calculus on Π [19], and u and α do not vary on the same scale; hence the integrals can be decoupled. The contribution of u to the energy is through the coefficient $b(\tau)$, which is minimum for $\tau = e^{2i\pi/3}$. On the other hand, the slow varying profile minimizing this approximate energy is an inverted parabola. A uniform distribution of zeroes inside the LLL provides a higher energy as computed in [8].

Using pseudodifferential calculus, one can show [19], when ε is small, that f^α is very close to αu: the error is at most like $\varepsilon^{1/4}$ if α is not more singular than an inverted parabola. In particular, when α is an inverted parabola, this implies that in the Thomas-Fermi region, the distribution of visible vortices is almost that of the triangular lattice since αu is a good approximation. Outside the support of the inverted parabola, where f^α is very small, the density of distribution of zeroes of f^α is very different from that of a regular lattice. Indeed, the Cauchy formula provides the number of zeroes of the holomorphic function $\Pi(\alpha f)$ in a ball of radius R,

$$N(R) = \frac{R}{2\pi\varepsilon}\int_0^{2\pi} d\theta \frac{\int e^{R\overline{z'}/\varepsilon} e^{-|z'|^2/\varepsilon}\overline{z'}\,\alpha(z')f(z'e^{-i\theta})d^2r'}{\int e^{R\overline{z'}/\varepsilon} e^{-|z'|^2/\varepsilon}\alpha(z')f(z'e^{-i\theta})d^2r'}. \tag{12}$$

The Laplace method yields that the ratio of the two integrals is bounded when R is large, hence $N(R)$ is proportional to R/ε. This is very different from the regular lattice case which would provide R^2/ε.

Contrary to what was explained in [7,20], it is not a small distortion of the lattice with this ansatz which results in large changes in the density distribution, but a very specific and far from uniform distribution of the invisible vortices (outside the Thomas-Fermi region) which allows to approximate an inverted parabola. This is consistent with the numerical simulations in [8].

The special shape of the inverted parabola comes out if one wants to approximate the equation of the minimizer of the energy: for any λ, we can prove that

$$\Pi((|z|^2 + G|f^\alpha|^2 e^{-|z|^2/\varepsilon} - \lambda)f^\alpha) + C\varepsilon^{1/4} \sim \Pi((|z|^2 + Gb(\tau)\langle |u|^2\rangle |\alpha|^2 - \lambda)\alpha f), \tag{13}$$

where C only depends on bounds on α. In other words, in the equation for f^α, one can separate in the term $|f^\alpha|^2 e^{-|z|^2/\varepsilon}$ the contributions due to the lattice and to the profile. The right-hand side of (13) is zero if α is the inverted parabola,

$$\alpha(z) = \sqrt{\frac{2}{\pi R_0^2\langle |u|^2\rangle}\left(1 - \frac{|z|^2}{R_0^2}\right)}, \quad R_0 = \left(\frac{2Gb(\tau)}{\pi}\right)^{1/4}, \tag{14}$$

and $\lambda = R_0^2$, so that f^α is almost a solution of (6), up to an error in $\varepsilon^{1/4}$.

This approach can be used to study the variations in energy due to deformations of the lattice. The triangular lattice, corresponding to $\tau^1 = e^{2i\pi/3}$, is such that the Hessian of $b(\tau)$ is isotropic ($\sim 0.63 Id$). Two lattices close to each other can be described by two close complex numbers τ^1 and τ^2; the difference in energy between $E(f^\alpha(\cdot,\tau^1))$ and $E(f^\alpha(\cdot,\tau^2))$ is at leading order,

$$\frac{G}{4}\frac{\partial^2 b}{\partial \tau_R^2}|\tau^1 - \tau^2|^2 \int |\alpha|^4\langle |u|^2\rangle^2 d^2r \sim \frac{0.63G}{3\pi R_0^2}|\tau^1 - \tau^2|^2. \tag{15}$$

This computation justifies the approach which consists in decoupling the lattice contribution from the profile contribution in the energy [20] but, given the definition of f^α using Π, it relies on strong deformations of the lattice for points far away from the Thomas-Fermi region. For a shear deformation for which u_{ij} are the components of the deformation tensor, $\tau^2 - \tau^1 = i\sqrt{3}u_{xy}$. The elastic coefficient C_2 is defined by the fact that the difference in energy should be $4C_2u_{xy}^2$. This separation of scales allows to compute $C_2 \sim 0.63G/(4\pi R_0^2)$ or equivalently $0.12G\langle |\psi|^2\rangle$ (see also Refs. [20,21]) and relate it in Bose-Einstein condensates

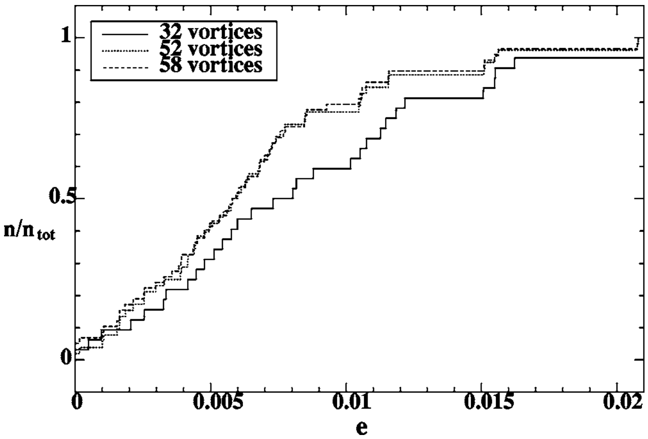

FIG. 1. Number of modes n/n_{tot} having a lower energy than a given energy e for $G=3$, $\Omega=0.999$.

(BEC) to the same one computed for the Abrikosov solution.

An interesting issue, especially for the computations of modes, is to get an estimate of the degree of the polynomial which could approximate f^{α}, since this function has an infinite number of zeroes. We can prove [19] that as the degree of the polynomial gets large, the minimum of the energy for the problem restricted to polynomials (and the computation of modes) is a good approximation of the full problem. The convergence rates that we obtain are not satisfactory yet. We believe that a good degree should be κ/ε, with $\kappa > R_0^2$ and R_0 is the radius of the inverted parabola. Given that the volume of the cell is $\pi\varepsilon$, $\kappa = R_0^2$ would correspond to having only the visible vortices. Numerical simulations indicate that a sufficient number of invisible vortices is needed to recreate the inverted parabola profile [8]. There are two types of invisible vortices: those close to the boundary of the inverted parabola which contribute to the bulk modes and those sufficiently far away which produce single particle excitations as explained in [12]. An open issue is to understand the location of these latter invisible vortices; some simulations suggest that they lie on concentric circles, but then the density of these circles should be very low to match our predicted global vortex density far away which behaves like $1/|z|$. We have performed numerical simulations with $\Omega=0.999$ and $G=3$: this fixes the number of visible vortices to 30, and we vary the number of total vortices N. One needs at least $N=52$ (that is 22 invisible vortices) to properly approximate the inverted parabola, the energy minimizer, and the bulk modes. The distortion of the lattice is small at the edges but large at large distances. For N too small, some modes do not appear (see Fig. 1), while for N very large, one expects higher modes that [12,22] interpret as single particle modes.

We have shown that for the minimizer of the Gross-Pitaevskii energy in the LLL, the lattice of vortices is infinite (but not uniform). Any slow varying profile can be approximated in the LLL by distorting the lattice. This is proved using an explicit expression for the projection onto the LLL. Our results also give an insight on the elastic coefficient C_2 and the approximation of the minimizer and modes by polynomials.

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