Optical theorem in N dimensions

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The quantum-mechanical optical theorem in N-dimensional space is derived in a simple way from S-matrix theory.

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One of the celebrated and most elegant results of quantum-mechanical scattering theory is the so-called optical theorem, stating that for plane-wave incidence in three dimensions the total cross section in the scattering process is linearly proportional to the imaginary part of the coherent scattered amplitude in the forward direction [1]. The physical basis for this proportionality is that any collision process that removes particles from an incident flux must be accompanied by interference between the incident plane wave and the scattered wave field in the forward direction, and hence must be a linear function of the forward scattered amplitude. For an arbitrary dimension N of the space, generalizations of the theorem referred to above have been given by Adawi [2] and Boya and Murray [3]. In the study [2] only radially symmetric scattering potentials were considered, and the construction of the generalized theorem was accomplished with the aid of the ultraspherical harmonics of Gegenbauer and the partial-wave decomposition technique. Nonsymmetric potentials were treated in the further work [3], where an optical-theorem-formula equivalent to that of Adawi was established by performing a saddle-point approximation in $N-1$ complex-dimensional space. Because the mathematical procedures developed in Refs. [2,3] appear to be more or less laborous and lengthy, it may be of interest to obtain a succinct proof of this important and many-faceted theorem via an alternative approach involving a rather simple calculation. In this contribution, we would like to present an almost elementary proof by utilizing the physically transparent language of the S-matrix formalism.

For a particle of mass m, incident momentum $\mathbf{p}=\hbar\mathbf{k}$, scattered in N dimensions by a short-range potential field $U(\mathbf{r})$, the stationary solution of the Schroedinger wave equation behaves asymptotically for large distances $\mathbf{r}$ as [3,4]

$$\psi_k(\mathbf{r}) \sim e^{i\mathbf{kr}} + \frac{e^{ikr}}{r^{(N-1)/2}} f(\mathbf{n}',\mathbf{n}). \tag{1}$$

Here, $\mathbf{n}=\mathbf{k}/k$ is a unit vector along the direction of incidence, $\mathbf{n}'=\mathbf{r}/r$ is a unit vector along the direction of observation, and $f(\mathbf{n}',\mathbf{n})$ is the scattered amplitude in a direction $\mathbf{n}'$ from the direction of $\mathbf{n}$. The scattering matrix S associated with Eq. (1) may be regarded as an integral operator with kernel [4]

$$s(\mathbf{n}',\mathbf{n}) = \delta(\mathbf{n}'-\mathbf{n}) + (2\pi)^{-(N-1)/2} i e^{i\pi(N-3)/4} k^{(N-1)/2} f(\mathbf{n}',\mathbf{n}), \tag{2}$$

where the Dirac δ function describes the propagation of the incoming wave.

In order to relate the total collisional cross section, $\sigma(\mathbf{n})=\int|f(\mathbf{n}',\mathbf{n})|^2 d\mathbf{n}'$, to the forward scattered amplitude, $f(\mathbf{n},\mathbf{n})$, let us combine the expression (2) with the isometric property [3] of the scattering matrix, that is, $S^{+}S=1$,

$$\int s^*(\mathbf{n}'',\mathbf{n}') s(\mathbf{n}'',\mathbf{n}) d\mathbf{n}'' = \delta(\mathbf{n}'-\mathbf{n}).$$

Above, the asterisk denotes complex conjugation, and the integration goes over all the directions of the unit vector $\mathbf{n}''$ in the hyperspace (in one-dimensional space, $\mathbf{n}''$ assumes only two discrete values, $\mathbf{n}''=\pm\mathbf{n}$). This step allows one to arrive at the key identity

$$\frac{f(\mathbf{n}',\mathbf{n})}{f_\lambda(N)} + \frac{f^*(\mathbf{n},\mathbf{n}')}{f_\lambda^*(N)} + \frac{1}{\sigma_\lambda(N)} \int f^*(\mathbf{n}'',\mathbf{n}') f(\mathbf{n}'',\mathbf{n}) d\mathbf{n}'' = 0, \tag{3a}$$

in which the natural units are provided by

$$f_\lambda(N) = (\lambda/i)^{(N-1)/2}, \quad \sigma_\lambda(N) = |f_\lambda(N)|^2 = \lambda^{N-1}, \tag{3b}$$

and $\lambda=2\pi/k$ is the de Broglie wavelength. When viewed on the forward direction $(\mathbf{n}'=\mathbf{n})$, Eq. (3a) gives immediately the final result of Boya and Murray for the generalized optical theorem:

$$\frac{\sigma(\mathbf{n})}{\sigma_\lambda(N)} + 2\,\mathrm{Re}\left\{\frac{f(\mathbf{n},\mathbf{n})}{f_\lambda(N)}\right\} = 0. \tag{4}$$

This completes the proof from the standpoint of S-matrix formulation. In three dimensions, Eq. (4) obtains the well-known form, $\sigma(\mathbf{n})=(4\pi/k)\mathrm{Im}\, f(\mathbf{n},\mathbf{n})$ [1].

Of direct interest for many practical applications is the weak-coupling scattering limit, which takes place when the typical strength U_0 of the potential field is sufficiently small. In this limit it is useful to develop the quantity $f(\mathbf{n}',\mathbf{n})$ as a multiple-scattering Born series,

$$f(\mathbf{n}',\mathbf{n}) = \sum_{m=1}^{\infty} f_m(\mathbf{n}',\mathbf{n}), \tag{5}$$

where the term of the mth-order is proportional to ${U_0}^m$. By inspecting this perturbation series with the help of construction (3a) and making order-of-magnitude comparisons, we find that the leading terms in the scattering amplitude satisfy the set of following requirements:

$$f_1(\mathbf{n}',\mathbf{n}) + i^{N-1} f_1^*(\mathbf{n},\mathbf{n}') = 0, \tag{6}$$

$$f_\lambda(N)[f_2(\mathbf{n}',\mathbf{n}) + i^{N-1}f_2^*(\mathbf{n},\mathbf{n}')] + \int f_1^*(\mathbf{n}'',\mathbf{n}')f_1(\mathbf{n}'',\mathbf{n})\mathbf{dn}'' = 0. \quad (7)$$

The condition (6) indicates that the first-order object $i^{(N-3)/2}f_1(\mathbf{n},\mathbf{n})$ has no imaginary part. In the case $N=3$, it also reproduces the familiar symmetry property of the first Born amplitude, $f_1(\mathbf{n}',\mathbf{n})=f_1^*(\mathbf{n},\mathbf{n}')$. Furthermore, one easily deduces from Eq. (7) that the forward scattered second Born amplitude establishes the total cross section of the single-scattering Born process $[\sigma_{\mathrm{Born}}(\mathbf{n})\equiv\int|f_1(\mathbf{n}',\mathbf{n})|^2 d\mathbf{n}']$ by means of a general relation

$$\sigma_{\mathrm{Born}}(\mathbf{n}) = 2\lambda^{(N-1)/2}\,\mathrm{Im}[i^{(N-3)/2}f_2(\mathbf{n},\mathbf{n})]. \quad (8)$$

It is to be noted at this point that for quasiclassical particles ($ka \gg 1$, where a is the characteristic range of the scattering potential), fundamental constraints analogous to Eqs. (6)–(8) can be conveniently obtained by employing instead of the Born series (5) the notion [5] of eikonal multiple-scattering series.

Next we concentrate on potential fields possessing a radial [2] or at least an inversion symmetry $[U(\mathbf{r})=U(-\mathbf{r})]$. Such fields convert the full amplitude into a completely symmetric object, $f(\mathbf{n}',\mathbf{n})=f(\mathbf{n},\mathbf{n}')$. After some rearrangements, Eq. (3a) now becomes

$$\mathrm{Im}[i^{(N-3)/2}f(\mathbf{n}',\mathbf{n})] = \frac{\lambda^{(1-N)/2}}{2}\int f^*(\mathbf{n}',\mathbf{n}'')f(\mathbf{n}'',\mathbf{n})d\mathbf{n}'', \quad (9)$$

yielding a generalized unitarity equation for the scattering amplitude. In the three-dimensional world, Eq. (9) leads directly to the Glauber-Schomaker relation [6],

$$\mathrm{Im}\,f(\mathbf{n}',\mathbf{n}) = \frac{k}{4\pi}\int f^*(\mathbf{n}',\mathbf{n}'')f(\mathbf{n}'',\mathbf{n})d\mathbf{n}''.$$

Suppose further that for a given λ the differential scattering cross section, $\sigma(\mathbf{n}',\mathbf{n})=|f(\mathbf{n}',\mathbf{n})|^2$, is known for all the scattering angles between $\mathbf{n}'$ and $\mathbf{n}$. Then, according to Eq. (9), the possibility arises to determine the phase $\alpha(\mathbf{n}',\mathbf{n})$ of the complex amplitude $f(\mathbf{n}',\mathbf{n})=[\sigma(\mathbf{n}',\mathbf{n})]^{1/2}\exp[i\alpha(\mathbf{n}',\mathbf{n})]$ as the solution of the following integral equation:

$$\sin[\alpha(\mathbf{n}',\mathbf{n}) + \pi(N-3)/4] = \frac{1}{2}\int\left(\frac{\sigma(\mathbf{n}',\mathbf{n}'')\sigma(\mathbf{n}'',\mathbf{n})}{\sigma_\lambda(N)\sigma(\mathbf{n}',\mathbf{n})}\right)^{1/2} \times\cos[\alpha(\mathbf{n}',\mathbf{n}'') - \alpha(\mathbf{n}'',\mathbf{n})]d\mathbf{n}''. \quad (10)$$

It is easily verified that this equation is invariant with respect to the replacement

$$\alpha \to (\pi/2)(1-N) - \alpha. \quad (11)$$

The inference to be drawn from this is that the phase of the scattering amplitude can be computed only with the accuracy of the transformation

$$f(\mathbf{n}',\mathbf{n}) \to i^{1-N}f^*(\mathbf{n}',\mathbf{n}). \quad (12)$$

Hence the knowledge of differential scattering cross section at all angles for a given collision energy enables to restore the scattering amplitude with the accuracy of the transformation (12).

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