Time refraction and the quantum properties of vacuum

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We present the classical and quantum theory of time refraction in a generic nonstationary medium. The classical approach leads to expressions for the temporal refraction coefficient, and the temporal Fresnel laws are given. The quantum formulation leads to the derivation of instantaneous Bogoliubov transformations and the evaluation of the number of photon pairs created from vacuum by the temporal changes in the medium. The influence of boundary conditions, the connection of this model with the dynamical Casimir effect, and radiation from superluminal nonaccelerated optical boundaries is also discussed.

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I. INTRODUCTION

In recent years there has been an increasing interest in the properties of the electromagnetic vacuum and, in particular, on the not yet experimentally observed effects of vacuum photon creation. Various specific models, driven by different physical motivations, have been explored, such as the dynamical Casimir effect [1–3], the Unruh-Davies radiation [4,5] and time refraction [6,7]. The first one results from an extension of the double plate geometry of the famous Casimir effect [8], which reveals on a macroscopical scale the energetic contents of vacuum. The Unruh-Davies radiation (more often simply called Unruh radiation) explores the equivalence between gravitation and acceleration, and demonstrates the existence of a thermal radiation spectrum produced by an accelerated boundary, in the same way as the gravitational field at the horizon of a black hole produces the Hawking radiation [9]. On the other hand, time refraction results from the symmetry between space and time, and extends the usual concept of refraction into the time domain. It can also be seen as the basic mechanism behind the processes of photon frequency shift, which are well known in plasma physics and non-linear optics, and are often called photon acceleration processes [10].

These various models and concepts explore in different ways the same physical properties of vacuum, even if the relation between them has not yet been completely established. Recently, Guerreiro *et al.* [11] have shown using the time refraction concept, that an non-accelerated but superluminal boundary would emit radiation that is similar to (but distinct from) the Unruh radiation. In the present work we explore the time refraction model even further, by considering an arbitrary optical medium. The most relevant specific feature of time refraction, as compared with the other two models for active vacuum, is that it is independent of boundary conditions. In that sense, it is a purely temporal effect, in contrast with the dynamic Casimir or the Unruh models which essentially depend on the spatial boundaries.

We study here the spectral changes due to an arbitrary time variation of the refractive index of the medium. We start with a classical description, and determine the temporal Fresnel formulae as well as the temporal Snell's law. By doing so we generalize our previous results, which were only valid for sudden changes of the refractive index as determined by one or several step functions [6,7]. We then consider the quantum description for an arbitrary temporal change of the medium, establish the associated instantaneous Bogoliubov transformations and calculate the number of photon pairs emitted from vacuum. We apply our generic results to physically relevant situations corresponding to irreversible and reversible changes in the medium. We also discuss the influence of the boundary conditions and establish a precise link with the dynamical Casimir effect. Finally, we revisit the problem of vacuum radiation from superluminal optical boundaries, by extending our previous results for sharp boundaries [11] to the case of superluminal boundaries with an arbitrary spatial profile.

II. CLASSICAL DESCRIPTION

Let us then consider an unbounded and nonstationary optical medium. Conversion to a bounded medium, is straightforward. The case of time refraction inside an optical fiber has already been discussed [12], and the case of a cavity will be discussed below. Let us also assume that the refractive index of the medium starts to change at time $t=0$. We can describe the variation of the refractive index by a generic function of time, $n(t)$, which can be approximated by a sequence of discrete steps of duration τ, where the continuous limit of $\tau\rightarrow 0$ can be taken afterwards. This discretization process can be analytically described by $n(t)=n_j$, for $(j-1)\tau < t \leqslant j\tau$, with j integer, where we can determine the successive values of the refractive index as $n_j=n_{j-1}+[dn(t)/dt]\tau$, for $\tau\rightarrow 0$. For a given mode of the electromagnetic waves propagating along the arbitrary Ox-direction in the medium, we can describe the electric field by

$$\vec{E}(x,t)=[\vec{E}(t)e^{-i\phi(t)}+\vec{E}'(t)e^{i\phi(t)}]e^{ikx}+\text{c.c.}, \tag{1}$$

with the phase function defined by

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$$\phi(t) = \int_0^t \omega(t')dt'. \quad (2)$$

Here, $\vec{E}$ and $\vec{E}'$ are the field amplitudes for waves propagating in the positive and negative Ox-directions, respectively. The time dependent value for the mode frequency ω will have to obey the linear instantaneous dispersion relation

$$\omega(t) = kc/n(t). \quad (3)$$

This expression can be seen as the temporal Snell's law [6], because it relates the wave frequencies at two different times, $\omega(t_1)n(t_1) = \omega(t_2)n(t_2)$, whereas the usual Snell's law for (space) refraction relates the wavevectors in two different media. Now, in order to establish the temporal evolution of the mode electric field, we consider the above discretized model for the refractive index, which corresponds to the alternative expression

$$\vec{E}_j(x,t) = [\vec{e}_j(t) + \vec{e}_j'(t)]e^{ikx} + \text{c.c.}, \quad (4)$$

where we now use

$$\vec{e}_j(t) = \vec{E}_j \exp(-i\omega_j t), \quad \vec{e}_j'(t) = \vec{E}_j' \exp(i\omega_j t) \quad (5)$$

and $\omega_j = kc/n_j$. These expressions are valid inside the interval $t_{j-1} < t < t_j$, with $t_j = j\tau$. Validity of Maxwell's equations for all times, including the discontinuity times $t = t_j$, implies the continuity of the displacement field $\vec{D} = \epsilon_0 n^2 \vec{E}$, and of the magnetic induction field $\vec{B} = -\int(\nabla \times \vec{E})dt$. Following the approach outlined in our previous work [7], we can derive

$$e_j = [A_{j-1}e_{j-1} - B_{j-1}e_{j-1}']e^{-i\omega_j\tau},$$

$$e_j' = [A_{j-1}e_{j-1}' - B_{j-1}e_{j-1}]e^{+i\omega_j\tau}, \quad (6)$$

where we have assumed that the fields e_j and e_j' were evaluated at the instant $t = t_j - 0$, and the fields e_{j-1} and e_{j-1}' at the instant $t = t_{j-1} - 0$. The coefficients A_{j-1} and B_{j-1} are determined by

$$A_{j-1} = 1 - \frac{3}{2}\frac{\tau}{n_{j-1}}\frac{dn}{dt}, \quad B_{J-1} = \frac{1}{2}\frac{\tau}{n_{j-1}}\frac{dn}{dt}. \quad (7)$$

Let us now use

$$e_j = e_{j-1} + \frac{de}{dt}\tau, \quad e_j' = e_{j-1}' + \frac{de'}{dt}\tau. \quad (8)$$

Taking the limit $\tau \to 0$, and noting that

$$\lim_{\tau\to 0}\frac{1}{\tau}[\exp(\pm i\omega_j\tau) - 1] = \pm i\omega_j, \quad (9)$$

we finally arrive at the evolution for the fields $e(t)$ and $e'(t)$ propagating with the same wave number k and a time dependent frequency $\omega(t)$, but in opposite directions

$$\frac{de}{dt} = -\frac{1}{2n}\frac{dn}{dt}(3e + e') - i\omega e,$$

$$\frac{de'}{dt} = -\frac{1}{2n}\frac{dn}{dt}(3e' + e) + i\omega e'. \quad (10)$$

We can also express these coupled equations in terms of the electric field amplitudes $E(t)$ and $E'(t)$, using the relations

$$e(t) = E(t)\exp\left(-i\int^t \omega(t')dt'\right),$$

$$e'(t) = E'(t)\exp\left(+i\int^t \omega(t')dt'\right). \quad (11)$$

This leads to the following equations:

$$\frac{dE}{dt} = -\frac{1}{2n}\frac{dn}{dt}\left[3E + E'\exp\left(+2i\int^t \omega(t')dt'\right)\right] \quad (12)$$

and

$$\frac{dE'}{dt} = -\frac{1}{2n}\frac{dn}{dt}\left[3E' + E\exp\left(-2i\int^t \omega(t')dt'\right)\right]. \quad (13)$$

In order to understand the physical meaning of these equations, let us consider the special case where initially we have a wave propagating along the positive direction that dominates over the wave propagating in the opposite direction, $|E| \gg |E'|$, and also that the temporal changes in the medium are very slow, which means that there is a weak coupling between these two waves. We can approximate the previous two equations by

$$\frac{dE}{dt} \simeq -\frac{3}{2n}\frac{dn}{dt}E \quad (14)$$

and

$$\frac{dE'}{dt} \simeq -\frac{1}{2n}\frac{dn}{dt}E\exp\left(-2i\int^t \omega(t')dt'\right). \quad (15)$$

Integration of the first of these equations leads to

$$E(t) \simeq E(0)\exp\left[-\frac{3}{2}\int_0^t \frac{1}{n(t')}\frac{dn}{dt'}dt'\right]. \quad (16)$$

For a very slowly varying medium, this can be expanded to give

$$E(t) \simeq \left[1 - \frac{3}{2}\int_0^t \frac{1}{n(t')}\frac{dn}{dt'}dt'\right]E(0) \equiv T(t)E(0), \quad (17)$$

where the temporal transmission coefficient is $T(t) \sim 1$. Considering now Eq. (15), and assuming that $E(t) \sim E(0) = \text{const}$, we can easily obtain the reflected field resulting from the nonstationarity of the medium, of the form

$$E'(t) = R(t)E(0), \quad (18)$$

with

$$R(t) \simeq -\frac{E(0)}{2}\int_0^t \frac{1}{n(t')}\frac{dn}{dt'} \exp\left(-2i\int^{t'} \omega(t'')dt''\right)dt'. \tag{19}$$

This expression for the temporal reflection coefficient $R(t)$ is formally analogous to the well known reflection coefficient for stationary by nonhomogenous media $R(x)$ [13], with the space coordinate along the gradient of the refractive index replaced by the time coordinate, which demonstrates the existence of symmetry between space and time, or between nonstationarity and nonhomogeneity.

III. QUANTUM THEORY

Instead of a classical field, let us now consider the electric field operator valid in the quantum description of the optical phenomena. For a uniform and stationary medium this field operator can generally be written as

$$\vec{E}(x,t) = i\int \sqrt{\hbar\omega_k/2n_k^2}[a(k,t)e^{ikx}\vec{\epsilon}(k) - a^\dagger(k,t)e^{-ikx}\vec{\epsilon}^*(k)]\frac{dk}{2\pi}, \tag{20}$$

where $\vec{\epsilon}(k)$ is the unit polarization vector, n_k the refractive index for the field mode k, and $\omega_k = kc/n_k$ is the corresponding mode frequency. Here we use the one-dimensional propagation along some generic axis Ox, but the generalization to three dimension is straightforward. We have also used the time dependent destruction and creation operators

$$a(k,t) = a(k)e^{-i\omega_k t}, \quad a^\dagger(k,t) = a^\dagger(k)e^{i\omega_k t}. \tag{21}$$

From the above general operator, we can extract the electric field operator associated with a specific field mode k, as

$$\vec{E}(x,k,t) = [e(t) + e'(t)]\vec{\epsilon}(k)e^{ikx}, \tag{22}$$

where we have used the following property of the polarization vector $\vec{\epsilon}(k) = \vec{\epsilon}^*(-k)$, which is valid for both the linear polarization and the circular polarization photon states. We have also used the following field operators:

$$e(t) = i\sqrt{\frac{\hbar\omega_k}{2n_k^2}}a(k,t), \quad e'(t) = -i\sqrt{\frac{\hbar\omega_k}{2n_k^2}}a^\dagger(-k,t). \tag{23}$$

This expression is formally analogous to that of the classical fields discussed in the previous section.

Let us now consider a time varying medium. In this case, the field operator (22) has to obey the same evolution equations (and thus the same continuity relations) as the classical field. This means that, for a nonstationary medium taking different values of the refractive index $n = n_j$ in different time intervals between $t = (j-1)\tau$ and $t = jt$, for $j = 0, 1, 2, \ldots$ we can establish the same kind of relations between successive field operators. Following the procedure of our previous work [7], we can establish the relation between the field operators valid at the beginning and at the end of the elementary time slab with duration τ, in the form

$$a_j = [A_{j-1}a_{j-1} + B_{j-1}a_{j-1}^\dagger]\exp(-i\omega_j\tau),$$

$$a_j^\dagger = [A_{j-1}a_{j-1}^\dagger + B_{j-1}a_{j-1}]\exp(i\omega_j\tau), \tag{24}$$

where we have assumed, to simplify the notation, that $a_j \equiv a(k, t_j - 0)$, and $a_j^\dagger \equiv a^\dagger(k, t_j - 0)$, and used similar notations for a_{j-1} and $a_{j-1}^\dagger$. The expressions defining the coefficients A and B, to the lowest order of the elementary time interval τ, can be written here as

$$B_{j-1} \simeq \frac{\tau}{2n_{j-1}}\frac{dn}{dt}, \quad A_{j-1} \simeq 1 + \frac{1}{2}\left(\frac{\tau}{2n_{j-1}}\frac{dn}{dt}\right)^2. \tag{25}$$

We can easily verify the hyperbolic character of the transformation (24), as stated by the relation

$$|A_j|^2 - |B_j|^2 = 1. \tag{26}$$

This means that we can define squeezing parameters r_j such that $A_j = \cosh r_j$, and $B_j = \sinh r_j$. Let us now consider the limit $\tau \to 0$, for which we can write

$$a_j = a_{j-1} + \frac{da}{dt}\tau, \quad a_j^\dagger = a_{j-1}^\dagger + \frac{da^\dagger}{dt}\tau. \tag{27}$$

Taking the limit $\tau \to 0$, we can transform the discrete relations (24) into the following differential equations for the time evolution of the destruction and creation photon operators

$$\frac{da}{dt} = -i\omega(t)a + \frac{1}{2n(t)}\frac{dn}{dt}a^\dagger,$$

$$\frac{da^\dagger}{dt} = i\omega(t)a^\dagger + \frac{1}{2n(t)}\frac{dn}{dt}a. \tag{28}$$

This is again quite similar, but not formally identical to the evolution equations for the classical field (10). In these new evolution equations, the first terms on the right hand side represent the rapid oscillations associated with the wave field with the frequency $\omega(t)$. The second term represents the slowly varying part of the operators associated with the change in the background medium. Notice that, in order for the concept of a wave mode k to make sense, the temporal changes in the refractive index $n(t)$, and consequently on the mode frequency $\omega(t)$ have to take place on a long time scale, much longer than the period $2\pi/\omega(t)$. This means that our present quantum field theoretical model is only physically meaningful if the second term in the above evolution equations is much smaller than the first one. In order to focus our attention on the slowly varying part of the operators $a(k,t)$ and $a^\dagger(-k,t)$, it is then useful to define

$$a(t) = a_{0,k}(t)\exp[-i\phi(t)], \quad a^\dagger = a_{0,-k}^\dagger(t)\exp[i\phi(t)], \tag{29}$$

where $a_{0,k}(t)$ and $a_{0,-k}^\dagger$ are now the slowly varying operator amplitudes, and the phase is determined by Eq. (2). Replacing these new definitions in the coupled evolution equations (28), we simply get

$$\frac{d}{dt}a_{0,k} = \nu(t)a_{0,-k}^\dagger, \quad \frac{d}{dt}a_{0,-k}^\dagger = \nu^*(t)a_{0,k}, \tag{30}$$

where the coupling coefficient $\nu(t)$ is determined by

$$\nu(t) = \frac{1}{2n(t)}\left(\frac{dn}{dt}\right)\exp[2i\phi(t)] \equiv f(t)\exp[2i\phi(t)]. \quad (31)$$

Before solving these equations we should notice that they describe the field operator changes on a time scale much longer than $2\pi/\omega(t)$. This means that only the slowly varying terms in the coupling coefficient $\nu(t)$, and its complex conjugate will contribute to the physically relevant solutions of the coupled equations. It is then appropriate to replace in the above equations, the quantity $\nu(t)$ by its average over the fast time scale $\bar{\nu}(t)$, as determined by

$$\bar{\nu}(t) = \frac{1}{T}\int_t^{(t+T)} f(t')\exp[2i\phi(t')]dt'. \quad (32)$$

The natural choice for the period of integration will be the instantaneous wave period $T=2\pi/\omega$, which can be considered constant over the integration interval. The resulting averaged coupling coefficient $\bar{\nu}(t)$ will preserve its time variation over periods much faster than T. Notice that, for optical field modes, the period is of the order of one femtosecond, which means that the "slow" time variation that is retained after the averaging process can be as fast as a few tens of femtoseconds. So, very fast time varying processes that can occur in experiments using very short laser pulses can still be described by $\bar{\nu}(t)$.

In order to obtain an explicit expression for this averaged coupling coefficient, we should notice that, inside the averaging interval, we can use $\phi(t)=\phi_0+\omega t'$, where ω can be considered constant. After integration by parts, we can easily obtain

$$\bar{\nu}(t) = \frac{1}{2\omega(t)}\left(\frac{df}{dt}\right)\exp[i(2\phi_0 + 3\pi/4)]. \quad (33)$$

Using the definition of $f(t)$ as stated in Eq. (31), choosing for convenience the arbitrary phase as $\phi_0=-3\pi/8$, and neglecting the second derivative of the refractive index (which is valid in the slow time scale approximation considered here), we get

$$\bar{\nu}(t) \simeq -\frac{1}{2\omega(t)}\left(\frac{1}{n(t)}\frac{dn}{dt}\right)^2. \quad (34)$$

Notice that this coefficient is always real and negative, independently of the sign of the derivative of the refractive index. This will have an importance physical significance, as discussed below. The coupled equations can then be replaced by a more physically relevant form, valid on time scales much larger than the wave mode period

$$\frac{d}{dt}a_{0,k} = \bar{\nu}(t)a_{0,-k}^{\dagger}, \quad \frac{d}{dt}a_{0,-k}^{\dagger} = \bar{\nu}(t)a_{0,k}. \quad (35)$$

The solution of these coupled equations, satisfying the appropriate initial conditions, are

$$a_{0,k}(t) = A_k(t)a_{0,k}(0) - B_k(t)a_{0,-k}^{\dagger}(0),$$
$$a_{0,-k}^{\dagger}(t) = A_k(t)a_{0,-k}^{\dagger}(0) - B_k(t)a_{0,k}(0), \quad (36)$$

with

$$A_k(t) = \cosh[r(t)], \quad B_k(t) = \sinh[r(t)] \quad (37)$$

and

$$r(t) = \int_0^t \bar{\nu}(t')dt' = \int_0^t \frac{1}{\omega(t')n^2(t')}\left(\frac{dn}{dt'}\right)^2 dt'. \quad (38)$$

Equations (36) take the form of a time-dependent Bogoliubov transformation with coefficients $A_k(t)$ and $B_k(t)$, with the instantaneous squeezing parameter $r(t)$. These solutions can now be used to calculate the number of photon pairs created from vacuum by the time evolution of the medium. At any time t, we can define the number operator for the photon mode k, as

$$N_k(t) = a_k^{\dagger}(t)a_k(t) = a_{0,k}^{\dagger}(t)a_{0,k}(t). \quad (39)$$

We can also define a time dependent vacuum state $|0\rangle_t$, such that $a_k(t)|0\rangle_t=0$. But, if we consider the initial vacuum $|0\rangle_{t=0}\equiv|0\rangle$, we will get

$$a_k(t)|0\rangle \neq 0, \quad a_k(t=0)|0\rangle = 0. \quad (40)$$

We can therefore define the average number of photons created from vacuum at time t, in the usual way, as

$$\langle N_k(t)\rangle = \sinh^2[r(t)]. \quad (41)$$

This has the usual hyperbolic-sine square shape, but the important thing here is that the argument $r(t)$ is given by an integral over the arbitrary temporal evolution of the unbounded medium, and results from a consistent exploration of the concept of time refraction [6].

Notice also that, for each photon created with wave number k along the arbitrary Ox-direction, there will be another one created with wave number $-k$, in order to preserve the total vacuum momentum: $\langle N_k(t)\rangle=\langle N_{-k}(t)\rangle$. Integrating over all the possible directions, we get the total number of photons created from vacuum at time t

$$N_{tot} = 2\int_{k(min)}^{k(max)} \langle N_k(t)\rangle \frac{d\vec{k}}{(2\pi)^3}, \quad (42)$$

where the factor of two accounts for the two possible polarization states, and the upper and lower wave number cutoffs $k(min)$ and $k(max)$ are imposed by physical constraints. If the medium is isotropic, this integration is simplified and the triple integration over $d\vec{k}$ is replaced by a simple integration over $4\pi k^2 dk$.

IV. DYNAMICAL CASIMIR EFFECT

Let us apply the results of the previous results to the case of a finite medium. Our aim here is to establish a link between our time refraction model and the dynamical Casimir effect that can take place inside an optical cavity. Such a link can be established in two steps, where the first one generalizes the usual dynamical Casimir effect to an arbitrary temporal change of the internal optical medium.

First of all, it should be remarked that the main difference of time refraction processes in a finite medium with that considered above of a homogeneous and unbounded medium

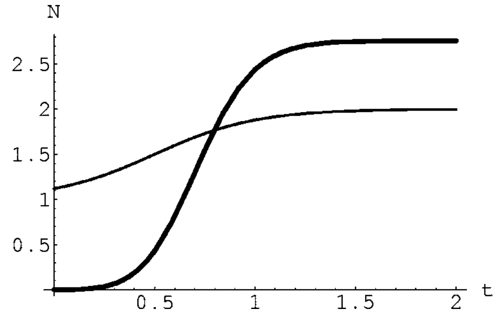

FIG. 1. Number of photon pairs created by an irreversible perturbation of the refractive index (in bold). The refractive index is also shown. We have taken $\omega=1$.

is that the range of allowed photon modes is reduced, because of the boundary conditions, and the dispersion relation is modified. The extension of time refraction to an optical fiber or a waveguide can be made in a straightforward way [12], where the axis Ox is no more arbitrary but coincides with the fiber axis, and the photon modes have a well defined transverse structure. Furthermore, if the optical fiber or waveguide has a finite length with reflecting boundaries, the continuum of modes k along this axis is reduced to a discrete set of cavity modes. The integration appearing in Eq. (42) is simply replaced by a summation over a discrete set of photon modes. In this sense, the time refraction model allows for an alternative view on the time varying optical cavity. In order to illustrate this, let us consider two cases, one where the refractive index of the medium inside the cavity changes from an initial value n_0 to a different value n_1 over a given time scale τ, and the other where the perturbation of the medium is reversible. These two cases were previous considered in the limit $\tau \to 0$ [6,7]. They can be described by the following evolution for the refractive index

$$n(t) = n_0 + (n_1 - n_0)\frac{1}{2}[1 + \tanh(t/\tau)] \tag{43}$$

and

$$n(t) = n_0 + \delta n \sin(t/\tau) \quad (0 < t < \tau/2). \tag{44}$$

The resulting photon number creation is represented in Figs. 1 and 2. These figures clearly show that the most interesting case is the second one, because the probability for photon

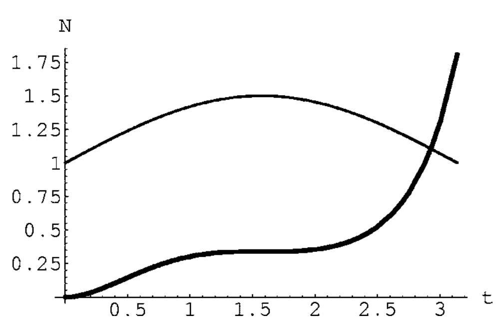

FIG. 2. Number of photon pairs created by a reversible perturbation of the refractive index (in bold). The refractive index is also shown. We have taken $\omega=1$.

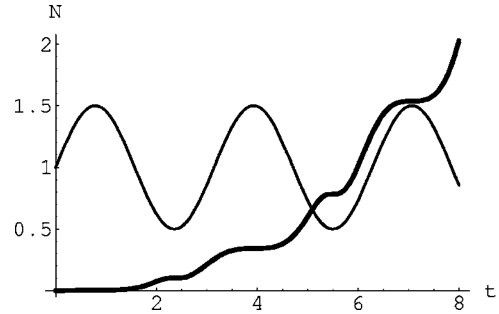

FIG. 3. Number of photon pairs created by a periodic perturbation of the refractive index (in bold). The refractive index is also shown. We have taken $\omega=1$.

creation accumulates, as long as the refractive index changes with time, irrespective of the sign of that change. An oscillating perturbation will then lead to an enhancement of the photon creation mechanism, as shown in Fig. 3. In the most favorable situations, the number of photons will increase exponentially with time, as discussed below. But, notice that the present discussion is only valid on a slow time scale. This means that the refractive index oscillations considered here have a characteristic frequency much smaller than the photon frequency. The case of fast time oscillations will be considered next.

The link between time refraction and the dynamical Casimir effect can easily be established by noting that a change in the refractive index is equivalent to a change in the optical length of an empty cavity, as illustrated in Fig. 4. This means that the two previous models are equivalent to the case of an empty cavity with a length that follows a similar temporal law. However, the traditional dynamical Casimir effect corresponds to an empty cavity with a rapidly oscillating optical length. In order to consider this case we have to modify the expression of the coupling coefficiente $\bar{\nu}$, because Eq. (34) was derived by assuming a slowly varying refractive index (equivalent to a slowly varying cavity length). Let us then go back to Eq. (31) and consider the Fourier transformation of the auxiliary function $f(t)=(dn/dt)/2n$. We can then write

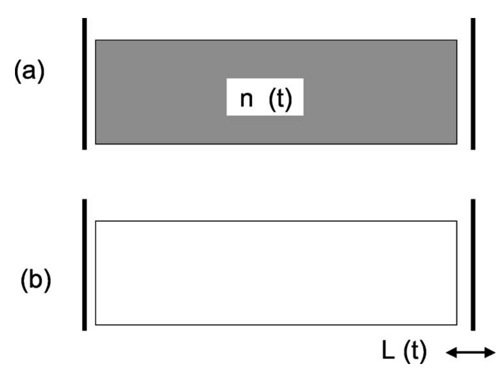

FIG. 4. Equivalence between a static cavity with a nonstationary medium with refractive index $n(t)$ (a) and an oscillating empty cavity with varying length $L(t)$ (b).

$$\nu(t) = f(t)e^{2i\phi(t)} = \int f(\omega')\exp[-i\omega' t + 2i\phi(t)]\frac{d\omega'}{2\pi}. \tag{45}$$

Noting that $\phi(t) \sim \omega t$, it can easily be realized that the dominant term in $\nu(t)$ will be determined by the Fourier component of the perturbation function $f(t)$ such that $\omega' = 2\omega$, because all the other terms will strongly oscillate in time and will give negligible contributions. So, for the rapidly oscillating cavity, we will have to replace Eq. (34) by

$$\bar{\nu}(t) = f[\omega' = 2\omega(t)], \tag{46}$$

where the time variation only occurs on a long time scale. In the usual dynamical Casimir effect, the optical length on the cavity oscillates with a frequency Ω, which means that coupling to the photon field will only efficiently occur for $\Omega = 2\omega$. In this case, and for very long times, the hyperbolic-sine can be approximated by an exponential, and we get from Eqs. (38) and (41)

$$\langle N_k(t)\rangle = 12\exp[2r(t)] \simeq \frac{2(\delta n)^2}{\Omega n_0^2(t')}t, \tag{47}$$

where δn is the amplitude of the perturbation of the refractive index n_0 at a frequency Ω. This shows an exponential growth of the number of photons created inside the cavity, which corresponds to the well known parametric instability predicted for the dynamical Casimir effect [1,3]. This result clearly states that that the dynamic Casimir effect of an oscillating cavity can be seen as a special case of the time refraction model. However, this model has much broader implications and can be used to describe many other physical configurations. Of particular interest is the case of superluminal moving boundaries, to be considered next.

V. SUPERLUMINAL OPTICAL BOUNDARIES

We have shown above how the state of light is altered by a generic nonstationary medium, where the refractive index varies uniformly, in unbounded space or inside a cavity. Recently, Guerreiro *et al.* [11] have demonstrated that a sharp optical boundary, moving with constant superluminal velocity would emit light from vacuum. It is therefore only natural to investigate whether a generic superluminal boundary profile would also give rise to a similar result, using the method introduced in this work.

Let us now consider an optical profile moving with an apparent velocity u along the Ox-direction, which changes the refractive index from n_0 to n_1, in a way similar to that considered in Ref. [11], but with a smooth shape as described by

$$n(x,t) = n_0 + \delta n h[(t - x/u)/\tau], \tag{48}$$

where $\delta n = n_1 - n_0$, and h is a continuous function in the range [0,1]. In this context, the apparent velocity u simply describes a delay in the change of refractive index between different points of space and does not refer to the actual velocity of the particles in the medium. Henceforth u can take arbitrarily large values, even larger than c. If this apparent velocity u is larger than c, we can always change from the laboratory frame S into a new frame S', with relative velocity $v_\infty = -c^2/u < c$ where this moving optical boundary will be perceived as moving with an infinite velocity

$$u' = \lim_{v\to v_\infty} \frac{u+v}{1+(vu)/c^2} = \infty \tag{49}$$

and Eq. (48) becomes formally analogous to Eq. (43). This is an important point, because, in the moving frame S' the superluminal boundary is perceived as a simple temporal change with spatial uniformity.

In this new frame S', the time boundary can be described as a four port device [7], coupling two initial electromagnetic field modes, evolving as $\exp(ik_i'x')$ and $\exp(ik_a'x')$, existing for $t' < 0$, with two final modes $\exp(ik_t'x')$ and $\exp(ik_r'x')$, existing for $t' > 0$. In this moving frame, the wave numbers of the two couples of modes are completely symmetrical, $k_a' = -k_i'$, and $k_r' = -k_t'$. This is a consequence of the total momentum conservation associated with purely temporal changes in the optical properties of the medium. But the same is not true in the laboratory frame S, where an asymmetry is introduced by different Doppler shifts, leading to

$$k_t(x,t) = k_i\frac{1 - s_i\beta/n_0}{1 - s_t\beta/n(x,t)} \tag{50}$$

and

$$k_r(x,t) = -k_i\frac{1 - s_i\beta/n_0}{1 - s_r\beta/n(x,t)}, \tag{51}$$

where $\beta = -v_\infty/c = -c/u < 1$, and we have introduced the signs $s_i = k_i/|k_i|$, $s_t = k_t/|k_t|$, and $s_r = k_r/|k_r|$. This means that, if in the frame S' the change of the refractive index occurs simultaneously at all points of space, in the frame S there is a delay between different points of space, which then introduced a delay in the change of the wave number, as expressed by the space and time dependence of the transmitted and reflected wave numbers, k_t and k_r.

Moreover, in the S' frame the refractive index has a different value as a direct consequence of the relativistic phase invariance (see [14]) and will be given by

$$n'(t') = s'n(t')\frac{s - \beta/n(t')}{1 - s\beta n(t')}, \tag{52}$$

where $s = k/|k|$, $s' = k'/|k'|$, and k and k' are the wave numbers of a given field mode expressed in the S and S' frames, respectively. We should then write the refractive index profile in the S' frame as $n'(t') = n_0' + \delta n' h(t'/\tau')$, with $\tau' = \gamma\tau$, and $\gamma = (1-\beta^2)^{-1/2}$. For small variations of the refractive index δn, we can write

$$\delta n' = s's\frac{1-\beta^2}{(1 - s\beta n_0)^2}\delta n + O(\delta n^2). \tag{53}$$

The above equations (30) can now be written in the frame S', as

$$\frac{d}{dt'}a_{0,k} = \nu'(t')a^{\dagger}_{0,-k}, \quad \frac{d}{dt}a^{\dagger}_{0,-k} = \nu'^{*}(t')a_{0,k}, \tag{54}$$

with

$$\nu'(t') = \frac{1}{2n'(t')}\frac{dn'}{dt'}\exp[2i\phi(t')] \simeq \frac{\delta n'}{2n_0}\frac{dh(t')}{dt'}\exp[2i\phi(t')]. \tag{55}$$

The relative change in the refractive index, appearing in this expression, is given by $\delta n'/n'_0 = \alpha\delta n/n_0$ with

$$\alpha = \frac{1-\beta^2}{(1-s\beta n_0)(s-\beta/n_0)}. \tag{56}$$

Finally, we can write $\nu'(t') = \alpha\nu(t-x/u)$, which implies for the squeezing parameter that $r'(t') = \alpha r(t-x/u)$. The number of photons produced from vacuum will then be determined by $\langle N_k(t-x/u)\rangle = \sinh^2[\alpha r(t-x/u)]$. Notice that, for $s\beta n_0 \to 1$, or for $\beta \to s n_0$, the change in the refractive index n' can, in principle, be arbitrarily large, as well as the number of photons emitted from vacuum. These resonant conditions are specially interesting for experimental purposes. Since the number of photons is a relativistic invariant, the same occurs in any reference frame, including the S frame. The existence of such resonances suggests that the quantum properties of the vacuum can be strongly enhanced by superluminal boundaries, increasing the prospects for future experimental evidence.

VI. CONCLUSIONS

In this paper we have described the phenomenon of time refraction in a nonstationary optical medium. The medium was considered uniform, but arbitrary temporal changes in the refractive index were considered. Both the classical and quantum theoretical descriptions were presented. The classical formulation allowed us to describe the effects of photon frequency shift that can be considered the basis for photon acceleration [14]. It also describes the coupling between counterpropagating waves, and confirms that (space) reflection is always associated with time refraction. Coupled mode equations for the counterpropagating signals were obtained. Under certain simplifying assumptions these equations lead to an expression for the temporal reflection coefficient that is the temporal analogue of the usual reflection coefficient in a nonhomogeneous but static medium. A clear symmetry is shown to exist between nonhomogeneity and nonstationarity, or between space and time changes in the optical medium.

However, such a symmetry is somewhat subtle, due to the nonexistence of time reflection, or reflection backwards in time. This means that, if the time refraction phenomena considered here always imply the occurrence of (space) reflection, the symmetric case of wave refraction due to spatial nonhomogeneity will never produce time reflection. Even so, a deep symmetry still holds because, if the first effect (time refraction) conserves momentum but not energy, the second one (the usual space refraction) conserves energy but not momentum.

On the other hand, the quantum theoretical description allowed us to derive temporal Bogoliubov transformations between time-dependent destruction and creation operators. The possible creation of photon pairs from vacuum was established for arbitrary time changes in the medium. This completes our previous derivation of time refraction, which were only valid for simple sudden changes in the medium [6,7], and extends this concept to more general temporal evolution laws.

The relation between time refraction and the dynamical Casimir effect in an oscillating cavity was also clarified. The usual conditions for a parametric instability in the time varying cavity were recovered. Our treatment allowed us to see these two different models in a more global perspective, and to demonstrate that the dynamical Casimir effect can be reduced to a special case of time refraction. Our approach is equally well suited for other physical configurations, such as that of reversible and irreversible changes in an optical fiber or in a cavity, as illustrated by numerical examples.

Finally, the results were extended to the case of an optical boundary with an arbitrary spatial profile and moving with a constant but superluminal velocity, generalizing our previous results for a sharp boundary [11] and showing that resonant excitation of photon pairs from vacuum can also in this case eventually occur. This new radiation process can be seen as a kind of Unruh radiation for superluminal boundaries. The existence of resonances in vacuum response to superluminal boundaries, as shown by Eq. (56), suggests that time refraction is an interesting candidate for experimental research, and can eventually be more favorable for evidence of vacuum radiation than the Unruh or the dynamical Casimir effects. In particular, the configuration first proposed by Yablonovich [15] to study the vacuum radiation processes can easily be adapted to the case of time refraction.

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