Topological objects in two-component Bose-Einstein condensates

Y. M. Cho*
Center for Theoretical Physics, College of Natural Sciences, Seoul National University, Seoul 151-742, Korea and C. N. Yang Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11790, USA

Hyojoong Khim† and Pengming Zhang‡
School of Physics, College of Natural Sciences, Seoul National University, Seoul 151-742, Korea
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We study the topological objects in two-component Bose-Einstein condensates. We compare two competing theories of two-component Bose-Einstein condensates, the popular Gross-Pitaevskii theory, and the recently proposed gauge theory of two-component Bose-Einstein condensate which has an induced vorticity interaction. We show that two theories produce very similar topological objects, in spite of the obvious differences in dynamics. Furthermore we show that the gauge theory of two-component Bose-Einstein condensates, with the $U(1)$ gauge symmetry, is remarkably similar to the Skyrme theory. Just like the Skyrme theory this theory admits the non-Abelian vortex, the helical vortex, and the vorticity knot. We construct the lightest knot solution in two-component Bose-Einstein condensates numerically, and discuss how the knot can be constructed in the spin-$\frac{1}{2}$ condensate of ^{87}Rb atoms.

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I. INTRODUCTION

Topological objects, in particular finite energy topological objects, have played important roles in physics [1,2]. In Bose-Einstein condensates (BEC) the best known topological objects are the vortices, which have been widely studied in the literature. Theoretically these vortices have successfully been described by the Gross-Pitaevskii Lagrangian. On the other hand, the recent advent of multicomponent BEC, in particular, the spin-$\frac{1}{2}$ condensate of ^{87}Rb atoms, has widely opened an interesting possibility for us to construct totally new topological objects in condensed matter physics. This is because the multicomponent BEC obviously has more interesting non-Abelian structure which does not exist in ordinary (one-component) BEC, and thus could admit new topological objects which are absent in ordinary BEC [3,4]. As importantly, the multicomponent BEC provides a rare opportunity to study the dynamics of the topological objects theoretically. The dynamics of multicomponent BEC could be significantly different from that of ordinary BEC. This is because the velocity field of the multicomponent BEC, unlike the ordinary BEC, in general, has a nonvanishing vorticity which could play an important role in the dynamics of the multicomponent BEC [5]. So the multicomponent BEC provides an excellent opportunity for us to study non-Abelian dynamics of the condensate theoretically and experimentally.

The purpose of this paper is to discuss the non-Abelian dynamics of two-component BEC. We first study the popular Gross-Pitaevskii theory of two-component BEC, and compare the theory with the recent gauge theory of two-component BEC which has a vorticity interaction [5]. We show that, in spite of the obvious dynamical differences, two theories are not much different physically. In particular, they admit remarkably similar topological objects, the helical vortex whose topology is fixed by $\pi_2(S^2)$ and the vorticity knot whose topology is fixed by $\pi_3(S^2)$. Moreover, we show that the vorticity knot is nothing but the vortex ring made of the helical vortex. Finally we show that the gauge theory of two-component BEC is very similar to the theory of two-gap superconductors, which implies that our analysis here can have an important implication in two-gap superconductors.

A prototype non-Abelian knot is the Faddeev-Niemi knot in Skyrme theory [6,7]. The vorticity knot in two-component BEC turns out to be surprisingly similar to the Faddeev-Niemi knot. So it is important for us to understand the Faddeev-Niemi knot first. The Faddeev-Niemi knot is described by a nonlinear sigma field $\hat{n}$ (with $\hat{n}^2=1$) which defines the Hopf mapping $\pi_3(S^2)$, the mapping from the compactified space S^3 to the target space S^2 of $\hat{n}$, in which the preimage of any point in the target space becomes a closed ring in S^3. When $\pi_3(S^2)$ becomes nontrivial, the preimages of any two points in the target space are linked, with the linking number fixed by the third homotopy of the Hopf mapping. In this case the mapping is said to describe a knot, with the knot quantum number identified by the linking number of two rings. And it is this Hopf mapping that describes the topology of the Faddeev-Niemi knot [6–8].

In this paper we show that the vorticity knot in two-component BEC has exactly the same topology as the Faddeev-Niemi knot. The only difference is that here the vorticity knot in two-component BEC has the extra dressing of the scalar field which represents the density of the condensation.

The paper is organized as follows. In Sec. II we review the Skyrme theory to emphasize its relevance in condensed matter physics. In Sec. III we review the topological objects in Skyrme theory in order to compare them with those in

*Electronic address: ymcho@yongmin.snu.ac.kr
†Electronic address: kimhj76@phya.snu.ac.kr
‡Electronic address: zhpm@phya.snu.ac.kr

two-component BEC. In Sec. IV we review the popular Gross-Pitaevskii theory of two-component BEC, and show that the theory is closely related to Skyrme theory. In Sec. V we discuss the helical vortex in the Gross-Pitaevskii theory of two-component BEC, and show that it is a twisted vorticity flux. In Sec. VI we discuss the gauge theory of two-component BEC which includes the vorticity interaction, and compare it with the Gross-Pitaevskii theory of two-component BEC. In Sec. VII we discuss the helical vortex in gauge theory of two-component BEC, and compare it with those in the Gross-Pitaevskii theory and Skyrme theory. We demonstrate that the helical vortex in all three theories are remarkably similar to one another. In Sec. VIII we present a numerical knot solution in the gauge theory of two-component BEC, and show that it is nothing but the vortex ring made of helical vorticity flux. Finally, in Sec. IX we discuss the physical implications of our result. In particular, we emphasize the similarity between the gauge theory of two-component BEC and the theory of two-gap superconductors.

II. SKYRME THEORY: A REVIEW

The Skyrme theory has long been interpreted as an effective field theory of strong interaction with a remarkable success [9]. However, it can also be interpreted as a theory of monopoles, in which the monopole-antimonopole pairs are confined through a built-in Meissner effect [6,8]. This suggests that the Skyrme theory could be viewed to describe very interesting condensed matter physics. Indeed the theory and the theory of two-component BEC have many common features. In particular, the topological objects that we discuss here are very similar to those in the Skyrme theory. To understand this we will review the Skyrme theory first.

Let ω and $\hat{n}$ (with $\hat{n}^2=1$) be the massless scalar field and the nonlinear sigma field in Skyrme theory, and let

$$U=\exp\left(\frac{\omega}{2i}\vec{\sigma}\cdot\hat{n}\right)=\cos\frac{\omega}{2}-i(\vec{\sigma}\cdot\hat{n})\sin\frac{\omega}{2},$$

$$L_\mu=U\partial_\mu U^\dagger. \tag{1}$$

With this one can write the Skyrme Lagrangian as [2]

$$\begin{aligned}\mathcal{L}&=\frac{\mu^2}{4}\mathrm{tr}L_\mu^2+\frac{\alpha}{32}\mathrm{tr}([L_\mu,L_\nu])^2\\&=-\frac{\mu^2}{4}\left[\frac{1}{2}(\partial_\mu\omega)^2+2\sin^2\frac{\omega}{2}(\partial_\mu\hat{n})^2\right]\\&\quad-\frac{\alpha}{16}\left[\sin^2\frac{\omega}{2}(\partial_\mu\omega\partial_\nu\hat{n}-\partial_\nu\omega\partial_\mu\hat{n})^2\right.\\&\quad\left.+4\sin^4\frac{\omega}{2}(\partial_\mu\hat{n}\times\partial_\nu\hat{n})^2\right],\end{aligned} \tag{2}$$

where μ and α are the coupling constants. The Lagrangian has a hidden local U(1) symmetry as well as a global SU(2) symmetry. From the Lagrangian one has the following equations of motion:

$$\begin{aligned}&\partial^2\omega-\sin\omega(\partial_\mu\hat{n})^2+\frac{\alpha}{8\mu^2}\sin\omega(\partial_\mu\omega\partial_\nu\hat{n}-\partial_\nu\omega\partial_\mu\hat{n})^2\\&\quad+\frac{\alpha}{\mu^2}\sin^2\frac{\omega}{2}\partial_\mu[(\partial_\mu\omega\partial_\nu\hat{n}-\partial_\nu\omega\partial_\mu\hat{n})\cdot\partial_\nu\hat{n}]\\&\quad-\frac{\alpha}{\mu^2}\sin^2\frac{\omega}{2}\sin\omega(\partial_\mu\hat{n}\times\partial_\nu\hat{n})^2=0,\\&\partial_\mu\left\{\sin^2\frac{\omega}{2}\hat{n}\times\partial_\mu\hat{n}+\frac{\alpha}{4\mu^2}\sin^2\frac{\omega}{2}[(\partial_\nu\omega)^2\hat{n}\times\partial_\mu\hat{n}\right.\\&\quad\left.-(\partial_\mu\omega\partial_\nu\omega)\hat{n}\times\partial_\nu\hat{n}]+\frac{\alpha}{\mu^2}\sin^4\frac{\omega}{2}(\hat{n}\cdot\partial_\mu\hat{n}\times\partial_\nu\hat{n})\partial_\nu\hat{n}\right\}\\&=0.\end{aligned} \tag{3}$$

Notice that the second equation can be interpreted as a conservation of the SU(2) current, which, of course, is a simple consequence of the global SU(2) symmetry of the theory.

With the spherically symmetric ansatz

$$\omega=\omega(r),\quad \hat{n}=\hat{r}, \tag{4}$$

(3) is reduced to

$$\begin{aligned}&\frac{d^2\omega}{dr^2}+\frac{2}{r}\frac{d\omega}{dr}-\frac{2\sin\omega}{r^2}+\frac{2\alpha}{\mu^2}\left[\frac{\sin^2(\omega/2)}{r^2}\frac{d^2\omega}{dr^2}+\frac{\sin\omega}{4r^2}\left(\frac{d\omega}{dr}\right)^2\right.\\&\quad\left.-\frac{\sin\omega\sin^2(\omega/2)}{r^4}\right]=0.\end{aligned} \tag{5}$$

Imposing the boundary condition

$$\omega(0)=2\pi,\quad \omega(\infty)=0, \tag{6}$$

one can solve the Eq. (5) and obtain the well-known skyrmion which has a finite energy. The energy of the skyrmion is given by

$$\begin{aligned}E&=\frac{\pi}{2}\mu^2\int_0^\infty\left\{\left(r^2+\frac{2\alpha}{\mu^2}\sin^2\frac{\omega}{2}\right)\left(\frac{d\omega}{dr}\right)^2\right.\\&\quad\left.+8\left(1+\frac{\alpha}{2\mu^2r^2}\sin^2\frac{\omega}{2}\right)\sin^2\frac{\omega}{2}\right\}dr\\&=\pi\sqrt{\alpha}\mu\int_0^\infty\left[x^2\left(\frac{d\omega}{dx}\right)^2+8\sin^2\frac{\omega}{2}\right]dx\simeq73\sqrt{\alpha}\mu,\end{aligned} \tag{7}$$

where $x=\mu/\sqrt{\alpha}$ is a dimensionless variable. Furthermore, it carries the baryon number [2,9]

$$Q_s=\frac{1}{24\pi^2}\int\varepsilon_{ijk}\mathrm{tr}(L_iL_jL_k)d^3r=1, \tag{8}$$

which represents the nontrivial homotopy $\pi_3(S^3)$ of the mapping from the compactified space S^3 to the SU(2) space S^3 defined by U in (1).

A remarkable point of (3) is that

$$\omega=\pi, \tag{9}$$

becomes a classical solution, independent of $\hat{n}$ [6]. So restricting ω to π, one can reduce the Skyrme Lagrangian (2) to the Skyrme-Faddeev Lagrangian

$$\mathcal{L} \to -\frac{\mu^2}{2}(\partial_\mu \hat{n})^2 - \frac{\alpha}{4}(\partial_\mu \hat{n} \times \partial_\nu \hat{n})^2, \tag{10}$$

whose equation of motion is given by

$$\hat{n} \times \partial^2 \hat{n} + \frac{\alpha}{\mu^2}(\partial_\mu H_{\mu\nu})\partial_\nu \hat{n} = 0,$$

$$H_{\mu\nu} = \hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = \partial_\mu C_\nu - \partial_\nu C_\mu. \tag{11}$$

Notice that $H_{\mu\nu}$ admits a potential C_μ because it forms a closed two form. Again the equation can be viewed as a conservation of SU(2) current,

$$\partial_\mu\left(\hat{n} \times \partial_\mu \hat{n} + \frac{\alpha}{\mu^2} H_{\mu\nu}\partial_\nu \hat{n}\right) = 0. \tag{12}$$

It is this equation that allows not only the baby skyrmion and the Faddeev-Niemi knot but also the non-Abelian monopole [6,8].

III. TOPOLOGICAL OBJECTS IN SKYRME THEORY

The Lagrangian (10) has non-Abelian monopole solutions [6]

$$\hat{n} = \hat{r}, \tag{13}$$

where $\hat{r}$ is the unit radial vector. This becomes a solution of (11) except at the origin, because

$$\partial^2 \hat{r} = -\frac{2}{r^2}\hat{r}, \quad \partial_\mu H_{\mu\nu} = 0. \tag{14}$$

This is very similar to the well-known Wu-Yang monopole in SU(2) Q.C.D. [6,10]. It has the magnetic charge

$$Q_m = \frac{1}{8\pi}\int \epsilon_{ijk} H_{ij} d\sigma_k = 1, \tag{15}$$

which represents the nontrivial homotopy $\pi_2(S^2)$ of the mapping from the unit sphere S^2 centered at the origin in space to the target space S^2.

The above exercise tells that we can identify $H_{\mu\nu}$ as a magnetic field and C_μ as the corresponding magnetic potential. As important, this tells that the skyrmion is nothing but a monopole dressed by the scalar field ω, which makes the energy of the skyrmion finite [6].

It has been well known that the Skyrme theory has a vortex solution known as the baby skyrmion [11]. Moreover, the theory also has a twisted vortex solution, the helical baby skyrmion [8]. To construct the desired helical vortex let (ϱ, φ, z) be the cylindrical coordinates, and choose the ansatz

$$\hat{n} = \begin{pmatrix} \sin f(\varrho)\cos(n\varphi + mkz) \\ \sin f(\varrho)\sin(n\varphi + mkz) \\ \cos f(\varrho) \end{pmatrix}. \tag{16}$$

With this we have (up to a gauge transformation)

$$C_\mu = -(\cos f + 1)(n\partial_\mu \varphi + mk\partial_\mu z), \tag{17}$$

and can reduce Eq. (11) to

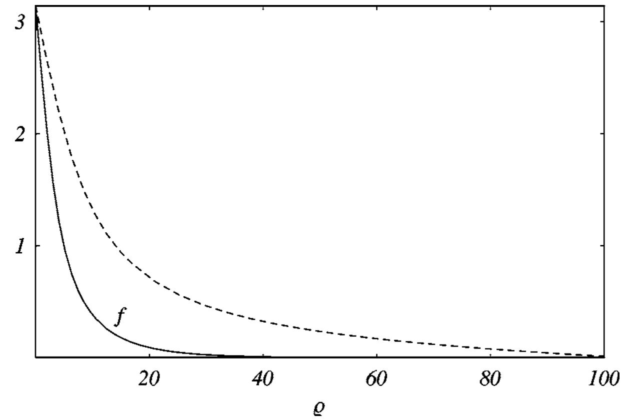

FIG. 1. The baby skyrmion (dashed line) with $m=0$, $n=1$ and the helical baby skyrmion (solid line) with $m=n=1$ in Skyrme theory. Here ϱ is in the unit $\sqrt{\alpha}/\mu$ and $k=0.8\ \mu/\sqrt{\alpha}$.

$$\left[1 + \frac{\alpha}{\mu^2}\left(\frac{n^2}{\varrho^2} + m^2k^2\right)\sin^2 f\right]\ddot{f} + \left[\frac{1}{\varrho} + \frac{\alpha}{\mu^2}\left(\frac{n^2}{\varrho^2} + m^2k^2\right) \times \dot{f}\sin f\cos f - \frac{\alpha}{\mu^2}\frac{1}{\varrho}\left(\frac{n^2}{\varrho^2} - m^2k^2\right)\sin^2 f\right] \times \dot{f} - \left(\frac{n^2}{\varrho^2} + m^2k^2\right)\sin f\cos f = 0. \tag{18}$$

So with the boundary condition

$$f(0) = \pi, \quad f(\infty) = 0, \tag{19}$$

we obtain the non-Abelian vortex solutions shown in Fig. 1. Notice that, when $m=0$, the solution describes the well-known baby skyrmion. But when m is not zero, it describes a helical vortex which is periodic in z coordinates [8]. In this case, the vortex has a nonvanishing magnetic potential C_μ not only around the vortex but also along the z axis.

Obviously the helical vortex has the helical magnetic field made of

$$H_{\hat{z}} = \frac{1}{\varrho}H_{\varrho\varphi} = \frac{n}{\varrho}\dot{f}\sin f,$$

$$H_{\hat{\varphi}} = -H_{\varrho z} = -mk\dot{f}\sin f, \tag{20}$$

which gives two quantized magnetic fluxes. It has a quantized magnetic flux along the z axis

$$\phi_{\hat{z}} = \int H_{\varrho\varphi} d\varrho\, d\varphi = -4\pi n, \tag{21}$$

and a quantized magnetic flux around the z axis (in a one period section from 0 to $2\pi/k$ in z coordinates)

$$\phi_{\hat{\varphi}} = -\int H_{\varrho z} d\varrho\, dz = 4\pi m. \tag{22}$$

Furthermore they are linked since $\phi_{\hat{z}}$ is surrounded by $\phi_{\hat{\varphi}}$. This point will be very important later when we discuss the knot.

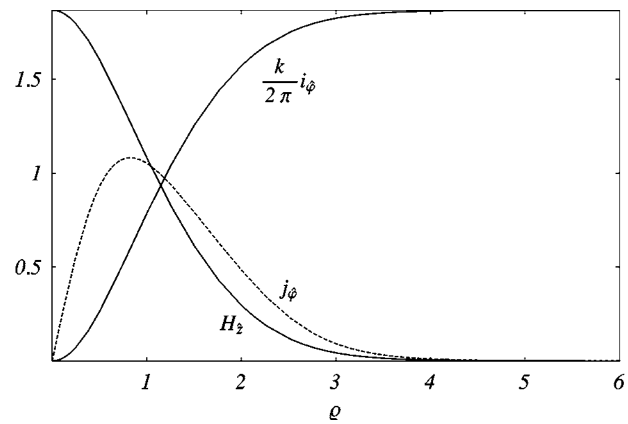

FIG. 2. The supercurrent $i_{\hat\varphi}$ (in one period section in z coordinates) and corresponding magnetic field $H_{\hat z}$ circulating around the cylinder of radius ϱ of the helical baby skyrmion with $m=n=1$. Here ϱ is in the unit $\sqrt{\alpha}/\mu$ and $k=0.8\ \mu/\sqrt{\alpha}$. The current density $j_{\hat\varphi}$ is represented by the dotted line.

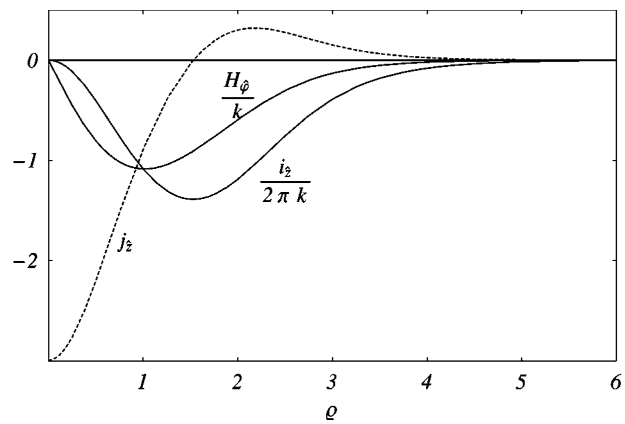

FIG. 3. The supercurrent $i_{\hat z}$ and corresponding magnetic field $H_{\hat\varphi}$ flowing through the disk of radius ϱ of the helical baby skyrmion with $m=n=1$. Here ϱ is in the unit $\sqrt{\alpha}/\mu$ and $k=0.8\ \mu/\sqrt{\alpha}$. The current density $j_{\hat z}$ is represented by the dotted line.

The vortex solutions implies the existence of the Meissner effect in Skyrme theory which confines the magnetic flux of the vortex [8]. To see how the Meissner effect comes about, notice that due to the U(1) gauge symmetry the Skyrme theory has a conserved current,

$$j_\mu = \partial_\nu H_{\mu\nu}, \quad \partial_\mu j_\mu = 0. \tag{23}$$

So the magnetic flux of the vortex can be thought to come from the helical electric current density

$$j_\mu = -\sin f\left[n\left(\ddot f + \frac{\cos f}{\sin f}\dot f^2 - \frac{1}{\varrho}\dot f\right)\partial_\mu\varphi - mk\left(\ddot f + \frac{\cos f}{\sin f}\dot f^2 + \frac{1}{\varrho}\dot f\right)\partial_\mu z\right]. \tag{24}$$

This produces the currents $i_{\hat\varphi}$ (per one period section in the z coordinate from $z=0$ to $z=2\pi/k$) around the z axis

$$i_{\hat\varphi} = -n\int_{\varrho=0}^{\varrho=\infty}\int_{z=0}^{z=2\pi/k} \sin f\left(\ddot f + \frac{\cos f}{\sin f}\dot f^2 - \frac{1}{\varrho}\dot f\right)\frac{d\varrho}{\varrho}dz = \frac{2\pi n}{k}\frac{\sin f}{\varrho}\dot f\bigg|_{\varrho=0}^{\varrho=\infty} = -\frac{2\pi n}{k}\dot f^2(0), \tag{25}$$

and $i_{\hat z}$ along the z axis

$$i_{\hat z} = -mk\int_{\varrho=0}^{\varrho=\infty} \sin f\left(\ddot f + \frac{\cos f}{\sin f}\dot f^2 + \frac{1}{\varrho}\dot f\right)\varrho d\varrho d\varphi = -2\pi mk\varrho\dot f\sin f\big|_{\varrho=0}^{\varrho=\infty} = 0. \tag{26}$$

Notice that, even though $i_{\hat z}=0$, it has a nontrivial current density which generates the net flux $\phi_{\hat\varphi}$.

The helical magnetic fields and currents are shown in Figs. 2 and 3. Clearly the helical magnetic fields are confined along the z axis, confined by the helical current. This is nothing but the Meissner effect, which confirms that the Skyrme theory has a built-in mechanism for the Meissner effect.

The helical vortex will become unstable and decay to the untwisted baby skyrmion unless the periodicity condition is enforced by hand. In this sense it can be viewed as unphysical. But for our purpose it plays a very important role, because it guarantees the existence of the Faddeev-Niemi knot in the Skyrme theory [6,8]. This is because we can naturally enforce the periodicity condition of the helical vortex making it a vortex ring by smoothly bending and connecting two periodic ends together. In this case the periodicity condition is automatically implemented, and the vortex ring becomes a stable knot.

The knot topology is described by the nonlinear sigma field $\hat n$, which defines the Hopf mapping from the compactified space S^3 to the target space S^2. When the preimages of two points of the target space are linked, the mapping $\pi_3(S^2)$ becomes nontrivial. In this case the knot quantum number of $\pi_3(S^2)$ is given by the Chern-Simon index of the magnetic potential C_μ,

$$Q_k = \frac{1}{32\pi^2}\int \varepsilon_{ijk}C_iH_{jk}d^3x = mn. \tag{27}$$

Notice that the knot quantum number can also be understood as the linking number of two magnetic fluxes of the vortex ring. This is because the vortex ring carries two magnetic fluxes linked together, m the unit of flux passing through the disk of the ring and n the unit of flux passing along the ring, whose linking number becomes mn. This linking number is described by the Chern-Simon index of the magnetic potential [8].

The knot has both topological and dynamical stability. Obviously the knot has a topological stability, because two flux rings linked together cannot be disconnected by any smooth deformation of the field.

The dynamical stability follows from the fact that the supercurrent (24) has two components, the one moving along the knot and the other moving around the knot tube. Clearly the current moving along the knot generates an angular momentum around the z axis which provides the centrifugal force preventing the vortex ring to collapse. Put it differently, the current generates the m unit of the magnetic flux trapped in the knot disk which cannot be squeezed out. And clearly, this flux provides a stabilizing repulsive force which prevent the collapse of the knot. This is how the knot acquires the dynamical stability. It is this remarkable interplay between topology and dynamics which assures the existence of the stable knot in Skyrme theory [8].

One could estimate the energy of the knot. Theoretically it has been shown that the knot energy has the following bound [12]

$$c\sqrt{\alpha}\mu Q^{3/4} \leqslant E_Q \leqslant C\sqrt{\alpha}\mu Q^{3/4}, \tag{28}$$

where $c=8\pi^2\times 3^{3/8}$ and C is an unknown constant that is no smaller than c. This suggests that the knot energy is proportional to $Q^{3/4}$. Indeed numerically, one finds [13]

$$E_Q \simeq 252\sqrt{\alpha}\mu Q^{3/4}, \tag{29}$$

up to $Q=8$. What is remarkable here is the sublinear Q dependence of the energy. This means that a knot with large Q cannot decay to knots with smaller Q.

IV. GROSS-PITAEVSKI THEORY OF TWO-COMPONENT BEC: A REVIEW

The creation of the multicomponent Bose-Einstein condensates of atomic gases has widely opened new opportunities for us to study the topological objects experimentally which so far have been only of theoretical interest. This is because the multicomponent BEC can naturally represent a non-Abelian structure, and thus can allow far more interesting topological objects. Already the vortices have successfully been created with different methods in two-component BECs [3,4]. But theoretically the multicomponent BEC has not been well-understood. In particular, it needs to be clarified how different the vortices in multicomponent BEC are from the well-known vortices in the single-component BEC. This is an important issue, because the new condensates could have a new interaction, the vorticity interaction, which is absent in single-component BECs. So in the following we first discuss the vortex in the popular Gross-Pitaevskii theory of two-component BEC, and compare it with that in the gauge theory of two-component BEC which has been proposed recently [5].

Let a complex doublet $\phi=(\phi_1,\phi_2)$ be the two-component BEC, and consider the nonrelativistic two-component Gross-Pitaevskii Lagrangian [14–17]

$$\begin{aligned}\mathcal{L} = {} & i\frac{\hbar}{2}\{[\phi_1^\dagger(\partial_t\phi_1) - (\partial_t\phi_1)^\dagger\phi_1] + [\phi_2^\dagger(\partial_t\phi_2) - (\partial_t\phi_2)^\dagger\phi_2]\} \\ & - \frac{\hbar^2}{2M}(|\partial_i\phi_1|^2 + |\partial_i\phi_2|^2) + \mu_1\phi_1^\dagger\phi_1 + \mu_2\phi_2^\dagger\phi_2 \\ & - \frac{\lambda_{11}}{2}(\phi_1^\dagger\phi_1)^2 - \lambda_{12}(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) - \frac{\lambda_{22}}{2}(\phi_2^\dagger\phi_2)^2, \end{aligned} \tag{30}$$

where μ_i are the chemical potentials and λ_{ij} are the quartic coupling constants which are determined by the scattering lengths a_{ij}

$$\lambda_{ij} = \frac{4\pi\hbar^2}{M}a_{ij}. \tag{31}$$

The Lagrangian (30) is a straightforward generalization of the single-component Gross-Pitaevskii Lagrangian to the two-component BEC. Notice that here we have neglected the trapping potential. This is justified if the range of the trapping potential is much larger than the size of topological objects we are interested in, and this is what we are assuming here. Clearly the Lagrangian has a global $U(1)\times U(1)$ symmetry.

One could simplify the Lagrangian (30) noticing the fact that experimentally the scattering lengths often have the same value. For example, for the spin-$\frac{1}{2}$ condensate of ^{87}Rb atoms, all a_{ij} have the same value of about 5.5 nm within 3% or so [3,4]. In this case one may safely assume

$$\lambda_{11} \simeq \lambda_{12} \simeq \lambda_{22} \simeq \bar{\lambda}. \tag{32}$$

With this assumption (30) can be written as

$$\begin{aligned}\mathcal{L} = {} & i\frac{\hbar}{2}[\phi^\dagger(\partial_t\phi) - (\partial_t\phi)^\dagger\phi] - \frac{\hbar^2}{2M}|\partial_i\phi|^2 - \frac{\bar{\lambda}}{2}\left(\phi^\dagger\phi - \frac{\mu}{\bar{\lambda}}\right)^2 \\ & - \delta\mu\phi_2^\dagger\phi_2, \end{aligned} \tag{33}$$

where

$$\mu = \mu_1, \quad \delta\mu = \mu_1 - \mu_2. \tag{34}$$

Clearly the Lagrangian has a global U(2) symmetry when $\delta\mu=0$. So the $\delta\mu$ interaction can be understood to be the symmetry breaking term which breaks the global U(2) symmetry to $U(1)\times U(1)$. Physically $\delta\mu$ represents the difference of the chemical potentials between ϕ_1 and ϕ_2 (Here one can always assume $\delta\mu\geqslant 0$ without loss of generality), so that it vanishes when the two condensates have the same chemical potential. Even when they differ the difference could be small, in which case the symmetry breaking interaction could be treated perturbatively. This tells us that the theory has an approximate global U(2) symmetry, even in the presence of the symmetry breaking term [18]. This is why it allows non-Abelian topological objects.

Normalizing ϕ to $(\sqrt{2M}/\hbar)\phi$ and parametrizing it by

$$\phi = \frac{1}{\sqrt{2}}\rho\zeta, \quad \left(|\phi| = \frac{1}{\sqrt{2}}\rho, \zeta^\dagger\zeta = 1\right) \tag{35}$$

we obtain the following Hamiltonian from the Lagrangian (33) in the static limit (in the natural unit $c=\hbar=1$),

$$\mathcal{H} = \frac{1}{2}(\partial_i\rho)^2 + \frac{1}{2}\rho^2|\partial_i\zeta|^2 + \frac{\lambda}{8}(\rho^2-\rho_0^2)^2 + \frac{\delta\mu^2}{2}\rho^2\zeta_2^*\zeta_2, \tag{36}$$

where

$$\lambda = 4M^2\bar{\lambda}, \quad \rho_0^2 = \frac{4\mu M}{\lambda}, \quad \delta\mu^2 = 2M\,\delta\mu. \tag{37}$$

Minimizing the Hamiltonian we have

$$\partial^2\rho - |\partial_i\zeta|^2\rho = \left(\frac{\lambda}{2}(\rho^2-\rho_0^2) + \delta\mu^2(\zeta_2^*\zeta_2)\right)\rho,$$

$$\left\{(\partial^2 - \zeta^\dagger\partial^2\zeta) + 2\frac{\partial_i\rho}{\rho}(\partial_i - \zeta^\dagger\partial_i\zeta) + \delta\mu^2(\zeta_2^*\zeta_2)\right\}\zeta_1 = 0,$$

$$\left\{(\partial^2 - \zeta^\dagger\partial^2\zeta) + 2\frac{\partial_i\rho}{\rho}(\partial_i - \zeta^\dagger\partial_i\zeta) - \delta\mu^2(\zeta_1^*\zeta_1)\right\}\zeta_2 = 0,$$

$$\zeta^\dagger\partial_i(\rho^2\partial_i\zeta) - \partial_i(\rho^2\partial_i\zeta^\dagger)\zeta = 0. \qquad (38)$$

The equation is closely related to Eq. (11) that we have in Skyrme theory, although on the surface it appears totally different from (11). To show this we let

$$\hat{n} = \zeta^\dagger\vec{\sigma}\zeta,$$

$$C_\mu = -2i\zeta^\dagger\partial_\mu\zeta, \qquad (39)$$

and find

$$(\partial_\mu\hat{n})^2 = 4(|\partial_\mu\zeta|^2 - |\zeta^\dagger\partial_\mu\zeta|^2) = 4|\partial_\mu\zeta|^2 - C_\mu^2,$$

$$\hat{n}\cdot(\partial_\mu\hat{n}\times\partial_\nu\hat{n}) = -2i(\partial_\mu\zeta^\dagger\partial_\nu\zeta - \partial_\nu\zeta^\dagger\partial_\mu\zeta) = \partial_\mu C_\nu - \partial_\nu C_\mu = H_{\mu\nu}. \qquad (40)$$

Notice that here $H_{\mu\nu}$ is precisely the closed two form which appears in (11). Moreover, from (39) we have the identity

$$\left[\partial_\mu + \frac{1}{2i}(C_\mu\hat{n} - \hat{n}\times\partial_\mu\hat{n})\cdot\vec{\sigma}\right]\zeta = 0. \qquad (41)$$

This identity plays an important role in the non-Abelian gauge theory, which shows that there exists a unique SU(2) gauge potential which parallelizes the doublet ζ [10]. For our purpose this allows us to rewrite the equation of the doublet ζ in (38) into a completely different form.

Indeed with the above identities we can express (38) in terms of $\hat{n}$ and C_μ. With (40) the first equation of (38) can be written as

$$\partial^2\rho - \tfrac{1}{4}[(\partial_i\hat{n})^2 + C_i^2]\rho = \left(\frac{\lambda}{2}(\rho^2-\rho_0^2) + \delta\mu^2(\zeta_2^*\zeta_2)\right)\rho. \qquad (42)$$

Moreover, with (41) the second and third equations of (38) can be expressed as

$$\frac{1}{2i}(A + \vec{B}\cdot\vec{\sigma})\zeta = 0,$$

$$A = \partial_i C_i + 2\frac{\partial_i\rho}{\rho}C_i + i(2\zeta_2^*\zeta_2 - 1)\delta\mu^2,$$

$$\vec{B} = \hat{n}\times\partial^2\hat{n} + 2\frac{\partial_i\rho}{\rho}\hat{n}\times\partial_i\hat{n} - C_i\partial_i\hat{n} - \left(\partial_i C_i + 2\frac{\partial_i\rho}{\rho}C_i\right)\hat{n} + i\delta\mu^2\hat{k}, \qquad (43)$$

where $\hat{k}=(0,0,1)$. This is equivalent to

$$A + \vec{B}\cdot\hat{n} = 0,$$

$$\hat{n}\times\vec{B} - i\hat{n}\times(\hat{n}\times\vec{B}) = 0, \qquad (44)$$

so that (43) is written as

$$\vec{n}\times\partial^2\vec{n} + 2\frac{\partial_i\rho}{\rho}\vec{n}\times\partial_i\vec{n} - C_i\partial_i\vec{n} = \delta\mu^2\hat{k}\times\hat{n}. \qquad (45)$$

Finally, the last equation of (38) is written as

$$\partial_i(\rho^2 C_i) = 0, \qquad (46)$$

which tells us that $\rho^2 C_i$ is solenoidal (i.e., divergenceless). So we can always replace C_i with another field B_i

$$C_i = \frac{1}{\rho^2}\varepsilon_{ijk}\partial_j B_k = -\frac{1}{\rho^2}\partial_i G_{ij},$$

$$G_{ij} = \varepsilon_{ijk}B_k, \qquad (47)$$

and express (45) as

$$\hat{n}\times\partial^2\hat{n} + 2\frac{\partial_i\rho}{\rho}\hat{n}\times\partial_i\hat{n} + \frac{1}{\rho^2}\partial_i G_{ij}\partial_j\hat{n} = \delta\mu^2\hat{k}\times\vec{n}. \qquad (48)$$

With this (38) can now be written as

$$\partial^2\rho - \frac{1}{4}[(\partial_i\hat{n})^2 + C_i^2]\rho = \left(\frac{\lambda}{2}(\rho^2-\rho_0^2) + \delta\mu^2(\zeta_2^*\zeta_2)\right)\rho,$$

$$\hat{n}\times\partial^2\hat{n} + 2\frac{\partial_i\rho}{\rho}\hat{n}\times\partial_i\hat{n} + \frac{1}{\rho^2}\partial_i G_{ij}\partial_j\hat{n} = \delta\mu^2\hat{k}\times\vec{n},$$

$$\partial_i G_{ij} = -\rho^2 C_j. \qquad (49)$$

This tells us that (38) can be transformed to a completely different form which has a clear physical meaning. The last equation tells us that the theory has a conserved U(1) current j_μ,

$$j_\mu = \rho^2 C_\mu, \qquad (50)$$

which is nothing but the Noether current of the global U(1) symmetry of the Lagrangian (33). The second equation tells that the theory has another partially conserved SU(2) Noether current $\vec{j}_\mu$,

$$\vec{j}_\mu = \rho^2(\hat{n}\times\partial_\mu\hat{n} - C_\mu\hat{n}), \qquad (51)$$

which comes from the approximate SU(2) symmetry of the theory broken by the $\delta\mu$ term.

More importantly this shows that (38) is not much different from Eq. (11) in the Skyrme theory. Indeed in the absence of ρ, (11) and (49) acquire an identical form when $\delta\mu^2=0$, except that here H_{ij} is replaced by G_{ij}. This reveals that the Gross-Pitaevskii theory of two-component BEC is closely related to the Skyrme theory, which is really remarkable.

The Hamiltonian (36) can be expressed as

$$\mathcal{H} = \lambda\rho_0^4\hat{\mathcal{H}},$$

$$\hat{\mathcal{H}} = \frac{1}{2}(\hat{\partial}_i\hat{\rho})^2 + \frac{1}{2}\hat{\rho}^2|\hat{\partial}_i\zeta|^2 + \frac{1}{8}(\hat{\rho}^2-1)^2,$$

$$+ \frac{\delta\mu}{4\mu}\hat{\rho}^2\zeta_2^*\zeta_2, \qquad (52)$$

where

$$\hat{\rho} = \frac{\rho}{\rho_0}, \quad \hat{\partial}_i = \kappa\partial_i, \quad \kappa = \frac{1}{\sqrt{\lambda}\rho_0}.$$

Notice that $\hat{\mathcal{H}}$ is completely dimensionless, with only one dimensionless coupling constant $\delta\mu/\mu$. This tells that the physical unit of the Hamiltonian is $\lambda\rho_0^4$, and the physical scale κ of the coordinates is $1/\sqrt{\lambda}\rho_0$. This is comparable to the correlation length $\bar{\xi}$,

$$\bar{\xi} = \frac{1}{\sqrt{2\mu M}} = \sqrt{2}\kappa. \qquad (53)$$

For ^{87}Rb we have

$$M \simeq 8.1 \times 10^{10}\ \text{eV}, \quad \bar{\lambda} \simeq 1.68 \times 10^{-7}\ \text{nm}^2,$$

$$\mu \simeq 3.3 \times 10^{-12}\ \text{eV}, \quad \delta\mu \simeq 0.1\mu, \qquad (54)$$

so that the density of ^{87}Rb atom is given by

$$\langle\phi^\dagger\phi\rangle = \frac{\mu}{\bar{\lambda}} \simeq 0.998 \times 10^{14}/\text{cm}^3. \qquad (55)$$

From (54) we have

$$\lambda \simeq 1.14 \times 10^{11}, \quad \rho_0^2 \simeq 3.76 \times 10^{-11}\ \text{eV}^2,$$

$$\delta\mu^2 \simeq 5.34 \times 10^{-2}\ \text{eV}^2. \qquad (56)$$

So the physical scale κ for ^{87}Rb becomes about 1.84×10^2 nm.

V. VORTEX SOLUTIONS IN THE GROSS-PITAEVSKII THEORY

The two-component Gross-Pitaevskii theory is known to have non-Abelian vortices [17,18]. To obtain the vortex solutions in the two-component Gross-Pitaevskii theory we first consider a straight vortex with the ansatz

$$\rho = \rho(\varrho), \quad \zeta = \begin{pmatrix} \cos\dfrac{f(\varrho)}{2}\exp(-in\varphi) \\ \sin\dfrac{f(\varrho)}{2} \end{pmatrix}. \qquad (57)$$

With the ansatz (38) is reduced to

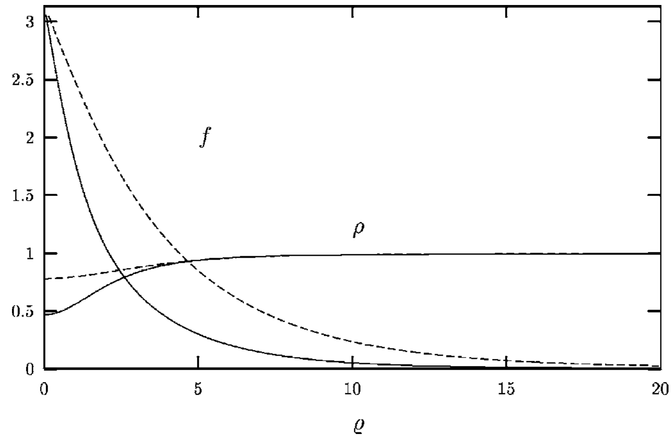

FIG. 4. The untwisted vortex in the Gross-Pitaevskii theory of two-component BEC. Here we have put $n=1$, and ϱ is in the unit of κ. Dashed and solid lines correspond to $\delta\mu/\mu=0.1$ and 0.2, respectively.

$$\ddot{\rho} + \frac{1}{\varrho}\dot{\rho} - \left[\frac{1}{4}\dot{f}^2 + \frac{n^2}{\varrho^2} - \left(\frac{n^2}{\varrho^2} - \delta\mu^2\right)\sin^2\frac{f}{2}\right]\rho = \frac{\lambda}{2}(\rho^2-\rho_0^2)\rho,$$

$$\ddot{f} + \left(\frac{1}{\varrho} + 2\frac{\dot{\rho}}{\rho}\right)\dot{f} + \left(\frac{n^2}{\varrho^2} - \delta\mu^2\right)\sin f = 0. \qquad (58)$$

Now, we choose the following ansatz for a helical vortex [18]

$$\rho = \rho(\varrho), \quad \zeta = \begin{pmatrix} \cos\dfrac{f(\varrho)}{2}\exp(-in\varphi) \\ \sin\dfrac{f(\varrho)}{2}\exp(imkz) \end{pmatrix}, \qquad (59)$$

and find that the equation (38) becomes

$$\ddot{\rho} + \frac{1}{\varrho}\dot{\rho} - \left[\frac{1}{4}\dot{f}^2 + \frac{n^2}{\varrho^2} - \left(\frac{n^2}{\varrho^2} - m^2k^2 - \delta\mu^2\right)\sin^2\frac{f}{2}\right]\rho$$

$$= \frac{\lambda}{2}(\rho^2-\rho_0^2)\rho,$$

$$\ddot{f} + \left(\frac{1}{\varrho} + 2\frac{\dot{\rho}}{\rho}\right)\dot{f} + \left(\frac{n^2}{\varrho^2} - m^2k^2 - \delta\mu^2\right)\sin f = 0. \qquad (60)$$

Notice that mathematically this equation becomes identical to the equation of the straight vortex (58), except that here $\delta\mu^2$ is replaced by $\delta\mu^2+m^2k^2$.

Now, with the boundary condition

$$\dot{\rho}(0) = 0, \quad \rho(\infty) = \rho_0,$$

$$f(0) = \pi, \quad f(\infty) = 0, \qquad (61)$$

we can solve (60). With $m=0$, $n=1$ we obtain the straight (untwisted) vortex solution shown in Fig. 4, but with $m=n=1$ we obtain the twisted vortex solution shown in Fig. 5. Of course (38) also admits the well-known Abelian vortices with $\zeta_1=0$ or $\zeta_2=0$. But obviously they are different from the non-Abelian vortices discussed here.

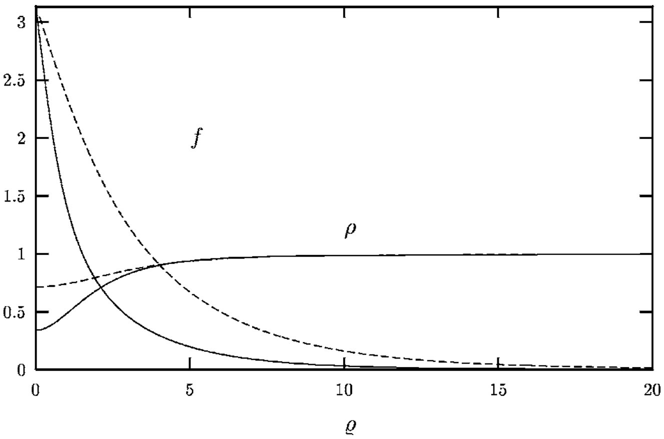

FIG. 5. The helical vortex in the Gross-Pitaevskii theory of two-component BEC. Here we have put $m=n=1$, $k=0.25/\kappa$, and ϱ is in the unit of κ. Dashed and solid lines correspond to $\delta\mu/\mu=0$, and 0.1, respectively.

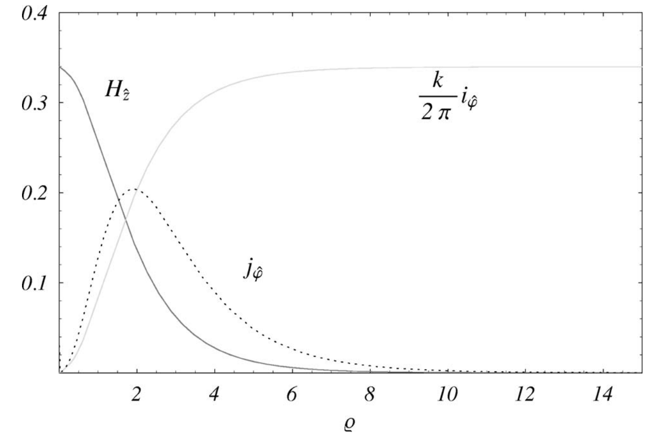

FIG. 6. The supercurrent $i_{\hat{\varphi}}$ (in one period section in z coordinates) and corresponding magnetic field $H_{\hat{z}}$ circulating around the cylinder of radius ϱ of the helical vortex in the Gross-Pitaevskii theory of two-component BEC. Here $m=n=1$, $k=0.25/\kappa$, $\delta\mu^2=0$, and ϱ is in the unit of κ. The current density $j_{\hat{\varphi}}$ is represented by the dotted line.

The untwisted non-Abelian vortex solution has been discussed before [16,17], but the twisted vortex solution here is new [18]. Although they look very similar on the surface, they are quite different. First, when $\delta\mu^2=0$, there is no untwisted vortex solution because in this case the vortex size (the penetration length of the vorticity) becomes infinite. However, the helical vortex exists even when $\delta\mu^2=0$. This is because the twisting reduces the size of the vortex tube. More importantly, they are physically different. The untwisted vortex is made of a single vorticity flux, but the helical vortex is made of two vorticity fluxes linked together [18].

In the Skyrme theory the helical vortex is interpreted as a twisted magnetic vortex, whose flux is quantized due to the topological reason. The helical vortex in the Gross-Pitaevskii theory is also topological, which can be viewed as a quantized vorticity flux [18]. To see this notice that, up to the overall factor 2, the potential C_μ introduced in (39) is nothing but the velocity potential V_μ (more precisely the momentum potential) of the doublet ζ [5,18]

$$V_\mu = -i\zeta^\dagger \partial_\mu \zeta = \frac{1}{2}C_\mu = -\frac{n}{2}(\cos f + 1)\partial_\mu \varphi - \frac{mk}{2}(\cos f - 1)\partial_\mu z, \tag{62}$$

which generates the vorticity

$$\bar{H}_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu = \frac{1}{2}H_{\mu\nu} = \frac{\dot{f}}{2}\sin f[n(\partial_\mu\varrho\partial_\nu\varphi - \partial_\nu\varrho\partial_\mu\varphi) + mk(\partial_\mu\varrho\partial_\nu z - \partial_\nu\varrho\partial_\mu z)]. \tag{63}$$

This has two quantized vorticity fluxes $\Phi_{\hat{z}}$ along the z axis

$$\Phi_{\hat{z}} = \int \bar{H}_{\hat{\varrho}\hat{\varphi}}\varrho\, d\varrho\, d\varphi = -2\pi n, \tag{64}$$

and $\Phi_{\hat{\varphi}}$ around the z axis (in one period section from $z=0$ to $z=2\pi/k$)

$$\Phi_{\hat{\varphi}} = \int_0^{2\pi/k} \bar{H}_{\hat{z}\hat{\varrho}} d\varrho dz = 2\pi m. \tag{65}$$

Clearly two fluxes are linked together.

Furthermore, just as in the Skyrme theory, these fluxes can be viewed to originate from the helical supercurrent which confines them with a built-in Meissner effect

$$\bar{j}_\mu = \partial_\nu \bar{H}_{\mu\nu} = -\sin f\left[n\left(\ddot{f} + \frac{\cos f}{\sin f}\dot{f}^2 - \frac{1}{\varrho}\dot{f}\right)\partial_\mu\varphi + mk\left(\ddot{f} + \frac{\cos f}{\sin f}\dot{f}^2 + \frac{1}{\varrho}\dot{f}\right)\partial_\mu z\right];$$

$$\partial_\mu \bar{j}_\mu = 0. \tag{66}$$

this produces the supercurrents $i_{\hat{\varphi}}$ (per one period section in z coordinates from $z=0$ to $z=2\pi/k$) around the z axis

$$i_{\hat{\varphi}} = -\frac{2\pi n}{k}\frac{\sin f}{\varrho}\dot{f}\bigg|_{\varrho=0}^{\varrho=\infty}, \tag{67}$$

and $i_{\hat{z}}$ along the z-axis

$$i_{\hat{z}} = -2\pi mk\varrho\dot{f}\sin f|_{\varrho=0}^{\varrho=\infty}. \tag{68}$$

The vorticity fluxes and the corresponding supercurrents are shown in Figs. 6 and 7. This shows that the helical vortex is made of two quantized vorticity fluxes, the $\Phi_{\hat{z}}$ flux centered at the core and the $\Phi_{\hat{\varphi}}$ flux surrounding it [18]. This is almost identical to what we have in Skyrme theory. Indeed the remarkable similarity between Figs. 6 and 7 and Figs. 4 and 5 in the Skyrme theory is unmistakable. This confirms that the

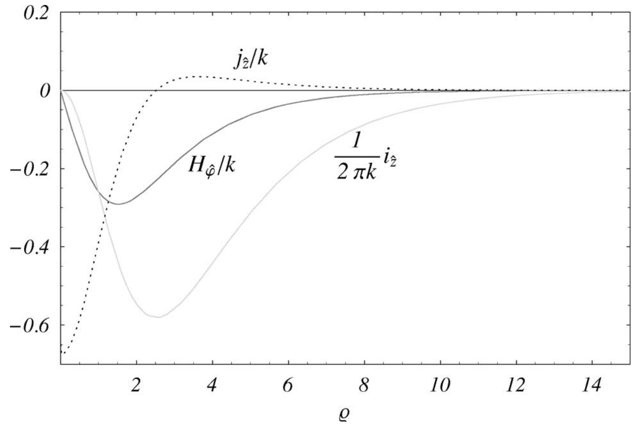

FIG. 7. The supercurrent $i_{\hat{z}}$ and corresponding magnetic field $H_{\hat{\varphi}}$ flowing through the disk of radius ϱ of the helical vortex in the Gross-Pitaevskii theory of two-component BEC. Here $m=n=1$, $k=0.25/\kappa$, $\delta\mu^2=0$, and ϱ is in units of κ. The current density $j_{\hat{z}}$ is represented by the dotted line.

helical vortex of two-component BEC is nothing but two quantized vorticity fluxes linked together. We emphasize that this interpretation holds even when the $\delta\mu^2$ is not zero.

The quantization of the vorticity (64) and (65) is due to the non-Abelian topology of the theory. To see this notice that the vorticity (63) is completely fixed by the nonlinear σ field $\hat{n}$ defined by ζ. Moreover, for the straight vortex $\hat{n}$ naturally defines a mapping $\pi_2(S^2)$ from the compactified two-dimensional space S^2 to the target space S^2. This means that the vortex in two-component BEC has exactly the same topological origin as the baby skyrmion in Skyrme theory. The only difference is that the topological quantum number here can also be expressed by the doublet ζ

$$Q_v = -\frac{i}{4\pi}\int \varepsilon_{ij}\partial_i\zeta^\dagger\partial_j\zeta\, d^2x = n. \tag{69}$$

Exactly the same topology assures the quantization of the twisted vorticity flux [18]. This clarifies the topological origin of the non-Abelian vortices of the Gross-Pitaevskii theory in two-component BEC.

The helical vortex will become unstable unless the periodicity condition is enforced by hand. But just as in the Skyrme theory we can make it a stable knot by making it a twisted vortex ring smoothly connecting two periodic ends. In this twisted vortex ring the periodicity condition of the helical vortex is automatically guaranteed, and the vortex ring becomes a stable knot. In this knot the n flux $\Phi_{\hat{z}}$ winds around m flux $\Phi_{\hat{\varphi}}$ of the helical vortex. Moreover the ansatz (57) tells us that $\Phi_{\hat{z}}$ is made of mainly the first component while $\Phi_{\hat{\varphi}}$ is made of mainly the second component of two-component BEC. So physically the knot can be viewed as two vorticity fluxes linked together, the one made of the first component and the other made of the second component which surrounds it.

As importantly the very twist which causes the instability of the helical vortex now ensures the stability of the knot. This is so because dynamically the momentum mk along the z axis created by the twist now generates a net angular momentum which provides the centrifugal repulsive force around the z axis preventing the knot to collapse.

Furthermore, this dynamical stability of the knot is now backed up by the topological stability. Again this is because the nonlinear σ field $\hat{n}$, after forming a knot, defines a mapping $\pi_3(S^2)$ from the compactified space S^3 to the target space S^2. So the knot acquires a nontrivial topology $\pi_3(S^2)$ whose quantum number is given by the Chern-Simon index of the velocity potential,

$$\begin{aligned} Q &= -\frac{1}{4\pi^2}\int \varepsilon_{ijk}\zeta^\dagger\partial_i\zeta(\partial_j\zeta^\dagger\partial_k\zeta)d^3x \\ &= \frac{1}{16\pi^2}\int \varepsilon_{ijk}V_i\bar{H}_{jk}d^3x = mn. \end{aligned} \tag{70}$$

This is precisely the linking number of two vorticity fluxes, which is formally identical to the knot quantum number of the Skyrme theory [6,7,18]. This assures the topological stability of the knot, because two fluxes linked together cannot be disconnected by any smooth deformation of the field configuration.

Similar knots in the Gross-Pitaevskii theory of two-component BEC have been discussed in the literature [14,15]. Our analysis here tells us that the knot in the Gross-Pitaevskii theory is a topological knot which can be viewed as a twisted vorticity flux ring linked together.

As we have argued our knot should be stable, dynamically as well as topologically. On the other hand, the familiar scaling argument indicates that the knot in the Gross-Pitaevskii theory of two-component BEC must be unstable. This has created a confusion on the stability of the knot in the literature [14,17]. To clarify the confusion it is important to realize that the scaling argument breaks down when the system is constrained. In our case the Hamiltonian is constrained by the particle number conservation which allows us to circumvent the no-go theorem and have a stable knot [17,18].

VI. GAUGE THEORY OF TWO-COMPONENT BEC

The above analysis tells that the non-Abelian vortex of the two-component Gross-Pitaevskii theory is nothing but a vorticity flux. And creating the vorticity costs energy. This implies that the Hamiltonian of two-component BEC must contain the contribution of the vorticity. This questions the wisdom of the Gross-Pitaevskii theory, because the Hamiltonian (36) has no such interaction. To make up this shortcoming a gauge theory of two-component BEC which can naturally accommodate the vorticity interaction has been proposed recently [5]. In this section we discuss the gauge theory of two-component BEC in detail.

Let us consider the following Lagrangian of U(1) gauge theory of two-component BEC [5]

$$\begin{aligned} \mathcal{L} = {} & i\frac{\hbar}{2}[\phi^\dagger(\tilde{D}_t\phi) - (\tilde{D}_t\phi)^\dagger\phi] - \frac{\hbar^2}{2M}|\tilde{D}_i\phi|^2 - \frac{\lambda}{2}\left(\phi^\dagger\phi - \frac{\mu}{\lambda}\right)^2 \\ & - \delta\mu\phi_2^\dagger\phi_2 - \frac{1}{4}\tilde{H}_{\mu\nu}^2, \end{aligned} \tag{71}$$

where $\tilde{D}_\mu=\partial_\mu+ig\tilde{C}_\mu$, and $\tilde{H}_{\mu\nu}$ is the field strength of the potential $\tilde{C}_\mu$. Two remarks are in order here. First, from now on we will assume

$$\delta\mu = 0, \tag{72}$$

since the symmetry breaking interaction can always be treated as a perturbation. With this the theory acquires a global U(2) symmetry as well as a local U(1) symmetry. Secondly, since we are primarily interested in the self-interacting (neutral) condensate, we treat the potential $\tilde{C}_\mu$ as a composite field of the condensate and identify $\tilde{C}_\mu$ with the velocity potential V_μ of the doublet ζ [5],

$$\tilde{C}_\mu = -\frac{i}{g}\zeta^\dagger \partial_\mu \zeta = \frac{1}{g}V_\mu. \tag{73}$$

With this the last term in the Lagrangian now represents the vorticity (63) of the velocity potential that we discussed before

$$\tilde{H}_{\mu\nu} = -\frac{i}{g}(\partial_\mu \zeta^\dagger \partial_\nu \zeta - \partial_\nu \zeta^\dagger \partial_\mu \zeta) = \frac{1}{g}\bar{H}_{\mu\nu}. \tag{74}$$

This shows that the gauge theory of two-component BEC naturally accommodates the vorticity interaction, and the coupling constant g here describes the coupling strength of the vorticity interaction [5]. This vorticity interaction distinguishes the gauge theory from the Gross-Pitaevskii theory.

At this point one might still wonder why one needs the vorticity in the Lagrangian (71), because in ordinary (one-component) BEC one has no such interaction. The reason is that in ordinary BEC the vorticity is identically zero, because there the velocity is given by the gradient of the phase of the complex condensate. Only a non-Abelian (multicomponent) BEC can have a nonvanishing vorticity. More importantly, it costs energy to create the vorticity in non-Abelian superfluids [19]. So it is natural that the two-component BEC (which is very similar to non-Abelian superfluids) has the vorticity interaction. Furthermore, here we can easily control the strength of the vorticity interaction with the coupling constant g. Indeed, if necessary, we could even turn off the vorticity interaction by putting $g=\infty$. This justifies the presence of the vorticity interaction in the Hamiltonian.

Another important difference between this theory and the Gross-Pitaevskii theory is the U(1) gauge symmetry. Clearly the Lagrangian (71) retains the U(1) gauge invariance, in spite of the fact that the gauge field is replaced by the velocity field (73). This has a deep impact. To see this notice that from the Lagrangian we have the following Hamiltonian in the static limit (again normalizing ϕ to $\sqrt{2M/\hbar}\phi$):

$$\mathcal{H} = \frac{1}{2}(\partial_i \rho)^2 + \frac{1}{2}\rho^2(|\partial_i \zeta|^2 - |\zeta^\dagger \partial_i \zeta|^2) + \frac{\lambda}{2}(\rho^2 - \rho_0^2)^2 - \frac{1}{4g^2}(\partial_i \zeta^\dagger \partial_j \zeta - \partial_j \zeta^\dagger \partial_i \zeta)^2,$$

$$\rho_0^2 = \frac{2\mu}{\lambda}. \tag{75}$$

Minimizing the Hamiltonian we have the following equation of motion:

$$\partial^2 \rho - (|\partial_i \zeta|^2 - |\zeta^\dagger \partial_i \zeta|^2)\rho = \frac{\lambda}{2}(\rho^2 - \rho_0^2)\rho,$$

$$\left\{(\partial^2 - \zeta^\dagger \partial^2 \zeta) + 2\left(\frac{\partial_i \rho}{\rho} + \frac{1}{g^2\rho^2}\partial_j(\partial_i \zeta^\dagger \partial_j \zeta - \partial_j \zeta^\dagger \partial_i \zeta) - \zeta^\dagger \partial_i \zeta\right) \times (\partial_i - \zeta^\dagger \partial_i \zeta)\right\}\zeta = 0. \tag{76}$$

But factorizing ζ by the U(1) phase γ and CP^1 field ξ with

$$\zeta = \exp(i\gamma)\xi, \tag{77}$$

we have

$$\zeta^\dagger \vec{\sigma}\zeta = \xi^\dagger \vec{\sigma}\xi = \hat{n},$$

$$|\partial_\mu \zeta|^2 - |\zeta^\dagger \partial_\mu \zeta|^2 = |\partial_\mu \xi|^2 - |\xi^\dagger \partial_\mu \xi|^2 = \tfrac{1}{4}(\partial_\mu \hat{n})^2,$$

$$-i(\partial_\mu \zeta^\dagger \partial_\nu \zeta - \partial_\nu \zeta^\dagger \partial_\mu \zeta) = -i(\partial_\mu \xi^\dagger \partial_\nu \xi - \partial_\nu \xi^\dagger \partial_\mu \xi) = \tfrac{1}{2}\hat{n}\cdot(\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = g\tilde{H}_{\mu\nu}, \tag{78}$$

so that we can rewrite (76) in terms of ξ

$$\partial^2 \rho - (|\partial_i \xi|^2 - |\xi^\dagger \partial_i \xi|^2)\rho = \frac{\lambda}{2}(\rho^2 - \rho_0^2)\rho,$$

$$\left\{(\partial^2 - \xi^\dagger \partial^2 \xi) + 2\left(\frac{\partial_i \rho}{\rho} + \frac{1}{g^2\rho^2}\partial_j(\partial_i \xi^\dagger \partial_j \xi - \partial_j \xi^\dagger \partial_i \xi) - \xi^\dagger \partial_i \xi\right) \times (\partial_i - \xi^\dagger \partial_i \xi)\right\}\xi = 0. \tag{79}$$

Moreover we can express the Hamiltonian (75) completely in terms of the non linear σ field $\hat{n}$ (or equivalently the CP^1 field ξ) and ρ as

$$\mathcal{H} = \frac{1}{2}(\partial_i \rho)^2 + \frac{1}{8}\rho^2(\partial_i \hat{n})^2 + \frac{\lambda}{2}(\rho^2 - \rho_0^2)^2 + \frac{1}{16g^2}(\partial_i \hat{n} \times \partial_j \hat{n})^2 = \lambda\rho_0^4\left\{\frac{1}{2}(\hat{\partial}_i \hat{\rho})^2 + \frac{1}{8}\hat{\rho}^2(\hat{\partial}_i \hat{n})^2 + \frac{1}{2}(\hat{\rho}^2 - 1)^2 + \frac{\lambda}{16g^2}(\hat{\partial}_i \hat{n} \times \hat{\partial}_j \hat{n})^2\right. . \tag{80}$$

This is because of the U(1) gauge symmetry. The U(1) gauge invariance of the Lagrangian (71) absorbs the U(1) phase γ of ζ, so that the theory is completely described by ξ. In other words, the Abelian gauge invariance of effectively reduces the target space S^3 of ζ to the gauge orbit space $S^2=S^3/S^1$, which is identical to the target space of the CP^1 field ξ. And since mathematically ξ is equivalent to the nonlinear σ field $\hat{n}$, one can express (75) completely in terms of $\hat{n}$.

This tells us that Eq. (76) can also be expressed in terms of $\hat{n}$. Indeed with (40), (41), and (78) we can obtain the following equation from (76) [5]:

$$\partial^2\rho - \tfrac{1}{4}(\partial_i\hat{n})^2\rho = \frac{\lambda}{2}(\rho^2-\rho_0^2)\rho,$$

$$\hat{n}\times\partial^2\hat{n} + 2\frac{\partial_i\rho}{\rho}\hat{n}\times\partial_i\hat{n} + \frac{2}{g\rho^2}\partial_i\widetilde{H}_{ij}\partial_j\hat{n} = 0. \quad (81)$$

This, of course, is the equation of motion that one obtains by minimizing the Hamiltonian (80). So we have two expressions, (76) and (81), which describe the equation of gauge theory of two-component BEC.

The above analysis clearly shows that our theory of two-component BEC is closely related to the Skyrme theory. In fact, in the vacuum

$$\rho^2 = \rho_0^2, \quad \frac{1}{g^2\rho_0^2} = \frac{\alpha}{\mu^2}, \quad (82)$$

Eqs. (11) and (81) become identical. Furthermore, this tells us that the equation (49) of the Gross-Pitaevskii theory is very similar to the above equation of the gauge theory of two-component BEC. Indeed, when $\delta\mu^2=0$, (49) and (81) become almost identical. This tells us that, in spite of different dynamics, the two theories are very similar to each other.

VII. TOPOLOGICAL OBJECTS IN GAUGE THEORY OF TWO-COMPONENT BEC

Now we show that, just like the Skyrme theory, the theory admits monopole, vortex, and knot. We start from the monopole. Let

$$\phi = \frac{1}{\sqrt{2}}\rho\xi \quad (\gamma=0),$$

$$\rho=\rho(r), \quad \xi = \begin{pmatrix} \cos\dfrac{\theta}{2}\exp(-i\varphi) \\ \sin\dfrac{\theta}{2} \end{pmatrix}, \quad (83)$$

and find

$$\hat{n} = \xi^\dagger\vec{\sigma}\xi = \hat{r}, \quad (84)$$

where (r,θ,φ) are the spherical coordinates. With this the second equation of (79) is automatically satisfied, and the first equation is reduced to

$$\ddot{\rho} + \frac{2}{r}\dot{\rho} - \frac{1}{2r^2}\rho = \frac{\lambda}{2}(\rho^2-\rho_0^2)\rho. \quad (85)$$

So with the boundary condition

$$\rho(0)=0, \quad \rho(\infty)=\rho_0, \quad (86)$$

we have a spherically symmetric solution shown in Fig. 8. Obviously this is a Wu-Yang type vorticity monopole dressed by the scalar field ρ [6,10]. In spite of the dressing, however, it has an infinite energy due to the singularity at the origin.

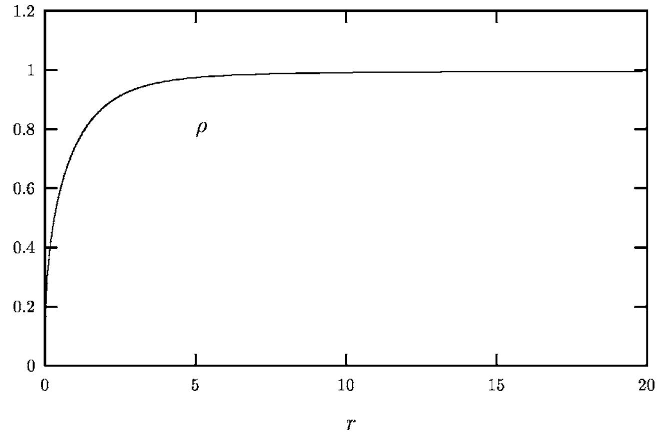

FIG. 8. The monopole solution in the gauge theory of two-component BEC. Here we have put $\lambda=1$ and r is in the unit of $1/\rho_0$.

Next we construct the vortex solutions. To do this we choose the ansatz in the cylindrical coordinates

$$\rho=\rho(\varrho), \quad \xi = \begin{pmatrix} \cos\dfrac{f(\varrho)}{2}\exp(-in\varphi) \\ \sin\dfrac{f(\varrho)}{2}\exp(imkz) \end{pmatrix}, \quad (87)$$

from which we have

$$\hat{n} = \begin{pmatrix} \sin f\cos(n\varphi+mkz) \\ \sin f\sin(n\varphi+mkz) \\ \cos f \end{pmatrix},$$

$$\widetilde{C}_\mu = -\frac{n}{2g}(\cos f+1)\partial_\mu\varphi - \frac{mk}{2g}(\cos f-1)\partial_\mu z. \quad (88)$$

With this Eq. (79) is reduced to

$$\ddot{\rho} + \frac{1}{\varrho}\dot{\rho} - \tfrac{1}{4}\left[\dot{f}^2 + \left(m^2k^2+\frac{n^2}{\varrho^2}\right)\sin^2 f\right]\rho = \frac{\lambda}{2}(\rho^2-\rho_0^2)\rho,$$

$$\left[1+\left(\frac{n^2}{\varrho^2}+m^2k^2\right)\frac{\sin^2 f}{g^2\rho^2}\right]\ddot{f} + \left[\frac{1}{\varrho}+2\frac{\dot{\rho}}{\rho}+\left(\frac{n^2}{\varrho^2}+m^2k^2\right)\times\frac{\sin f\cos f}{g^2\rho^2}\dot{f} - \frac{1}{\varrho}\left(\frac{n^2}{\varrho^2}-m^2k^2\right)\frac{\sin^2 f}{g^2\rho^2}\right]\dot{f} - \left(\frac{n^2}{\varrho^2}+m^2k^2\right)\sin f\cos f = 0. \quad (89)$$

Notice that the first equation is similar to what we have in the Gross-Pitaevskii theory, but the second one is remarkably similar to the helical vortex equation in the Skyrme theory. Now with the boundary condition

$$\dot{\rho}(0)=0, \quad \rho(\infty)=\rho_0,$$

$$f(0)=\pi, \quad f(\infty)=0, \quad (90)$$

we obtain the non-Abelian vortex solution shown in Fig. 9.

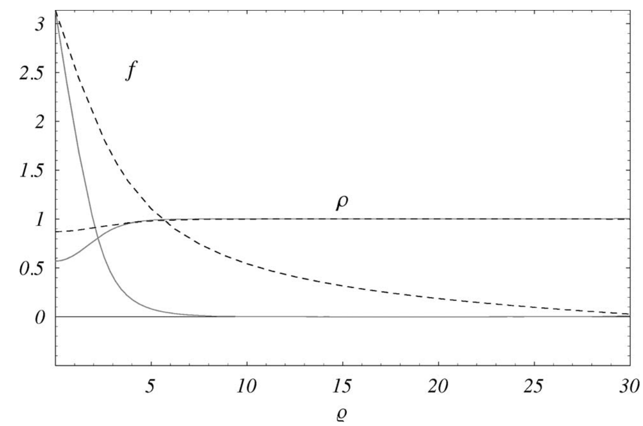

FIG. 9. The non-Abelian vortex (dashed line) with $m=0$, $n=1$ and the helical vortex (solid line) with $m=n=1$ in the gauge theory of two-component BEC. Here we have put $\lambda/g^2=1$, $k=0.64/\kappa$, and ϱ is in the unit of κ.

The solution is similar to the one we have in the Gross-Pitaevskii theory. First, when $m=0$, the solution describes the straight non-Abelian vortex. But when m is not zero, it describes a helical vortex which is periodic in z coordinates [5]. In this case, the vortex has a nonvanishing velocity current (not only around the vortex but also along the z axis). Secondly, the doublet ξ starts from the second component at the core, but the first component takes over completely at the infinity. This is due to the boundary condition $f(0)=\pi$ and $f(\infty)=0$, which assures us that our solution describes a genuine non-Abelian vortex. This confirms that the vortex solution is almost identical to what we have in the Gross-Pitaevskii theory shown in Fig. 5. All the qualitative features are exactly the same. This implies that physically the gauge theory of two-component BEC is very similar to the Gross-Pitaevskii theory, in spite of the obvious dynamical differences.

Clearly our vortex has the same topological origin as the vortex in Gross-Pitaevskii theory. This tells us that, just as in the Gross-Pitaevskii theory, the non-Abelian helical vortex here is nothing but the twisted vorticity flux of the CP^1 field ξ confined along the z axis by the velocity current, whose flux is quantized due to the topological reason. The only difference here is the profile of the vorticity, which is slightly different from that of the Gross-Pitaevskii theory. Indeed the solution has the following vorticity:

$$\tilde{H}_{\hat{z}} = \frac{1}{\varrho}\tilde{H}_{\varrho\varphi} = \frac{n}{2g\varrho}\dot{f}\sin f,$$

$$\tilde{H}_{\hat{\varphi}} = -\tilde{H}_{\varrho z} = -\frac{mk}{2g}\dot{f}\sin f, \tag{91}$$

which gives two quantized vorticity fluxes, a flux along the z axis

$$\phi_{\hat{z}} = \int \tilde{H}_{\varrho\varphi} d\varrho\, d\varphi = -\frac{2\pi i}{g}\int (\partial_\varrho \xi^\dagger \partial_\varphi \xi - \partial_\varphi \xi^\dagger \partial_\varrho \xi) d\varrho = -\frac{2\pi n}{g}, \tag{92}$$

and a flux around the z axis (in a one period section from 0 to $2\pi/k$ in z coordinates)

$$\phi_{\hat{\varphi}} = -\int \tilde{H}_{\varrho z} d\varrho\, dz = \frac{2\pi i}{g}\int (\partial_\varrho \xi^\dagger \partial_z \xi - \partial_z \xi^\dagger \partial_\varrho \xi)\frac{d\varrho}{k} = \frac{2\pi m}{g}. \tag{93}$$

This tells us that the vorticity fluxes are quantized in the unit of $2\pi/g$.

Just like the Gross-Pitaevskii theory the theory has a built-in Meissner effect which confines the vorticity flux. The current which confines the flux is given by

$$\tilde{j}_\mu = \partial_\nu \tilde{H}_{\mu\nu} = -\frac{\sin f}{2g}\left[n\left(\ddot{f} + \frac{\cos f}{\sin f}\dot{f}^2 - \frac{1}{\varrho}\dot{f}\right)\partial_\mu \varphi + mk\left(\ddot{f} + \frac{\cos f}{\sin f}\dot{f}^2 + \frac{1}{\varrho}\dot{f}\right)\partial_\mu z\right],$$

$$\partial_\mu \tilde{j}_\mu = 0. \tag{94}$$

This produces the supercurrents $i_{\hat{\varphi}}$ (per a one period section in z coordinates from $z=0$ to $z=2\pi/k$) around the z axis

$$i_{\hat{\varphi}} = -\frac{\pi n}{gk}\; \frac{\sin f}{\varrho}\dot{f}\Big|_{\varrho=0}^{\varrho=\infty}, \tag{95}$$

and $i_{\hat{z}}$ along the z axis

$$i_{\hat{z}} = -\pi\;\frac{mk}{g}\varrho \dot{f}\sin f\Big|_{\varrho=0}^{\varrho=\infty}. \tag{96}$$

The helical vorticity fields and supercurrents are shown in Figs. 10 and 11. The remarkable similarity between these and those in Skyrme theory (Figs. 2 and 3) and Gross-Pitaevskii theory (Figs. 6 and 7) is unmistakable.

With the ansatz (88) the energy (per a one periodic section) of the helical vortex is given by

$$\begin{aligned} E &= \frac{4\pi^2}{k}\int_0^\infty \left\{ \frac{1}{2}\dot{\rho}^2 + \frac{1}{8}\rho^2\left(\left[1 + \frac{1}{g^2\rho^2}\left(\frac{n^2}{\varrho^2} + m^2k^2\right)\sin^2 f\right]\dot{f}^2 + \left(\frac{n^2}{\varrho^2} + m^2k^2\right)\sin^2 f\right) + \frac{\lambda}{8}(\rho^2 - \rho_0^2)^2 \right\}\varrho\, d\varrho \\ &= 4\pi^2\frac{\rho_0^2}{k}\int_0^\infty \left\{ \frac{1}{2}\left(\frac{d\hat{\rho}}{dx}\right)^2 + \frac{1}{8}\hat{\rho}^2\left(\left[1 + \frac{\lambda}{g^2\hat{\rho}^2} \times\left(\frac{n^2}{x^2} + m^2\kappa^2k^2\right)\sin^2 f\right]\left(\frac{df}{dx}\right)^2 + \left(\frac{n^2}{x^2} + m^2\kappa^2k^2\right)\sin^2 f\right] + \frac{1}{8}(\hat{\rho}^2 - 1)^2 \right\} x\, dx . \end{aligned} \tag{97}$$

One could calculate the energy of the helical vortex numerically. With $m=n=1$ and $k=0.64\kappa$ we find that the energy in

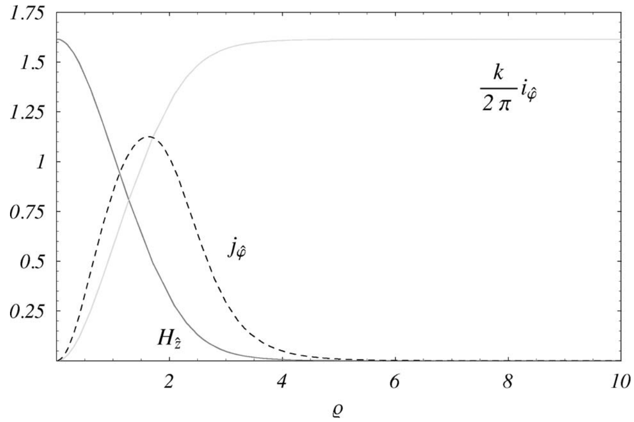

FIG. 10. The supercurrent $i_{\hat{\varphi}}$ (in a one period section in z coordinates) and corresponding magnetic field $H_{\hat{z}}$ circulating around the cylinder of radius ϱ of the helical vortex in the gauge theory of two-component BEC. Here $m=n=1$, $\lambda/g^2=1$, $k=0.64/\kappa$, and ϱ is in the unit of κ. The current density $j_{\hat{\varphi}}$ is represented by the dotted line.

one period section of the helical vortex in ^{87}Rb is given by

$$E \simeq 51\frac{\rho_0}{\sqrt{\lambda}} \simeq 4.5 \times 10^{-10}\ \text{eV} \simeq 0.7\ \text{MHz}, \tag{98}$$

which will have an important meaning later.

VIII. VORTICITY KNOT IN TWO-COMPONENT BEC

The existence of the helical vortex predicts the existence of a topological knot in the gauge theory of two-component BEC, for exactly the same reason that the helical vortices in the Skyrme theory and the Gross-Pitaevskii theory assure the existence of knots in these theories. To demonstrate the existence of knots in the gauge theory of two-component BEC we introduce the toroidal coordinates (η,γ,φ) defined by

$$x = \frac{a}{D}\sinh\,\eta\cos\,\varphi, \quad y = \frac{a}{D}\sinh\,\eta\sin\,\varphi,$$

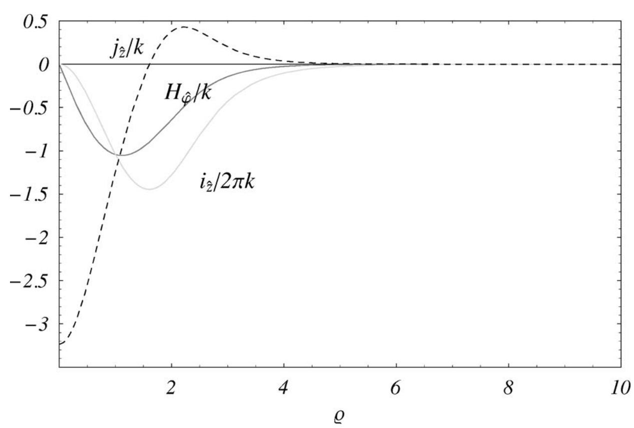

FIG. 11. The supercurrent $i_{\hat{z}}$ and corresponding magnetic field $H_{\hat{\varphi}}$ flowing through the disk of radius ϱ of the helical vortex in the gauge theory of two-component BEC. Here $m=n=1$, $\lambda/g^2=1$, $k=0.64/\kappa$, and ϱ is in the unit of κ. The current density $j_{\hat{z}}$ is represented by the dotted line.

$$z = \frac{a}{D}\sin\,\gamma,$$

$$D = \cosh\,\eta - \cos\,\gamma,$$

$$ds^2 = \frac{a^2}{D^2}(d\eta^2 + d\gamma^2 + \sinh^2\,\eta\, d\varphi^2),$$

$$d^3x = \frac{a^3}{D^3}\sinh\,\eta\, d\eta\, d\gamma\, d\varphi, \tag{99}$$

where a is the radius of the knot defined by $\eta=\infty$. Notice that in toroidal coordinates, $\eta=\gamma=0$ represents spatial infinity of R^3, and $\eta=\infty$ describes the torus center.

Now we choose the following ansatz:

$$\phi = \frac{\rho(\eta,\gamma)}{\sqrt{2}}\begin{pmatrix} \cos\frac{f(\eta,\gamma)}{2}\exp[-in\omega(\eta,\gamma)] \\ \sin\frac{f(\eta,\gamma)}{2}\exp(im\varphi) \end{pmatrix}. \tag{100}$$

With this we have the velocity potential

$$\tilde{C}_\mu = -\frac{m}{2g}(\cos f - 1)\partial_\mu\varphi - \frac{n}{2g}(\cos f + 1)\partial_\mu\omega, \tag{101}$$

which generates the vorticity

$$\tilde{H}_{\mu\nu} = \partial_\mu\tilde{C}_\nu - \partial_\nu\tilde{C}_\mu,$$

$$\tilde{H}_{\eta\gamma} = \frac{n}{2g}K\sin f, \quad \tilde{H}_{\gamma\varphi} = \frac{m}{2g}\sin f\,\partial_\gamma f,$$

$$\tilde{H}_{\varphi\eta} = -\frac{m}{2g}\sin f\,\partial_\eta f, \tag{102}$$

where

$$K = \partial_\eta f\partial_\gamma\omega - \partial_\gamma f\partial_\eta\omega. \tag{103}$$

Notice that, in the orthonormal frame $(\hat{\eta},\hat{\gamma},\hat{\varphi})$, we have

$$\tilde{C}_{\hat{\eta}} = -\frac{nD}{2ga}(\cos f + 1)\partial_\eta\omega,$$

$$\tilde{C}_{\hat{\gamma}} = -\frac{nD}{2ga}(\cos f + 1)\partial_\gamma\omega,$$

$$\tilde{C}_{\hat{\varphi}} = -\frac{mD}{2ga\sinh\,\eta}(\cos f - 1), \tag{104}$$

and

$$\tilde{H}_{\hat{\eta}\hat{\gamma}} = \frac{nD^2}{2ga^2}K\sin f,$$

$$\tilde{H}_{\hat{\gamma}\hat{\varphi}} = \frac{mD^2}{2ga^2\sinh\,\eta}\sin f\partial_\gamma f,$$

$$\tilde{H}_{\hat{\varphi}\hat{\eta}} = -\frac{mD^2}{2ga^2\sinh\eta}\sin f\partial_\eta f. \tag{105}$$

From the ansatz (100) we have the following equations of motion:

$$\left[\partial_\eta^2+\partial_\gamma^2+\left(\frac{\cosh\eta}{\sinh\eta}-\frac{\sinh\eta}{D}\right)\partial_\eta-\frac{\sin\gamma}{D}\partial_\gamma\right]\rho-\frac{1}{4}\left[(\partial_\eta f)^2+(\partial_\gamma f)^2+\left(n^2[(\partial_\eta\omega)^2+(\partial_\gamma\omega)^2]+\frac{m^2}{\sinh^2\eta}\right)\sin^2 f\right]\rho=\frac{\lambda a^2}{2D^2}(\rho^2-\rho_0^2)\rho,$$

$$\left[\partial_\eta^2+\partial_\gamma^2+\left(\frac{\cosh\eta}{\sinh\eta}-\frac{\sinh\eta}{D}\right)\partial_\eta-\frac{\sin\gamma}{D}\partial_\gamma\right]f-\left(n^2[(\partial_\eta\omega)^2+(\partial_\gamma\omega)^2]+\frac{m^2}{\sinh^2\eta}\right)\sin f\cos f+\frac{2}{\rho}(\partial_\eta\rho\partial_\eta f+\partial_\gamma\rho\partial_\gamma f)=-\frac{1}{g^2\rho^2}\frac{D^2}{a^2}(A\cos f+B\sin f)\sin f,$$

$$\left[\partial_\eta^2+\partial_\gamma^2+\left(\frac{\cosh\eta}{\sinh\eta}-\frac{\sinh\eta}{D}\right)\partial_\eta-\frac{\sin\gamma}{D}\partial_\gamma\right]\omega+2(\partial_\eta f\partial_\eta\omega+\partial_\gamma f\partial_\gamma\omega)\frac{\cos f}{\sin f}+\frac{2}{\rho}(\partial_\eta\rho\partial_\eta\omega+\partial_\gamma\rho\partial_\gamma\omega)=\frac{1}{g^2\rho^2}\frac{D^2}{a^2}C, \tag{106}$$

where

$$A=n^2K^2+\frac{m^2}{\sinh^2\eta}[(\partial_\eta f)^2+(\partial_\gamma f)^2],$$

$$B=n^2\partial_\eta K\partial_\gamma\omega-n^2\partial_\gamma K\partial_\eta\omega+n^2K\left[\left(\frac{\cosh\eta}{\sinh\eta}+\frac{\sinh\eta}{D}\right)\partial_\gamma\omega-\frac{\sin\gamma}{D}\partial_\eta\omega\right]+\frac{m^2}{\sinh^2\eta}\left[\partial_\eta^2+\partial_\gamma^2-\left(\frac{\cosh\eta}{\sinh\eta}-\frac{\sinh\eta}{D}\right)\partial_\eta+\frac{\sin\gamma}{D}\partial_\gamma\right]f,$$

$$C=\partial_\eta K\partial_\gamma f-\partial_\eta f\partial_\gamma K+K\left[\left(\frac{\cosh\eta}{\sinh\eta}+\frac{\sinh\eta}{D}\right)\partial_\gamma-\frac{\sin\gamma}{D}\partial_\eta\right]f.$$

Since $\eta=\gamma=0$ represents spatial infinity of R^3 and $\eta=\infty$ describes the torus center, we can impose the following boundary condition:

$$\rho(0,0)=\rho_0,\quad \dot\rho(\infty,\gamma)=0,$$
$$f(0,\gamma)=0,\quad f(\infty,\gamma)=\pi,$$
$$\omega(\eta,0)=0,\quad \omega(\eta,2\pi)=2\pi, \tag{107}$$

to obtain the desired knot. From the ansatz (100) and the boundary condition (107) we can calculate the knot quantum number

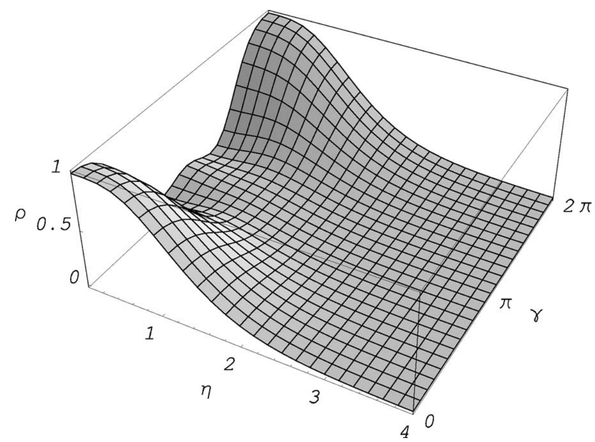

FIG. 12. The ρ profile of the BEC knot with $m=n=1$. Here we have put $\lambda/g^2=1$, and the scale of the radius a is κ.

$$Q=\frac{mn}{8\pi^2}\int K\sin f\,d\eta\,d\gamma\,d\varphi=\frac{mn}{4\pi}\int\sin f\,df\,d\omega=mn, \tag{108}$$

where the last equality comes from the boundary condition. This tells us that our ansatz describes the correct knot topology.

Of course, an exact solution of (106) with the boundary conditions (107) is extremely difficult to find [7,15]. But here we can obtain the knot profile of ρ, f, and ω which minimizes the energy numerically. We find that, for $m=n=1$, the radius of knots which minimizes the energy is given by

$$a\simeq 1.6\kappa. \tag{109}$$

From this we obtain the following solution of the lightest axially symmetric knot in the gauge theory of two-component BEC (with $m=n=1$) shown in Figs. 12–14. With this we obtain a three-dimensional energy profile of the lightest knot shown in Fig. 15.

We can calculate the vorticity flux of the knot. Since the flux is helical, we have two fluxes, the flux $\Phi_{\hat\gamma}$ passing

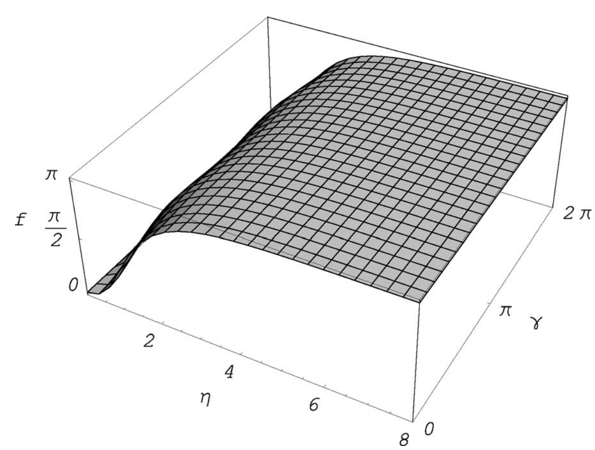

FIG. 13. The f profile of the BEC knot with $m=n=1$. Here we have put $\lambda/g^2=1$.

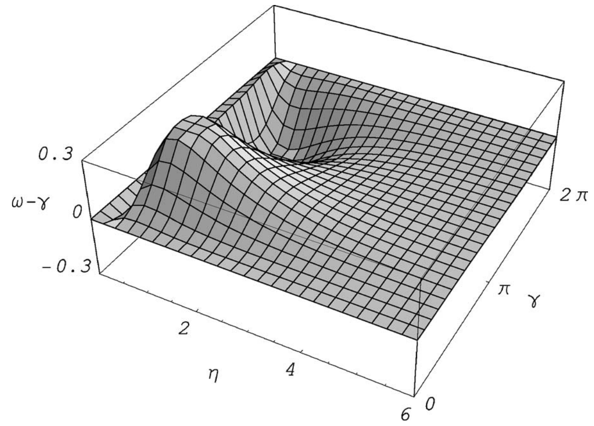

FIG. 14. The ω profile of the BEC knot with $m=n=1$. Notice that here we have plotted $\omega-\gamma$. Here again we have put $\lambda/g^2=1$.

through the knot disk of radius a in the xy plane and the flux $\Phi_{\hat{\varphi}}$ which surrounds it. From (105) we have

$$\Phi_{\hat{\gamma}}=\int_{\gamma=\pi}\tilde{H}_{\hat{\gamma}}\frac{a^2\sinh\eta}{D^2}d\eta\,d\varphi$$

$$=-\frac{m}{2g}\int_{\gamma=\pi}\sin f\partial_\eta f\,d\eta\,d\varphi=-\frac{2\pi m}{g}, \qquad (110)$$

and

$$\Phi_{\hat{\varphi}}=\int\tilde{H}_{\hat{\varphi}}\frac{a^2}{D^2}d\eta\,d\gamma,$$

$$=\frac{n}{2g}\int K\sin f\,d\eta\,d\gamma=\frac{2\pi n}{g}. \qquad (111)$$

This confirms that the flux is quantized in the unit of $2\pi/g$. As importantly this tells that the two fluxes are linked, whose linking number is fixed by the knot quantum number.

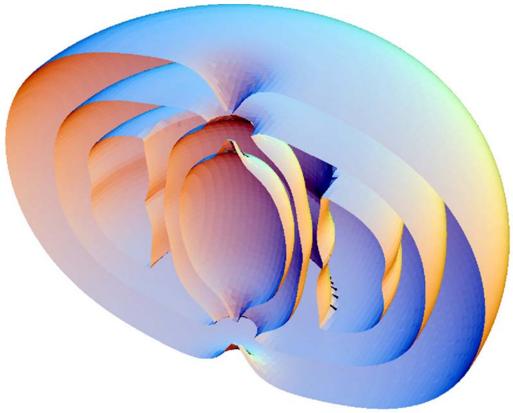

FIG. 15. (Color online). The energy profile of the BEC knot with $m=n=1$ for the energetically stable vorticity knot. Here again we have put $\sqrt{\lambda}/g^2=1$.

Just as in Gross-Pitaevskii theory the vorticity flux here is generated by the helical vorticity current which is conserved

$$\tilde{j}_\mu=\frac{nD^2}{2ga^2}\sin f\left(\partial_\gamma+\frac{\sin\gamma}{D}\right)K\partial_\mu\eta-\frac{nD^2}{2ga^2}\sin f\times\left(\partial_\eta+\frac{\cosh\eta}{\sinh\eta}+\frac{\sinh\eta}{D}\right)K\partial_\mu\gamma-\frac{mD^2}{2ga^2}\times\left[\left(\partial_\eta-\frac{\cosh\eta}{\sinh\eta}+\frac{\sinh\eta}{D}\right)\times\sin f\partial_\eta f+\left(\partial_\gamma+\frac{\sin\gamma}{D}\right)\sin f\partial_\gamma f\right]\partial_\mu\varphi,$$

$$\partial_\mu\tilde{j}_\mu=0. \qquad (112)$$

Clearly this supercurrent generates a Meissner effect which confines the vorticity flux.

From (75) and (100) we have the following Hamiltonian for the knot:

$$\mathcal{H}=\frac{D^2}{2a^2}\left\{(\partial_\eta\rho)^2+(\partial_\gamma\rho)^2+\frac{\rho^2}{4}\left[(\partial_\eta f)^2+(\partial_\gamma f)^2+\left(n^2[(\partial_\eta\omega)^2+(\partial_\gamma\omega)^2]+\frac{m^2}{\sinh^2\eta}\right)\sin^2 f\right]\right\}+\frac{\lambda}{8}(\rho^2-\rho_0^2)^2+\frac{D^4}{8g^2a^4}A\sin^2 f. \qquad (113)$$

With this the energy of the knot is given by

$$E=\int\mathcal{H}\frac{a^3}{D^3}\sinh\eta\,d\eta\,d\gamma\,d\varphi$$

$$=\frac{\rho_0}{\sqrt{\lambda}}\int\hat{\mathcal{H}}\frac{a^3}{\kappa^3D^3}\sinh\eta\,d\eta\,d\gamma\,d\varphi, \qquad (114)$$

where

$$\hat{\mathcal{H}}=\frac{\kappa^2D^2}{2a^2}\left\{(\partial_\eta\hat{\rho})^2+(\partial_\gamma\hat{\rho})^2+\frac{\hat{\rho}^2}{4}\left[(\partial_\eta f)^2+(\partial_\gamma f)^2+\left(n^2[(\partial_\eta\omega)^2+(\partial_\gamma\omega)^2]+\frac{m^2}{\sinh^2\eta}\right)\sin^2 f\right]\right\}+\frac{1}{8}(\hat{\rho}^2-1)^2+\frac{\lambda}{8g^2}\frac{\kappa^4D^4}{a^4}A\sin^2 f. \qquad (115)$$

Minimizing the energy we reproduce the knot equation (106).

From this we can estimate the energy of the axially symmetric knots. For the lightest knot (with $m=n=1$) we find the following energy:

$$E\simeq 54\frac{\rho_0}{\sqrt{\lambda}}\simeq 4.8\times 10^{-10}\text{ eV}\simeq 0.75\text{ MHz}. \qquad (116)$$

One should compare this energy with the energy of the helical vortex (98). Notice that the lightest knot has the radius $r\simeq 1.6\kappa$. In our picture this knot can be constructed bending a helical vortex with $k\simeq 0.64/\kappa$. So we expect that the en-

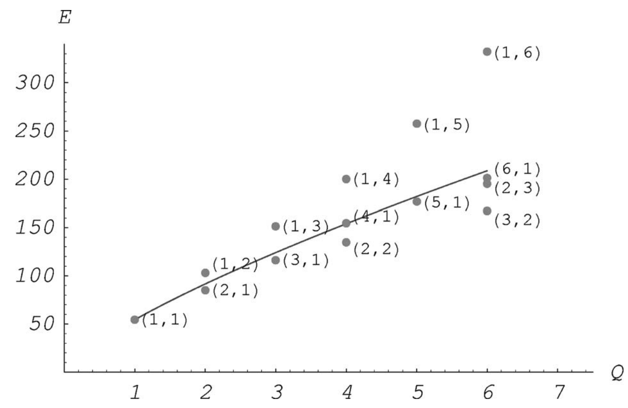

FIG. 16. The Q dependence of axially symmetric knots. The solid line corresponds to the function $E_0Q^{3/4}$ with $E_0=E(1,1)$, and the dots represent the energy $E(m,n)$ with the different $Q=mn$.

ergy of the lightest knot should be comparable to the energy of the helical vortex with $k\simeq 0.64/\kappa$. And we have already estimated the energy of the helical vortex with $k\simeq 0.64/\kappa$ in (98). The fact that two energies are of the same order assures us that the knot can indeed be viewed as a twisted vorticity flux ring.

As we have remarked the Q dependence of the energy of Faddeev-Niemi knot is proportional to $Q^{3/4}$ [12,13]. An interesting question is whether we can have a similar Q dependence of energy for the knots in BEC. With our ansatz we have estimated the energy of knots numerically for different m and n up to $Q=6$. The result is summarized in Fig. 16, which clearly tells us that the energy depends crucially on m and n. Our result suggests that, for the minimum energy knots, we have a similar (sublinear) Q dependence of energy for the knots in two-component BEC. It would be very interesting to establish such Q dependence of energy mathematically.

IX. DISCUSSION

In this paper we have discussed two competing theories of two-component BEC, the popular Gross-Pitaevskii theory, and the U(1) gauge theory which has the vorticity interaction. Although dynamically different two theories have remarkably similar topological objects, the helical vortex and the knots, which both have a nontrivial non-Abelian topology.

We have shown that the U(1)×U(1) symmetry of two-component BEC can be viewed as a broken U(2) symmetry. This allows us to interpret the vortex and knot in two-component BEC as non-Abelian topological objects. Furthermore, we have shown that these topological objects are the vorticity vortex and vorticity knot.

A major difference between the Gross-Pitaevskii theory and the gauge theory is the vorticity interaction. In spite of the fact that the vorticity plays an important role in two-component BEC, the Gross-Pitaevskii theory has no vorticity interaction. In comparison, the gauge theory of two-component BEC naturally accommodates the vorticity interaction in the Lagrangian. This makes the theory very similar to the Skyrme theory. More significantly, the explicit U(1) gauge symmetry makes it very similar to the theory of two-gap superconductors. The only difference is that the two-component BEC is a neutral system which is not charged, so that the gauge interaction has to be an induced interaction. On the other hand, the two-gap superconductor is made of charged condensates, so that it has a real (independent) electromagnetic interaction [20].

As importantly the gauge theory of two-component BEC, with the vorticity interaction, could play an important role in describing multicomponent superfluids [5,19]. In fact we believe that the theory could well describe both non-Abelian BEC and non-Abelian superfluids.

In this paper we have constructed a numerical solution of knots in the gauge theory of two-component BEC. Our result confirms that it can be identified as a vortex ring made of a helical vorticity vortex. Moreover our result tells that the knot can be viewed as two quantized vorticity fluxes linked together, whose linking number becomes the knot quantum number. This makes the knot very similar to the Faddeev-Niemi knot in Skyrme theory. We close with the following remarks:

(1) Recently a number of authors have also established the existence of knots identified as the "skyrmions" in the Gross-Pitaevskii theory of two-component BEC [14,15], which we believe is identical to our knot in the Gross-Pitaevskii theory. In this paper we have clarified the physical meaning of the knot. The knot in the Gross-Pitaevskii theory is also of topological origin. Moreover, it can be identified as a vorticity knot, a twisted vorticity flux ring, in spite of the fact that the Gross-Pitaevskii Lagrangian has neither the velocity $\tilde{C}_\mu$ nor the vorticity $\tilde{H}_{\mu\nu}$ which can be related to the knot.

(2) Our analysis tells us that at the center of the topological vortex and knot in two-component BEC lies the baby skyrmion and the Faddeev-Niemi knot. In fact they are the prototype of the non-Abelian topological objects that one can repeatedly encounter in physics [5,6,8,21]. This suggests that the Skyrme theory could also play an important role in condensed matter physics. Ever since Skyrme proposed his theory, the Skyrme theory has always been associated to nuclear and/or high energy physics. This has lead people to believe that the topological objects in Skyrme theory can only be realized at high energy, at the GeV scale. But our analysis opens up a new possibility for us to construct them in a completely different environment, at the eV scale, in two-component BEC [5,8]. This is really remarkable.

(3) From our analysis there should be no doubt that the non-Abelian vortices and knots must exist in two-component BEC. If so, the challenge now is to verify the existence of these topological objects experimentally. Constructing the knots might not be a simple task at the present moment. But the construction of the non-Abelian vortices could be rather straightforward, and might have already been done [4,22]. Identifying them, however, may be a tricky business because the two-component BEC can admit both the Abelian and non-Abelian vortices. To identify them, one has to keep the following in mind. First, the non-Abelian vortices must have a nontrivial profile of $f(\varrho)$. This is a crucial point which

distinguishes them from the Abelian vortices. Secondly, the energy of the non-Abelian vortices must be bigger than that of the Abelian counterparts, again because they have extra energy coming from the nontrivial profile of f. With this in mind, one should be able to construct the non-Abelian vortices in the new condensates without much difficulty. We strongly urge the experimentalists to meet the challenge.

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