Bound-state eigenenergy outside and inside the continuum for unstable multilevel systems

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The eigenvalue problem for the dressed bound state of unstable multilevel systems is examined both outside and inside the continuum, based on the N-level Friedrichs model, which describes the couplings between the discrete levels and the continuous spectrum. It is shown that a bound-state eigenenergy always exists below each of the discrete levels that lie outside the continuum. Furthermore, by strengthening the couplings gradually, the eigenenergy corresponding to each of the discrete levels inside the continuum finally emerges. On the other hand, the absence of the eigenenergy inside the continuum is proved in weak but finite coupling regimes, provided that each of the form factors that determine the transition between some definite level and the continuum does not vanish at that energy level. An application to the spontaneous emission process for the hydrogen atom interacting with the electromagnetic field is demonstrated.

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I. INTRODUCTION

The theoretical description of the unstable quantum system often refers to the system of a finite level coupled with the spectral continuum. In weak coupling regimes, the initial state localized at the finite level undergoes exponential decay [1]. However, by changing the couplings to stronger regimes, instead of the total decay a partial one can occur [2]. This means that the superposition between the states localized at the finite level and the continuum forms the dressed bound state, that is, a bound eigenstate extended over the total Hilbert space. The formation of the bound eigenstate is of great interest in the study of the various systems having to do with such matters as the photodetachment of electrons from negative ions [3,4] and the spontaneous emission of photons from atoms in photonic crystals [5–7]. It is then clarified that the energy of the bound eigenstate depends not only on the strength of the couplings but also on the relative location between the electron bound-energy and the detachment threshold [3,4], or between the energy of the atomic frequency and the continuum edge of the radiation frequency [6,7]. Further research has been directed to those eigenstates, aiming at the decoherence control [8,9].

In these analyses, however, single-level systems are treated often, while multilevel systems are examined less. In the latter, some peculiar time evolutions are theoretically observed; steplike decay [10], decaying oscillation [11], and various long-time nonexponential decays [12,13]. These peculiarities are never found in single-level approaches. Furthermore, to the author's knowledge, the possibility of a bound-state eigenenergy "inside" the continuum has not been studied except in a special multilevel case where all form factors are assumed to be identical [8,14].

In the present paper, we attempt to examine the eigenvalue problem for the dressed bound state in multilevel cases, based on the N-level Friedrichs model [15,16], allowing some class of form factors, including identical cases. We show that for the discrete energy levels lying outside the continuum, the bound-state eigenenergy always remains below each of them. Moreover, by increasing the couplings, the bound-state eigenenergy corresponding to each of the discrete levels inside the continuum can emerge out of that continuum. For the bound-state eigenenergy inside the continuum, we can only prove its absence in weak coupling cases under the condition that the form factors do not vanish at the energy of each level. This result is just an extension of the lemma already proved for a system with identical form factors [14]. An upper bound of the coupling constant for the case of no bound-state eigenenergy being inside the continuum is also obtained explicitly. We apply this result to the spontaneous emission process for the hydrogen atom under the four-level approximation.

In Sec. II, we introduce the N-level Friedrichs model and its eigenvalue problem. In Sec. III, we consider the eigenvalues outside the continuum, with resort to the perturbation theory about the eigenvalue of the Hermitian matrix. The discussion developed here helps us to undertake the problem for the inside case, which is argued in Sec. IV. Concluding remarks are given in Sec. V. We also present an Appendix where both the small and large energy behaviors of the energy shift are studied in detail.

II. THE N-LEVEL FRIEDRICHS MODEL AND THE EIGENVALUE PROBLEM

The N-level Friedrichs model describes the N-level system coupled with the continuum system. The total Hamiltonian H is defined by

$$H = H_0 + \lambda V, \tag{1}$$

where $\lambda \in \mathbb{R}$ is the coupling constant. We here define the free Hamiltonian H_0 as

$$H_0 = \sum_{n=1}^{N} \omega_n |n\rangle\langle n| + \int_{\Omega} \omega |\omega\rangle\langle\omega| \rho(\omega) d\omega, \tag{2}$$

where it was assumed that $\omega_1 \leqslant \omega_2 \leqslant \cdots \leqslant \omega_N$. $|n\rangle$ and $|\omega\rangle$ satisfy the orthonormality condition: $\langle n|n'\rangle = \delta_{nn'}$, $\langle\omega|\omega'\rangle = \delta(\omega-\omega')/\rho(\omega)$, and $\langle n|\omega\rangle = 0$, where $\delta_{nn'}$ is Kronecker's

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delta and $\delta(\omega-\omega')$ is Dirac's delta function. $\rho(\omega)$ is a nonnegative function interpreted as, e.g., an electromagnetic density of mode, and $\Omega=\{\omega|\rho(\omega)\neq 0\}$ is a specific region, like the energy band allowed by the electromagnetic mode. The interaction Hamiltonian V describing the couplings between $|n\rangle$ and $|\omega\rangle$ is

$$V=\sum_{n=1}^{N}\int_{\Omega}[v_n(\omega)|\omega\rangle\langle n|+v_n^*(\omega)|n\rangle\langle\omega|]\rho(\omega)d\omega, \quad (3)$$

where the asterisk denotes the complex conjugate and $v_n(\omega)$ is the form factor characterizing the transition between $|n\rangle$ and $|\omega\rangle$. We here assumed that $v_n\in L^2(0,\infty)$, i.e.,

$$\int_{\Omega}|v_n(\omega)|^2\rho(\omega)d\omega<\infty. \quad (4)$$

For clarity of discussion below, we assume that $\rho(\omega)=1$ for $\omega\geqslant 0$ and 0 otherwise, so that $\Omega=[0,\infty)$. Then, we merely write $\int_\Omega$ by $\int_0^\infty$, and the outside of the continuum means the half line $(-\infty,0)$. An extension of Ω to more general cases, such as gap structures, is not difficult, however, our facilitation could extract the essential of the matter.

Let us next set up the eigenvalue problem for this model. We suppose that the eigenstate corresponding to the eigenvalue E is of the form $|u_E\rangle=\sum_{n=1}^{N}c_n|n\rangle+\int_0^\infty f(\omega)|\omega\rangle d\omega$, and it is normalizable, i.e., [17]

$$\langle u_E|u_E\rangle=\sum_{n=1}^{N}|c_n|^2+\int_0^\infty|f(\omega)|^2d\omega<\infty. \quad (5)$$

Then, the eigenequation $H|u_E\rangle=E|u_E\rangle$ is equivalent to the following ones:

$$\omega_n c_n+\lambda\int_0^\infty v_n^*(\omega)f(\omega)d\omega=Ec_n, \quad \forall\ n=1,\ldots,N, \quad (6)$$

$$\omega f(\omega)+\lambda\sum_{n=1}^{N}c_n v_n(\omega)=Ef(\omega). \quad (7)$$

Equation (7) immediately implies

$$f(\omega)=-\lambda\frac{\sum_{n=1}^{N}c_n v_n(\omega)}{\omega-E}. \quad (8)$$

By setting this into Eq. (5), we have the normalization condition

$$\int_0^\infty|f(\omega)|^2d\omega=\lambda^2\int_0^\infty\frac{\left|\sum_{n=1}^{N}c_n v_n(\omega)\right|^2}{|\omega-E|^2}d\omega<\infty, \quad (9)$$

which is the essential of the localization of the dressed bound state.

III. BOUND-STATE EIGENENERGY OUTSIDE THE CONTINUUM

We first review the results on the negative-eigenvalue problem for $N=1$, the single-level case [18]. If $E<0$, the integral in Eq. (9) always converges under the condition (4). In fact,

$$|c_1|^2\int_0^\infty\frac{|v_1(\omega)|^2}{|\omega+|E||^2}d\omega\leqslant\frac{|c_1|^2}{|E|^2}\int_0^\infty|v_1(\omega)|^2d\omega<\infty. \quad (10)$$

Thus, the substitution of $f(\omega)$ into Eq. (6) is allowed. By introducing the function $\kappa(E)$ as

$$\kappa(E)=\omega_1-\lambda^2\int_0^\infty\frac{|v_1(\omega)|^2}{\omega-E}d\omega, \quad (11)$$

Eq. (6) reads [19]

$$\kappa(E)=E, \quad (12)$$

which is either an algebraic or transcendental equation of E, depending on $v_1(\omega)$. $\kappa(E)$ has two important properties as follows:

$$\kappa(E')\geqslant\kappa(E)\ \text{and}\ \kappa(E)\leqslant\omega_1, \quad (13)$$

for all E and E' satisfying $E'\leqslant E<0$. The former means that $\kappa(E)$ is monotone decreasing in E. Therefore, there is only one solution (negative eigenvalue) E of Eq. (12) if and only if

$$\lim_{E\uparrow 0}\kappa(E)=\omega_1-\lim_{E\uparrow 0}\lambda^2\int_0^\infty\frac{|v_1(\omega)|^2}{\omega-E}d\omega<0. \quad (14)$$

When $E>0$, E should be a zero of $v_1(\omega)$ so that Eq. (9) holds. This is discussed in detail in Sec. IV.

Let us now turn to the N-level case. Corresponding to Eq. (10), this time we have that

$$\int_0^\infty\frac{\left|\sum_{n=1}^{N}c_n v_n(\omega)\right|^2}{|\omega+|E||^2}d\omega\leqslant\frac{\sum_{n=1}^{N}\int_0^\infty|v_n(\omega)|^2d\omega}{|E|^2}<\infty, \quad (15)$$

and Eq. (9) is satisfied again, where we used that $\sum_{n=1}^{N}|c_n|^2\leqslant 1$. Substituting Eq. (8) into (6), one obtains

$$\sum_{n'=1}^{N}[\omega_n\delta_{nn'}-\lambda^2 S_{nn'}(E)]c_{n'}=Ec_n, \quad (16)$$

where

$$S_{nn'}(z)=\int_0^\infty\frac{v_n^*(\omega)v_{n'}(\omega)}{\omega-z}d\omega, \quad (17)$$

with $z\in\mathbb{C}\backslash[0,\infty)$. For convenience later, we introduce an $N\times N$ matrix $S(z)$ with the components $S_{nn'}(z)$. Note that $S(E)$ for $E<0$ turns out to be a Gram matrix [20], which is positive semidefinite. One obtains the following property of $S(E)$:

Lemma III.1. $S(E') \leqslant S(E)$ for $E' \leqslant E < 0$.

Proof: We have that

$$S_{nn'}(E) - S_{nn'}(E') = (E - E')T_{nn'}(E,E'), \tag{18}$$

for all E and E' satisfying $E' \leqslant E < 0$. We here introduce the matrix $T(E,E')$ whose components are

$$T_{nn'}(E,E') := \int_0^\infty \frac{v_n^*(\omega)v_{n'}(\omega)}{(\omega - E)(\omega - E')}d\omega. \tag{19}$$

Note that since $T(E,E')$ is a Gram matrix, it is positive semidefinite. Therefore the proof is completed. ∎

We also introduce the matrices K_0 and $K(E) = K_0 - \lambda^2 S(E)$ with components

$$K_{0nn'} := \omega_n \delta_{nn'}, \tag{20}$$

and

$$K_{nn'}(E) := \omega_n \delta_{nn'} - \lambda^2 S_{nn'}(E), \tag{21}$$

respectively. For any $E<0$, $K(E)$ becomes a Hermitian matrix, and thus there are N eigenvalues of $K(E)$. We denote them by $\{\kappa_n(E)\}_{n=1}^N$, where $\kappa_1(E) \leqslant \kappa_2(E) \leqslant \cdots \leqslant \kappa_N(E)$. The existence of a nontrivial solution $\{c_n\}$ of Eq. (16) is guaranteed if and only if there exists a negative E to satisfy

$$\kappa_n(E) = E, \tag{22}$$

for a certain integer n. As in the former part of Eq. (13), $\kappa_n(E)$ has the following property:

Lemma III.2. For any fixed n, $\kappa_n(E') \geqslant \kappa_n(E)$ for $E' \leqslant E < 0$.

Proof: We see from Eq. (18) that

$$K(E) - K(E') = -(E - E')\lambda^2 T(E,E') \leqslant 0, \tag{23}$$

for $E' \leqslant E < 0$. Then, by using the Theorem 4.3.1 in Ref. [20], the following inequality between the eigenvalues of $K(E)$, $K(E')$, and $T(E,E')$ holds [21]:

$$\kappa_n(E') - (E - E')\lambda^2 \tau_N(E,E') \leqslant \kappa_n(E) \leqslant \kappa_n(E') - (E - E')\lambda^2 \tau_1(E,E'), \tag{24}$$

where $\tau_n(E,E')$ denotes the nth eigenvalue of $T(E,E')$. Note that since $T(E,E') \geqslant 0$, all $\tau_n(E,E') \geqslant 0$. Then, $-(E-E')\tau_1(E,E') \leqslant 0$ for $E \geqslant E'$, and the inequality

$$\kappa_n(E) \leqslant \kappa_n(E'), \tag{25}$$

immediately follows from the last part of Eq. (24). ∎

We also have the statement below, which corresponds to the latter part of Eq. (13).

Lemma III.3. For any fixed n, $\kappa_n(E) \leqslant \omega_n$ for all $E<0$, and $\lim_{E\to-\infty} \kappa_n(E) = \omega_n$.

Proof: From Eq. (21) and Theorem 4.3.1 in Ref. [20] again, one obtains that

$$\omega_n - \lambda^2 \sigma_N(E) \leqslant \kappa_n(E) \leqslant \omega_n - \lambda^2 \sigma_1(E), \tag{26}$$

where $\sigma_n(E)$ denotes the nth eigenvalue of $S(E)$. If we recall the fact that $S(E) \geqslant 0$ implies $\sigma_n(E) \geqslant 0$ for every n, the above inequality reads

$$0 \leqslant \lambda^2 \sigma_1(E) \leqslant \omega_n - \kappa_n(E) \leqslant \lambda^2 \sigma_N(E). \tag{27}$$

Asymptotic behavior of the right-hand side of the above can be evaluated from Eq. (17) as

$$\sigma_N(E) \leqslant \mathrm{tr}[S(E)] \leqslant \frac{1}{|E|}\sum_{n=1}^N \int_0^\infty |v_n(\omega)|^2 d\omega \to 0, \tag{28}$$

as $E \to -\infty$, and thus the lemma is proved. ∎

Therefore, summarizing Lemmas III.2 and III.3, we obtain the following theorem.

Theorem III.4. If $\lim_{E\uparrow 0} \kappa_n(E) < 0$ up to $n=M$, then each of the $\kappa_n(E)$ for $n=1,\ldots,M$ intersects E only once, so that M negative eigenenergies of H exist. In particular, if H_0 has N_- negative eigenenergies, i.e., $\omega_n<0$ up to $n=N_-$, then N_- negative eigenenergies of H, denoted by E_n, exist and satisfy $E_n \leqslant \omega_n$.

We also see from Eq. (26) that

$$\kappa_n(E) \leqslant \omega_n - \lambda^2 \sigma_1(E). \tag{29}$$

This means that when $|\lambda|$ is large enough, every $\kappa_n(E)$, even originating from a positive ω_n, becomes negative, unless $\sigma_1(E)=0$, i.e., the $v_n(\omega)$'s are linearly dependent [20]. More precisely, the following statement holds:

Proposition III.5. Suppose that only N_{ind} form factors are linearly independent among them. Then, it follows that for any $E<0$

$$-\lambda^2 \sigma_{N+1-n}(E) + \omega_1 \leqslant \kappa_n(E) \leqslant -\lambda^2 \sigma_{N+1-n}(E) + \omega_N, \tag{30}$$

and $\sigma_{N+1-n}(E) \neq 0$ for $n=1,\ldots,N_{\mathrm{ind}}$, while

$$\omega_1 \leqslant \kappa_n(E) \leqslant \omega_N, \tag{31}$$

for $n=N_{\mathrm{ind}}+1,\ldots,N$. Therefore, only the first N_{ind} eigenvalues of $K(E)$ are ensured to be negative as $|\lambda|$ goes to infinity without regard to the location of $\{\omega_n\}_{n=1}^N$.

Proof: Taking $-\lambda^2 S(E)$ as the unperturbed part of $K(E)$, we obtain Eq. (30) for all n. Note that if only N_{ind} form factors are linearly independent, it holds that $\sigma_m(E)=0$ for $m=1,\ldots,N-N_{\mathrm{ind}}$ and otherwise does not vanish. Then, the assertion is proved straightforwardly. ∎

To illustrate the emergence of the negative eigenenergies, described in Theorem III.4 and Proposition III.5, let us consider the three-level system especially in the case where, $\omega_1<0$ while $\omega_2>0$ and $\omega_3>0$. We also choose three form factors, such as

$$v_n(\omega) = \Lambda^{1/2} \frac{\sqrt{\omega/\Lambda}[1 + a_n(\omega/\Lambda)^{2(n-1)}]}{[1 + (\omega/\Lambda)^2]^{1+n}}, \tag{32}$$

where Λ is the cutoff constant, and a_n is a parameter. The form factors described by such algebraic functions are often found in various systems involving the process of the spontaneous emission of photons from the hydrogen atom [22,23], the photodetachment of electrons from negative ions [3,4,24], and quantum dots [25]. In the calculation depicted in Figs. 1 and 2, we have chosen a set of parameters $\omega_1/\Lambda=-0.01$, $\omega_2/\Lambda=0.01$, and $\omega_3/\Lambda=0.02$, and $a_1=0.0$, $a_2=2.0$, and $a_3=1.0$. These choices for a_n guarantee linear independency among v_n's, so that $N_{\mathrm{ind}}=3$.

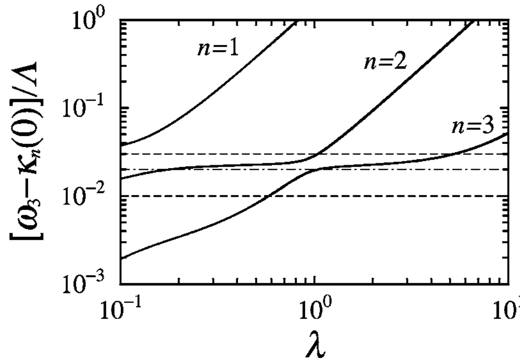

FIG. 1. $\omega_3-\kappa_n(0)$ for $n=1,2,3$ (three solid lines) for a three-level system with form factors (32), plotted against λ, and $\omega_3-\omega_1$, $\omega_3-\omega_2$ (two dashed lines), and ω_3 (dot-dashed line), for reference, where $\omega_3-\omega_1>\omega_3>\omega_3-\omega_2$. Three different regions are distinguished, corresponding to the number of solid lines satisfying $\omega_3-\kappa_n(0)>\omega_3$, that is, just the number of negative eigenenergies of H, by Theorem III.4.

Figure 1 shows $\omega_3-\kappa_n(0)$ for $n=1,2,3$, changing λ from 0.1 to 10.0, and $\omega_3-\omega_1$, $\omega_3-\omega_2$ (two dashed lines) and ω_3 (dot-dashed line) for reference. The latter satisfy the relation that $\omega_3-\omega_1>\omega_3>\omega_3-\omega_2>0$. One may recognize three different regions in this figure: for small $\lambda\lesssim 0.2$, one inequality $\omega_3-\kappa_1(0)>\omega_3$, i.e., $\kappa_1(0)<0$, holds. In the next region $0.2\lesssim\lambda\lesssim 1.0$, two inequalities, $\omega_3-\kappa_1(0)>\omega_3$ and $\omega_3-\kappa_2(0)>\omega_3$, hold. For $\lambda\gtrsim 1.0$ the last region, three inequalities, $\omega_3-\kappa_n(0)>\omega_3$ for all $n=1,2,3$, are satisfied. Therefore, according to Theorem III.4, one sees that one, two, and three negative eigenenergies of H exist in the first, second, and third regions, respectively. It is worth noting that the appearance of the negative eigenenergy in the first region merely occurs from the fact that $\omega_1<0$ (see, the latter part of Theorem III.4), whereas that in other regions could be understood as a strong-coupling effect (Proposition III.5).

Figure 2 shows three curves of $\omega_3-\kappa_n(E)$ for $n=1,2,3$ (three solid lines) and ω_3-E (short dashed line), plotted against E. An intersection of the former and the latter means an emergence of a negative eigenenergy. We also plot the asymptotes $\omega_3-\omega_1$ and $\omega_3-\omega_2$ (two dashed lines), to which $\omega_3-\kappa_1(E)$ and $\omega_3-\kappa_2(E)$ are close from above as $E\to-\infty$, respectively (see Lemma III.3). Figures 2(a)–2(c), are in the cases where $\lambda=0.1$, belongs to the first region, $\lambda=0.7$, of the second one, and $\lambda=10.0$, of the last one, respectively (see Fig. 1). It is seen in Fig. 2(a) that ω_3-E intersects $\omega_3-\kappa_1(E)$ only, so that there is one negative eigenenergy. In Fig. 2(b), one distinguishes the two intersections between ω_3-E and $\omega_3-\kappa_1(E)$, and between ω_3-E and $\omega_3-\kappa_2(E)$. Thus two negative eigenenergies appear. The intersection between the latter pair still lies around $E=0.0$. In Fig. 2(c), where a relatively large λ was chosen, ω_3-E finally intersects all three lines, $\omega_3-\kappa_n(E)$ for $n=1,2,3$, which tells us three negative eigenenergies exist.

IV. ABSENCE OF BOUND-STATE EIGENENERGY INSIDE THE CONTINUUM

Let us next examine the non-negative-eigenvalue problem for Eqs. (6) and (7). In this case, the normalization condition

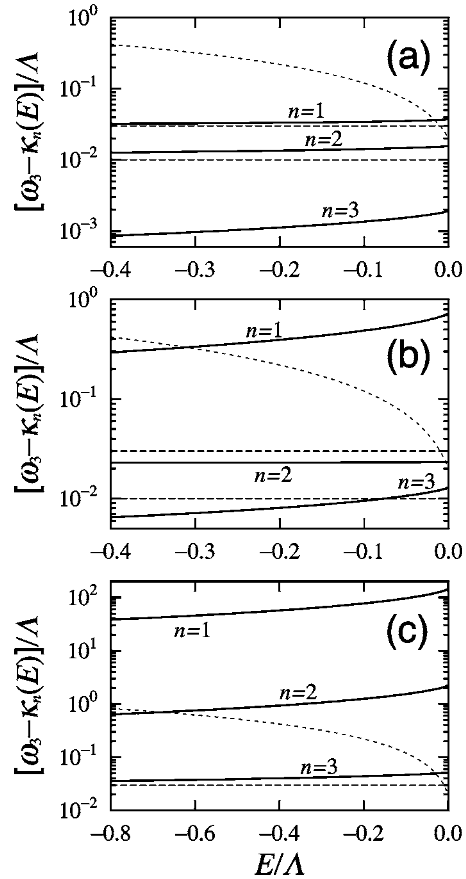

FIG. 2. $\omega_3-\kappa_n(E)$ for $n=1,2,3$ (three solid lines) for a three-level system of the form factors (32), ω_3-E (short-dashed line), and $\omega_3-\omega_1$ and $\omega_3-\omega_2$ (two dashed lines). We plot them in $\lambda=0.1$, 0.7, and 10.0, in (a), (b), and (c), respectively, choosing the parameters as $\omega_1/\Lambda=-0.01$, $\omega_2/\Lambda=0.01$, and $\omega_3/\Lambda=0.02$. (a) For a relatively small λ, only $\omega_3-\kappa_1(E)$ intersects ω_3-E at $E/\Lambda\simeq-0.02$, which is predicted by Theorem III.4. Thus there is one negative eigenenergy. One also sees that $\omega_3-\kappa_1(E)$ and $\omega_3-\kappa_2(E)$ still lie closely above the asymptotes $\omega_3-\omega_1$ and $\omega_3-\omega_2$, respectively. (b) Both $\omega_3-\kappa_1(E)$ and $\omega_3-\kappa_2(E)$ intersect ω_3-E in the vicinity of $E=-0.3$ and 0.0, respectively, so that there are two negative eigenenergies. (c) All $\omega_3-\kappa_n(E)$ for $n=1$, 2, and 3 intersect ω_3-E, and thus three negative eigenenergies exist. In this figure, only two intersections for $n=2$ and 3 are depicted.

(9) does not hold automatically, unlike the case where $E<0$, because of a possible divergence of $f(\omega)$ at $\omega=E$. Before going to the N-level case, let us first observe the single level one. Except in the trivial case where $c_1=0$, the condition (9) for an eigenvalue $E\geqslant 0$, if any, imposes the nontrivial condition or constraint that

$$v_1(E)=0, \tag{33}$$

where we assume some extent of the smoothness of $v_1(\omega)$ [26]. Then, $f(\omega)=-\lambda c_1 v_1(\omega)/(\omega-E)$ is ensured to be square integrable, and Eq. (6) reads

$$\omega_1-\lambda^2\int_0^\infty \frac{|v_1(\omega)|^2}{\omega-E}d\omega=E. \tag{34}$$

To find the solution E of Eq. (34), one may attempt to interpret it as an intersection between the left-hand and the right-hand sides, as in Eq. (12). However, this approach seems impossible at first, because the left-hand side of Eq. (34) is not well defined for a general E except such points satisfying Eq. (33). This matter can be solved by alternatively considering the following equation:

$$\omega_1 - \lambda^2 \mathrm{P}\int_0^\infty \frac{|v_1(\omega)|^2}{\omega - E} d\omega = E, \tag{35}$$

that is obtained from Eq. (34) by replacing $\int_0^\infty [|v_1(\omega)|^2/(\omega-E)]d\omega$ with its principal value $\mathrm{P}\int_0^\infty [|v_1(\omega)|^2/(\omega-E)]d\omega$. In this case, the left-hand side can make sense for a general E, and we can treat E as an independent variable. If we find the solution for E of Eq. (35), and furthermore if it satisfies Eq. (33), it becomes a true solution of the original Eq. (34). Indeed, in such a situation, we have that $\int_0^\infty [|v_1(\omega)|^2/(\omega-E)]d\omega = \mathrm{P}\int_0^\infty [|v_1(\omega)|^2/(\omega-E)]d\omega$, and thus Eq. (35) just reproduces Eq. (34).

In the N-level cases, the condition (9) for an eigenvalue $E \geqslant 0$, if any, can be translated into the equivalent condition for both the coefficients $\{c_n\}_{n=1}^N$ and E, that is,

$$\sum_{n=1}^N c_n v_n(E) = 0. \tag{36}$$

Under this condition, we can safely substitute Eq. (8) into Eq. (6). However, similarly to Eq. (35), we consider the alternative equation in the N-level cases as

$$\sum_{n'=1}^N [\omega_n \delta_{nn'} - \lambda^2 D_{nn'}(E)] c_{n'} = E c_n, \tag{37}$$

for $n=1,2,\ldots,N$, where

$$D_{nn'}(E) \coloneqq \mathrm{P}\int_0^\infty \frac{v_n^*(\omega) v_{n'}(\omega)}{\omega - E} d\omega, \tag{38}$$

which are the components of the Hermitian matrix $D(E)$ defined for all $E \geqslant 0$. One sees that Eq. (37) has the same form as Eq. (16), except the point that $S(E)$ $(E<0)$ is replaced by $D(E)$ $(E \geqslant 0)$. Then, we can implement a formulation in the matrix form, just as in the preceding section. In fact, the solutions of Eq. (37) can be connected with those of Eq. (6) under the condition (36). We first note that

$$\mathrm{P}\int_0^\infty \frac{v_n^*(\omega) \sum_{n'=1}^N c_{n'} v_{n'}(\omega)}{\omega - E} d\omega = \sum_{n'=1}^N c_{n'} \mathrm{P}\int_0^\infty \frac{v_n^*(\omega) v_{n'}(\omega)}{\omega - E} d\omega, \tag{39}$$

which is always valid for all E. Then, substituting this relation into Eq. (37), we have

$$\omega_n c_n - \lambda^2 \mathrm{P}\int_0^\infty \frac{v_n^*(\omega) \sum_{n'=1}^N c_{n'} v_{n'}(\omega)}{\omega - E} d\omega = E c_n, \tag{40}$$

for $n=1,2,\ldots,N$. For a comparison, see Eq. (6) again. Therefore, if the solutions E and $\{c_n\}_{n=1}^N$ of Eq. (40), i.e., Eq. (37), satisfy the condition (36), Eq. (40) can reproduce Eq. (6), so that the solutions of Eq. (40) become the true ones of Eq. (6).

Our procedure for finding the coefficients $\{c_n\}_{n=1}^N$ and the non-negative eigenvalue E of H that satisfy Eqs. (6) and (7) consists of the two steps; we first solve Eq. (37), and then we check whether the solutions satisfy the condition (36). For a later convenience, we introduce the Hermitian matrix $K(E)$ for $E \geqslant 0$ whose components are defined by

$$K_{nn'}(E) \coloneqq \omega_n \delta_{nn'} - \lambda^2 D_{nn'}(E). \tag{41}$$

Then, the existence of a nontrivial solution of Eq. (37) is ensured if and only if there exists a non-negative E to satisfy

$$\kappa_n(E,\lambda) = E, \tag{42}$$

for a certain integer n, where $\{\kappa_n(E,\lambda)\}_{n=1}^N$ are the eigenvalues of $K(E)$, arranged in increasing order. To summarize again, if an eigenvalue of $K(E)$ is E, then it is an eigenvalue of H, provided that it also satisfies the condition (36).

It is worth noting that the condition (36) seems not necessarily to require the existence of a zero of $v_n(\omega)$, unlike the single-level case of Eq. (33). However, the following statement means that if $v_n(\omega_n) \neq 0$ for all $\omega_n > 0$, the weak-coupling condition results in no positive eigenvalue of H strictly.

Theorem IV.1. Suppose that H_0 has N_+ positive eigenvalues without any degeneracy, and each $v_n(\omega)$ is an L^2 function of the form, $v_n(\omega) = \omega^{p_n} f_n(\omega)$, where $p_n > 0$ and $f_n(\omega)$ is a C^1 function in $[0,\infty)$. Furthermore, it is assumed that there is some $\delta_0 > 0$ such that $\sup_{\omega > \delta_0} |v_n^*(\omega) v_{n'}(\omega)| < \infty$ and $\sup_{\omega > \delta_0} |d[v_n^*(\omega) v_{n'}(\omega)]/d\omega| < \infty$ for all n and n'. Then, if λ is sufficiently small but not zero and the condition that $v_n(\omega_n) \neq 0$ for all $n \geqslant N - N_+ + 1$ is satisfied, H has no positive eigenvalues.

Proof: Under the assumption that $E>0$, we first consider the eigenvalue problem

$$\sum_{j=1}^N K_{ij}(E) c_{nj} = \kappa_n(E,\lambda) c_{ni}, \tag{43}$$

for $i=1,2,\ldots,N$, where $\{c_{ni}\}_{i=1}^N$ is the normalized eigenvector corresponding the nth eigenvalue $\kappa_n(E,\lambda)$ of $K(E)$. Then, by Theorem 4.3.1 in Ref. [20], one sees that

$$\begin{aligned} |\kappa_n(E,\lambda) - \omega_n| &\leqslant \lambda^2 \max\{|\delta_1(E)|, |\delta_N(E)|\} \\ &= \lambda^2 \|D(E)\| \leqslant \lambda^2 \sup_{E>0} \|D(E)\|, \end{aligned} \tag{44}$$

for all n, where $\delta_n(E)$ is the nth eigenvalue of $D(E)$. Note that from the assumption in the theorem and the propositions in the Appendix, it holds that $\sup_{E>0}\|D(E)\| < \infty$. Therefore, by choosing λ so that $|\lambda| < \lambda_a$, we have the fact that $|\kappa_n(E,\lambda) - \omega_n| < R_a$ for all $E>0$ and all n, and in particular $\kappa_n(E,\lambda) > 0$ for all $E>0$ for all $n \geqslant N - N_+ + 1$, where $\lambda_a = [R_a / \sup_{E>0}\|D(E)\|]^{1/2}$ and

$$R_a = \min\left\{ \omega_{N-N_+ +1}/3, \min_{n,m}\left\{ |\omega_n - \omega_m|/3 | n \neq m \right\} \right\}. \quad (45)$$

The latter means that $\kappa_n(E,\lambda)$ only for $n \geqslant N-N_+ +1$ becomes a candidate for positive eigenvalue of H. Note that κ_{N-N_+} cannot be such a candidate even if $\omega_{N-N_+}=0$. Because in such a case, putting $\lambda_b=[R_b/\sup_{E>0}\|D(E)\|]^{1/2}$, we find from Eq. (44) that for $|\lambda|<\lambda_b$, $|\kappa_{N-N_+}(E,\lambda)|<R_b$ for all $E>0$. We here choose such a R_b as to satisfy that $D(E') \geqslant 0$ for all positive $E'<R_b$. Existence of such an R_b is ensured by Eq. (A.2) in Proposition 1. Then, from Theorem 4.3.1 in Ref. [20] again, we have the estimation that

$$-\lambda^2 \delta_N(E) \leqslant \kappa_{N-N_+}(E,\lambda) \leqslant -\lambda^2 \delta_1(E) \leqslant 0, \quad (46)$$

for all $E<R_b$. Hence, we conclude that if $|\lambda|<\lambda_b$, it holds that $\kappa_{N-N_+}(E,\lambda)<E$ for all $E>0$ [27].

However, we can show that if we choose λ sufficiently small, any such a $\kappa_n(E,\lambda)$ and eigenvector $\Sigma_i c_{ni}|i\rangle$ cannot satisfy Eq. (36), no matter how well they satisfy Eqs. (37) and (42). To this end, let us look at Eq. (36), which is rewritten as

$$\left| \sum_{i=1}^N c_{ni} v_i(\kappa_n) \right|^2 = \left\| P_n(E,\lambda) \sum_{i=1}^N v_i^*(\kappa_n)|i\rangle \right\|^2 \quad (47)$$

$$= |v_n^*(\kappa_n)|^2 + \sum_{i=1}^N \langle i| v_i(\kappa_n)[P_n(E,\lambda) - |n\rangle\langle n|] \sum_{i'=1}^N v_{i'}^*(\kappa_n)|i'\rangle, \quad (48)$$

where $P_n(E,\lambda)$ denotes the projection operator associated with the nth eigenvalue $\kappa_n(E,\lambda)$. One sees that the first term on the right-hand side of Eq. (48) behaves as

$$\lim_{\lambda\to 0} |v_n^*(\kappa_n(E,\lambda))|^2 = |v_n^*(\omega_n)|^2, \quad (49)$$

for all $E>0$ uniformly, because of Eq. (44). From the assumption of the theorem, $|v_n^*(\omega_n)|^2$ does not vanish. For the second term on the right-hand side of Eq. (48), we can use the result on the perturbation of the projection operator [28], which leads to the fact that

$$P_n(E,\lambda) = |n\rangle\langle n| + \sum_{j=1}^\infty \lambda^{2j} P_n^{(j)}, \quad (50)$$

with

$$P_n^{(j)} := -\frac{1}{2\pi i}\oint_{\Gamma_n} (K_0-\zeta)^{-1}[D(E)(K_0-\zeta)^{-1}]^j d\zeta, \quad (51)$$

where Γ_n is the closed positively oriented circle around $\zeta=\omega_n$ with radius $\min_{m(\neq n)}\{|\omega_n-\omega_m|/3\}$. Series (50) is ensured to converge uniformly for all λ such that $|\lambda|<\min\{\lambda_a,\lambda_b\}$, because

$$\sup_{\zeta\in\Gamma_n} |\lambda|^2 \|D(E)\| \, \|(K_0-\zeta)^{-1}\| < \lambda_a^2/\lambda_n^2 \leqslant 1, \quad (52)$$

where

$$\lambda_n = \left[\min_{m(\neq n)} \{|\omega_n-\omega_m|/3\} \Big/ \sup_{E>0}\|D(E)\| \right]^{1/2}. \quad (53)$$

From the assumption of no degeneracy among $\{\omega_n\}_{n=1}^N$ and the discussion after Eq. (44), for such a λ, all Γ_n's are disconnected from each other, and there should be only one eigenvalue of K in each circle. This leads to $\dim[P_n(E,\lambda)\mathbb{C}^N]=\dim[|n\rangle\langle n|\mathbb{C}^N]=1$, so that $\lambda=0$ is not an exceptional point [28]. It is worth noting that λ_n does not depend on E. Thus, the second term on the right-hand side of Eq. (48) is estimated as

$$\sum_{i=1}^N \langle i|v_i(\kappa_n)[P_n(E,\lambda)-|n\rangle\langle n|]\sum_{i'=1}^N v_{i'}^*(\kappa_n)|i'\rangle$$

$$\leqslant \left\| \sum_{i=1}^N v_i^*(\kappa_n)|i\rangle \right\|^2 \|P_n(E,\lambda)-|n\rangle\langle n|\| \quad (54)$$

$$\leqslant \left[\sum_{i=1}^N \sup_{|\omega-\omega_n|<R_a} |v_i(\omega)|^2 \right] \frac{(\lambda/\lambda_n)^2}{1-(\lambda/\lambda_n)^2} \to 0, \quad (55)$$

as $\lambda\to 0$, for all $E>0$ uniformly, where it was used that $\sup_{\zeta\in\Gamma_n}\oint_{\Gamma_n}\|(K_0-\zeta)^{-1}\|\,|d\zeta| \leqslant 2\pi$. Equation (48) with the results (49) and (55) means that Eq. (36) is never satisfied for sufficiently small λ with $|\lambda|<\min\{\lambda_a,\lambda_b\}$, even if $\kappa_n(E,\lambda)=E$ holds. ■

It is worth considering the opposite condition that $v_n(\omega_n)=0$. In this case, we could infer the existence of an eigenvalue inside the continuum, from the decay process arising from the pole $z_{\mathrm{p},n}$. Indeed, if we recall the explicit form of the decay rate [29], if the opposite condition holds, the decay rate comes small so that a much slower decay occurs. Then, one may associate such a behavior with the presence of a bound state [30], though it is not obvious whether this pole actually becomes an eigenenergy of H.

Let us now evaluate an explicit value of λ for which there is no positive eigenvalue of H. Under the assumption of the analyticity of v_n, one sees that if $|\lambda|<\min\{\lambda_a,\lambda_b\}$, Eq. (49) is rewritten by using Eqs. (44) and (53) as

$$\big||v_n^*[\kappa_n(E,\lambda)]|^2 - |v_n^*(\omega_n)|^2\big|$$

$$\leqslant \sup_{|\omega-\omega_n|<R_a} \left| \frac{d|v_n(\omega)|^2}{d\omega} \right| |\kappa_n(E,\lambda)-\omega_n| \quad (56)$$

$$= \frac{\lambda^2}{\lambda_n^2} \min_{m(\neq n)}\{|\omega_n-\omega_m|/3\} \sup_{\omega>0} \left| \frac{d|v_n(\omega)|^2}{d\omega} \right|. \quad (57)$$

Therefore, by setting Eqs. (55) and (57) into Eq. (47), the left-hand side of Eq. (47) is ensured to be positive, and no positive eigenenergy of H exists, providing that λ is chosen to satisfy the $N_+ + 1$ inequalities,

$$|\lambda| < \min\{\lambda_a,\lambda_b\}, \quad (58)$$

and

$$|v_n^*(\omega_n)|^2 > \frac{\lambda^2}{\lambda_n^2} \min_{m(\neq n)}\{|\omega_n-\omega_m|/3\} \sup_{\omega>0}\left|\frac{d|v_n(\omega)|^2}{d\omega}\right| + \left[\sum_{i=1}^{N} \sup_{|\omega-\omega_n|<R_a} |v_i(\omega)|^2\right] \frac{\lambda^2/\lambda_n^2}{1-\lambda^2/\lambda_n^2}, \qquad (59)$$

for $n=N-N_+ +1,\ldots,N$. By solving Eq. (59) for λ explicitly, Eqs. (58) and (59) are reduced into the single inequality

$$|\lambda| < \min\{\lambda_a, \lambda_b, \bar{\lambda}_{N-N_+ +1}, \ldots, \bar{\lambda}_N\}, \qquad (60)$$

with

$$\bar{\lambda}_n = \sqrt{\frac{\lambda_n^2}{2\beta_n}\left[\alpha_n+\beta_n+\gamma_n-\sqrt{(\alpha_n+\beta_n+\gamma_n)^2-4\alpha_n\beta_n}\right]} < \lambda_n, \qquad (61)$$

where

$$\alpha_n = |v_n^*(\omega_n)|^2,$$

$$\beta_n = \min_{m(\neq n)}\{|\omega_n-\omega_m|/3\} \sup_{\omega>0}|d|v_n(\omega)|^2/d\omega|,$$

and

$$\gamma_n = \sum_{i=1}^{N} \sup_{|\omega-\omega_n|<R_a} |v_i(\omega)|^2.$$

In order to demonstrate Theorem IV.1, we apply it to the spontaneous emission process for the hydrogen atom interacting with the electromagnetic field [23]. We suppose that $|n\rangle$ is the product state between the $(n+1)p$ state of the atom and the vacuum state of the field, and also $|\omega\rangle$ the product state between the $1s$ state of the atom and the one-photon state. Then, an initially excited atom is expected to make a transition to the ground state by emitting a photon. We treat the atom as a four-level system composed of the ground state and the three excited states: the $2p$, $3p$, and $4p$ state. The form factors corresponding to the $2p$-$1s$, $3p$-$1s$, and $4p$-$1s$ transitions were obtained as follows [13,22,23]:

$$v_1^*(\omega) = i\Lambda_1^{1/2}\frac{(\omega/\Lambda_1)^{1/2}}{[1+(\omega/\Lambda_1)^2]^2}, \qquad (62)$$

$$v_2^*(\omega) = i81\Lambda_1^{1/2}\frac{(\omega/\Lambda_2)^{1/2}[1+2(\omega/\Lambda_2)^2]}{128\sqrt{2}[1+(\omega/\Lambda_2)^2]^3}, \qquad (63)$$

$$v_3^*(\omega) = i54\sqrt{3}\Lambda_1^{1/2}(\omega/\Lambda_3)^{1/2}\,\frac{45+146(\omega/\Lambda_3)^2+125(\omega/\Lambda_3)^4}{15\,625[1+(\omega/\Lambda_3)^2]^4}, \qquad (64)$$

where $\Lambda_1 = 8.498\times 10^{18}\ \mathrm{s}^{-1}$, $\Lambda_2 = (8/9)\Lambda_1\ \mathrm{s}^{-1}$, and $\Lambda_3 = (10/12)\Lambda_1\ \mathrm{s}^{-1}$ are the cut-off constants. One sees that these form factors satisfy all conditions required in Theorem IV.1. The coupling constant is also given by $\lambda^2 = 6.435\times 10^{-9}$. The eigenvalues of H_0 are given by $\omega_n = \frac{4}{3}\Omega[1-(n+1)^{-2}]$ with $\Omega = 1.55\times 10^{16}\ \mathrm{s}^{-1}$, all of which are embedded in the energy continuum. The Hamiltonian (1) is then derived under the four-level approximation (i.e., $N=N_+=3$) and the rotating wave approximation. The various parameters are numerically obtained as follows: $R_a = |\omega_2-\omega_3|/3 = (7/324)\Omega$, $\sup_{E>0}\|D(E)\| = -\delta_1(E) = 11.332\Lambda_1$ at $E=0.6145\Lambda_1$, $\lambda_1^2 = 5.45\times 10^{-3}\Omega/\Lambda_1$, $\lambda_2^2=\lambda_3^2=\lambda_a^2 = 1.91\times 10^{-3}\Omega/\Lambda_1$, $\alpha_1 = 1.82\times 10^{-3}\Lambda_1$, $\alpha_2 = 4.87\times 10^{-4}\Lambda_1$, $\alpha_3 = 1.99\times 10^{-4}\Lambda_1$, $\beta_1 = 6.17\times 10^{-2}\Omega$, $\beta_2 = 4.87\times 10^{-3}\Omega$, $\beta_3 = 1.88\times 10^{-3}\Omega$, $\gamma_1 = 2.45\times 10^{-3}\Lambda_1$, $\gamma_2 = 3.04\times 10^{-3}\Lambda_1$, $\gamma_3 = 2.45\times 10^{-3}\Lambda_1$, from which Eq. (61) reads $\bar{\lambda}_1^2 = 4.18\times 10^{-6}$, $\bar{\lambda}_2^2 = 5.01\times 10^{-7}$, and $\bar{\lambda}_3^2 = 2.14\times 10^{-7}$. Then, it follows that

$$\min\{\lambda_a^2, \bar{\lambda}_1^2, \bar{\lambda}_2^2, \bar{\lambda}_3^2\} = \bar{\lambda}_3^2 > \lambda^2, \qquad (65)$$

and thus Eq. (60) holds. This conclusion indicates that the intrinsic values of the parameters characterizing the system does not allow any bound state. In fact, we have not observed any such state. It is worth noticing that the upper bound estimated in Eq. (65) is dominated by the factor λ_3^2, roughly speaking, the minimum level-spacing over the maximum cut-off constant.

V. CONCLUDING REMARKS

We have considered the eigenvalue problem for unstable multilevel systems, on the basis of the N-level Friedrichs model, where the eigenenergies are supposed outside or possibly inside the continuum. The outside case is essentially determined by the location of the discrete level ω_n of the free Hamiltonian and the strength of the coupling constant λ. If ω_n lies outside the continuum, the corresponding eigenvalue always lies below ω_n. If ω_n lies inside the continuum, by choosing a λ large enough the eigenvalue originating from ω_n can emerge from the continuum. Such behaviors are similar to those seen in single-level cases, however, this is not the case if the form factors v_n are linearly dependent. On the other hand, we have shown the absence of the eigenvalue lying inside the continuum in the weak coupling cases, under the condition that $v_n(\omega_n)\neq 0$ if ω_n lies inside the continuum. This statement is just an extension of Lemma 2.1 in Ref. [14], where only identical form factors were considered, and the upper bound for $|\lambda|$ required in the lemma was not estimated. We have evaluated this upper bound in our case, which proves to be proportional to the minimum level-spacing over the maximum cut off constant. Hence, comparing this value with the actual λ, one can check at least the absence of the eigenvalue, even in the case that one cannot evaluate the reduced resolvent explicitly. At first sight, the normalization condition, i.e., Eq. (36), seems not necessarily to require the zeros of the form factors for a presence of a bound-state eigenenergy inside the continuum, though it is misplaced in weak-coupling regimes. However, we still do not have a definite answer to this matter in other coupling regimes where the multilevel effect may allow a presence of a bound-state eigenenergy inside the continuum without zeros of the form factors.

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APPENDIX

In this section, we present Propositions 1 and 2. The former and the latter state that the behavior of the energy shift $D(E)$ at small and large energies is quite regular without any divergence, respectively, under some form-factor conditions that are often satisfied by actual systems.

Proposition 1. Suppose that the function $\eta(\omega)$ belonging to $L^1([0,\infty))$ is of the form

$$\eta(\omega) := \omega^p r(\omega), \tag{A.1}$$

where $p>0$ and $r(\omega)$ is a C^1-function defined in $[0,\infty)$. It then holds that $\eta(\omega)/\omega \in L^1([0,\infty))$ and

$$\int_0^\infty \frac{\eta(\omega)}{\omega}d\omega = \lim_{E\uparrow 0}\int_0^\infty \frac{\eta(\omega)}{\omega - E}d\omega = \lim_{E\downarrow 0} \mathrm{P}\int_0^\infty \frac{\eta(\omega)}{\omega - E}d\omega. \tag{A.2}$$

Proof: From the proof of Proposition 3.2.2 in Ref. [16], the principal value of the integral on the right-hand side is written by the absolutely integrable function as follows:

$$\mathrm{P}\int_0^\infty \frac{\eta(\omega)}{\omega - E}d\omega = \int_0^\infty \frac{\eta(\omega) - \eta(E)\varphi_\delta(\omega - E)}{\omega - E}d\omega, \tag{A.3}$$

for all $E>0$, where $\varphi_\delta(\omega)$ is a C_0^∞ function with support $[-\delta,\delta]$ $(0<\forall\,\delta<E)$, even with respect to the origin, and such that $\varphi_\delta(0)=1$. In the following, we choose $\varphi_\delta(\omega)=\exp\{1-1/[1-(\omega/\delta)^2]\}$ for $\omega\in(-\delta,\delta)$ or 0 otherwise, and $\delta=E/2$. On the other hand, since from the assumption (A.1) $\eta(\omega)/\omega$ is absolutely integrable, the first equality in Eq. (A.2) is obvious. Therefore, it is sufficient to show that

$$\lim_{E\downarrow 0}\int_0^\infty \left[\frac{\eta(\omega)}{\omega} - \frac{\eta(\omega) - \eta(E)\varphi_\delta(\omega - E)}{\omega - E}\right]d\omega = 0. \tag{A.4}$$

Note that the above integrand can be rewritten as

$$\frac{\eta(\omega)}{\omega} - \frac{\eta(\omega) - \eta(E)\varphi_\delta(\omega - E)}{\omega - E}$$
$$= -E\frac{\eta(\omega)}{\omega(\omega - E)} + \frac{\eta(E)\varphi_\delta(\omega - E)}{\omega - E} \tag{A.5}$$
$$= \frac{\eta(E)\varphi_\delta(\omega - E)}{\omega} - E\frac{\eta(\omega) - \eta(E)\varphi_\delta(\omega - E)}{\omega(\omega - E)}. \tag{A.6}$$

Let us first consider the case where $\omega\in I := (0,E/2]\cup[3E/2,\infty)$. Then, since $\varphi_\delta(\omega-E)=0$, we can use Eq. (A.5) to estimate the integrand

$$\left|E\frac{\eta(\omega)}{\omega(\omega - E)}\right| \leqslant 2\left|\frac{\eta(\omega)}{\omega}\right|, \tag{A.7}$$

where the right-hand side is absolutely integrable and independent of E. Furthermore, it follows that $\lim_{E\downarrow 0} E\chi_I(\omega)\eta(\omega)/[\omega(\omega-E)]=0$ for every $\omega\in(0,\infty)$, where $\chi_I(\omega)=1$ $(\omega\in I)$ or 0 $(\omega\notin I)$, being the characteristic function. Thus, by the dominated convergence theorem, we can see that

$$\lim_{E\downarrow 0}\left(\int_0^{E/2} + \int_{3E/2}^\infty\right)E\frac{\eta(\omega)}{\omega(\omega - E)}d\omega = 0. \tag{A.8}$$

For $\omega\in(E/2,3E/2)$, we can use Eq. (A.6). The integration of the first term of Eq. (A.6) is estimated by

$$\left|\int_{E/2}^{3E/2}\frac{\eta(E)\varphi_\delta(\omega - E)}{\omega}d\omega\right| \leqslant \frac{\eta(E)}{E/2}\int_{E/2}^{3E/2}\varphi_\delta(\omega - E)d\omega$$
$$= \eta(E)\int_{-1}^{1}\varphi_1(x)dx \to 0, \tag{A.9}$$

as $E\downarrow 0$. The second term of Eq. (A.6) is also estimated by

$$|\eta(\omega) - \eta(E)\varphi_\delta(\omega - E)| \leqslant |\eta(\omega) - \eta(E)|$$
$$+ |\eta(E)||1 - \varphi_\delta(\omega - E)|. \tag{A.10}$$

The integration of the first term on the right-hand side right-hand side of the above is evaluated as

$$\int_{E/2}^{3E/2} E\frac{|\eta(\omega) - \eta(E)|}{\omega|\omega - E|}d\omega \leqslant (\ln 3)E\sup_{E/2\leqslant\omega\leqslant 3E/2}|\eta'(\omega)| \tag{A.11}$$

$$\leqslant (\ln 3)E\left[pE^{p-1}\max\left\{\left(\frac{1}{2}\right)^{p-1},\left(\frac{3}{2}\right)^{p-1}\right\}\sup_{\omega\in[0,3E/2]}|r(\omega)| + \left(\frac{3E}{2}\right)^p\sup_{\omega\in[0,3E/2]}|r'(\omega)|\right] \to 0 \text{ as } E\downarrow 0, \tag{A.12}$$

where the prime on $\eta'(\omega)$ implies the differentiation of $\eta(\omega)$ and so on. The integral corresponding to the last term on the right-hand side of Eq. (A.10) is also estimated as

$$\int_{E/2}^{3E/2} E\frac{|\eta(E)||1 - \varphi_\delta(\omega - E)|}{\omega|\omega - E|}d\omega$$
$$\leqslant (\ln 3)E|\eta(E)|\sup_{E/2\leqslant\omega\leqslant 3E/2}|\varphi_\delta'(\omega - E)| \tag{A.13}$$
$$= 2(\ln 3)|\eta(E)|\sup_{|x|\leqslant 1}|\varphi_1'(x)| \to 0\ (E\downarrow 0). \tag{A.14}$$

Thus, we can obtain

$$\lim_{E\downarrow 0}\int_{E/2}^{3E/2} E\frac{\eta(\omega) - \eta(E)\varphi_\delta(\omega - E)}{\omega(\omega - E)}d\omega = 0. \tag{A.15}$$

Equations (A.8), (A.9), and (A.15) mean the completion of the proof of (A.4) ■

Proposition 2. Suppose that the function $\eta(\omega)$ belongs to $L^1([0,\infty))\cap C^1([0,\infty))$, and satisfies that $\sup_{\omega\geqslant\delta_0}|\eta(\omega)|<\infty$ and $\sup_{\omega\geqslant\delta_0}|\eta'(\omega)|<\infty$ for some $\delta_0>0$. Then,

$$\sup_{E>\delta_0}\left|\mathrm{P}\int_0^\infty \frac{\eta(\omega)}{\omega-E}d\omega\right| < \infty. \tag{A.16}$$

Proof: To examine this integral, we use the expression (A.3) and divide the interval $[0,\infty)$ into $I_{\delta,E}=[E-\delta,E+\delta]$ and $\overline{I_{\delta,E}}=[0,\infty)\backslash I_{\delta,E}$, again, where we assume $\delta_0>\delta>0$. In the latter interval, it is estimated that $\chi_{\overline{I_{\delta,E}}}(\omega)|\eta(\omega)/(\omega-E)| \leqslant |\eta(\omega)|/\delta \in L^1([0,\infty))$. Then,

$$\sup_{E>\delta_0}\left|\int_0^\infty \chi_{\overline{I_{\delta,E}}}(\omega)\frac{\eta(\omega)}{\omega-E}d\omega\right| \leqslant \frac{1}{\delta}\int_0^\infty |\eta(\omega)|d\omega < \infty. \tag{A.17}$$

In the former interval, the integrand in Eq. (A.3) is evaluated as

$$\left|\frac{\eta(\omega)-\eta(E)\varphi_\delta(\omega-E)}{\omega-E}\right| \leqslant \sup_{\omega\in I_{\delta,E}}|\eta'(\omega)| + |\eta(E)|\sup_{|\omega|\leqslant\delta}|\varphi_\delta{}'(\omega)|, \tag{A.18}$$

which results in

$$\sup_{E>\delta_0}\left|\int_0^\infty \chi_{I_{\delta,E}}(\omega)\frac{\eta(\omega)-\eta(E)\varphi_\delta(\omega-E)}{\omega-E}\right| \leqslant 2\delta\left[\sup_{E>\delta_0}|\eta'(E)| + \sup_{E>\delta_0}|\eta(E)|\sup_{|\omega|\leqslant\delta}|\varphi_\delta{}'(\omega)|\right] < \infty, \tag{A.19}$$

where we used the assumption for $\eta(\omega)$ in the statement. Incorporating Eq. (A.17) with Eqs. (A.19) and (A.16) is obtained. ■

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[26] Note that when $v_1(\omega)$ is differentiable, the condition (33) means that $v_1(\omega)$ comes to 0 smoothly around $\omega=E$. Such zeros of $v_1(\omega)$ could be yielded instead by the "pointlike" gaps of $\rho(\omega)$. If $v_1(\omega)$ behaves like $\lim_{\omega\to E}v_1(\omega)\neq 0$ but $v_1(E)=0$ discontinuously, Eq. (9) never holds.

[27] This argument does not exclude the possibility of $\kappa_{N-N_+}(0,\lambda)$ being the zero-energy eigenvalue of H for $|\lambda|<\min\{\lambda_a,\lambda_b\}$. In this case, Eq. (46) implies that $\delta_1(0)=\sigma_1(0)=0$, so that the linear dependency among $v_n(\omega)$ is at least required. In addition, the discussion in Sec. III tells us that $\kappa_{N-N_+}(E)=0$ for all $E<0$ should hold.

[28] T. Kato, *Perturbation theory for linear operators* (Springer-Verlag, Berlin, 1966).

[29] In weak-coupling cases, the pole $z_{\mathrm{p},n}$ originating from ω_n is of the form $z_{\mathrm{p},n}=\omega_n-\lambda^2D_{nn}(\omega_n)-i\pi\lambda^2|v_n(\omega_n)|^2+O(\lambda^3)$ [11,16]. The decay rate is then given by $-2\,\mathrm{Im}\,z_{\mathrm{p},n}$.

[30] For a system with identical form factors, which have a common zero, we can find an appropriate λ to make such a zero an actual eigenvalue.