Structure of the sets of mutually unbiased bases for *N* qubits

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For a system of N qubits, living in a Hilbert space of dimension $d=2^N$, it is known that there exists $d+1$ mutually unbiased bases. Different construction algorithms exist, and it is remarkable that different methods lead to sets of bases with different properties as far as separability is concerned. Here we derive four sets of nine bases for three qubits, and show how they are unitarily related. We also briefly discuss the four-qubit case, give the entanglement structure of 16 sets of bases, and show some of them and their interrelations, as examples. The extension of the method to the general case of N qubits is outlined.

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I. INTRODUCTION

Every quantum system is associated with some state (pure or mixed) in a Hilbert space. It is possible to ascertain this quantum state by performing a series of measurements on an ensemble consisting of many identical members. Each measurement will modify the measured ensemble member in such a way that it is, in general, not possible to get any additional information about the original state. Several techniques, such as state tomography [1,2], maximum likelihood [3–5], and maximum-entropy methods [6] (or combinations thereof [7]), have been devised for efficient state estimation.

When the Hilbert space is finite, it has been shown that the optimal approach to get the information is related to a special set of states that are "mutually unbiased" [8–10], for which the uncertainty spread of the inferred state is minimized. Note, however, that we are ignoring more general measurements, such as joint measurements on all the members of the ensemble [11–13] or adaptive measurements [14,15], which surpass the ability of *a priori* fixed, single-copy measurements.

Let us denote basis sets by $A=1,2,\ldots$ and states within a basis by $|A,a\rangle$, with $a=1,2,\ldots,d$, d being the dimension of the Hilbert space. We recall that two bases $|A,a\rangle$ and $|B,b\rangle$ are said to be mutually unbiased bases (MUBs) if a system prepared in any element of A has a uniform probability distribution of being found in any element of B, that is

$$|\langle A,a|B,b\rangle|^2 = 1/d,$$

where orthonormality among states of the same basis is assumed. These MUBs are central to the formulation of the discrete Wigner function [16–19]. They have also been used in cryptographic protocols [20,21] due to the complete uncertainty about the outcome of a measurement in some basis after the preparation of the system in another, if the bases are mutually unbiased. MUBs are also used for quantum error correction codes [22,23] and recently they have also found uses in quantum game theory, in particular to provide a solution to the mean king problem [24–29].

It has been shown that the maximum number of MUBs can be at most $d+1$ [8]. Actually, it is known that if d is prime or the power of prime, the maximal number of MUBs can be achieved [8,10]. Remarkably though, there is no known answer for any other values of d, although there are some attempts to find a solution to this problem in some simple cases, such as $d=6$ or when d is a nonprime integer squared [30–32]. Recent works have suggested that the answer to this question may well be related with the nonexistence of finite projective planes of certain orders [33,34] or with the problem of mutually orthogonal Latin squares in combinatorics [35,36].

Experimental quantum information and computation have already moved from single-qubit protocols to several qubits (at present, around eight [37]), so there is a need to extend our knowledge, specially about entanglement properties of several qubits. This also includes extensions of measurement techniques to systems with more than two qubits. Therefore, a new problem related with MUBs naturally appears, namely that for more than two qubits, different MUB structures exist, where the word "structure" refers to the entanglement properties of the bases. We are already aware of the existence of three MUB structures for three qubits [10,38]. In this paper we will show that, in fact, there exists exactly four different MUB structures in this space. We will also show how they are interrelated. For the experimentalist, this information is very important, because the complexity of an implementation of two or more MUBs will, of course, greatly depend on how many of the qubits need to be entangled. We will also briefly discuss the four-qubit case and show that in this space there exist sixteen different MUB structures. We will exhibit which they are and derive some of them explicitly. It is then possible to continue and analyze the general N-qubit MUBs much in the same manner, although, for brevity and simplicity, we will stop at four qubits.

II. MUBS FOR ONE AND TWO QUBITS

Because states belonging to the same basis are usually taken to be orthonormal, to study the property of "mutually unbiasedness" it is possible to use either mutually unbiased bases or the operators which have the basis states as eigenvectors. We thus need d^2-1 operators to obtain the whole set

TABLE I. Five sets of three operators defining a (3,2) MUB.

1	$\hat{\sigma}_z\hat{\sigma}_z$	$\hat{1}\hat{\sigma}_z$	$\hat{\sigma}_z\hat{1}$	2
2	$\hat{\sigma}_x\hat{\sigma}_x$	$\hat{1}\hat{\sigma}_x$	$\hat{\sigma}_x\hat{1}$	2
3	$\hat{\sigma}_y\hat{\sigma}_y$	$\hat{1}\hat{\sigma}_y$	$\hat{\sigma}_y\hat{1}$	2
4	$\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\sigma}_z\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_z$	1
5	$\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_z$	1

of states. In the case of power of prime dimension, this set can be constructed as $d+1$ classes of $d-1$ commuting operators, which is related with the grading of a Lie algebra [39,40]. For N qubits (2^N-dimensional case), we need N commuting operators to define uniquely a pure state [41,42].

In finite-dimensional systems is also possible to define a discrete phase space, and when the dimension of the system is either a prime or a power of prime the phase space is a finite geometry. The above operators are related with translations in this phase space and (without a phase factor) they are the so-called displacement operators, which satisfy the covariant property of the discrete Wigner function defined there [18].

The two-dimensional Hilbert vector space (one qubit) is spanned, e.g., by the two orthonormal eigenvectors of the spin $1/2$ observable $\hat{\sigma}_z$, which will be used in the following as our computational basis. In this Hilbert space, the MUB set of $2^1+1=3$ bases is given by the eigenvectors of the Pauli matrices $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$. Any unitary operation preserves the angles between the axes of the transformed operators, so we can redefine our coordinates to have a new set of Pauli matrices. We can then say that the structure of the MUBs remains invariant under any unitary transformation. This is akin to saying that only one MUB structure exists in the two-dimensional Hilbert space.

A similar result also holds for two qubits, although, in addition, the extra feature of entanglement appears. Several methods have been presented for the explicit construction of MUBs [8,10,23,31,43–45]. Here we will only focus on one of them, which is based on the use of the finite Fourier transform, employing the operators $\hat{\sigma}_x$, $\hat{\sigma}_z$, and tensor products [46,47]. Because this work attempts to delineate the structure and interrelation between MUBs, and not their explicit mathematical construction, we omit such a discussion and refer the interested readers to the aforementioned work. If we follow the algorithm in Ref. [47], we get a table with five rows of three mutually commuting (tensor products of) operators, shown in Table I, which reproduces Eqs. (3.30) and (3.32)–(3.35) in Ref. [47]. We have suppressed the tensor multiplication sign in all the tables.

By construction, the algorithm guarantees that the simultaneous eigenstates of the operators in each row give a complete basis, and each basis is mutually unbiased to each other. The number on the left enumerates the bases, while the number on the right denotes how many subsystems the bases can be factorized into.

It is easy to see that the three first bases are fully separable (the three operators in each of the first three rows commute for each of the two subsystems, separately), and that the last two bases are not separable. In fact, their simultaneous eigenstates are all maximally entangled states. We call this MUB construction with three (bi-)separable and two nonseparable MUBs a (3,2) construction. This is equivalent, under local unitary transformations, to the construction given in Fig. 1 of Ref. [38].

The algorithm imposes several characteristic features of the table, which is composed of binary tensor products of the four operators $\hat{\sigma}_z$, $\hat{\sigma}_x$, $\hat{\sigma}_y=i\hat{\sigma}_x\hat{\sigma}_z=-i\hat{\sigma}_z\hat{\sigma}_x$, and $\hat{1}=\hat{\sigma}_x^2=\hat{\sigma}_y^2=\hat{\sigma}_z^2$. In all, there exists $4^2=16$ combinations of such products, but the operator $\hat{1}\otimes\hat{1}$ must be excluded because it commutes with every operator in the set. Each of the $15=3\times 5$ remaining operators is represented once. Moreover, the table is uniquely defined by the four entries in the two first columns of the first two rows. All other operators $O_{r,c}$ are determined by the relations $O_{r,c}=O_{r,c-2}O_{r,c-1}$, and $O_{r,c}=O_{2,c}O_{1,c+r-3}$ for $r>2$, where the indices r and c denote the row and the column of the operator, respectively, and must be taken modulo four.

Noting that each separable basis (i.e., the first three rows) has two eigenoperators containing the identity, that a nonseparable basis cannot have any eigenoperator containing the identity, and that there must be six entries, $\hat{1}\otimes\hat{\sigma}_x$, $\hat{1}\otimes\hat{\sigma}_y,\ldots,\hat{\sigma}_y\otimes\hat{1},\hat{\sigma}_z\otimes\hat{1}$ containing the identity in the table, we can conclude that the (3,2) set is the only possible construction in this space. That is, any nonlocal unitary transformation that yields either separable or maximally entangled basis (that is, a transformation from the Clifford group), will yield an isomorphic table with respect to the separability, except, perhaps, for some row permutations.

III. MUB STRUCTURES FOR THREE QUBITS

Lawrence, Brukner, and Zeilinger [38] have shown explicitly two different sets of MUBs in the case $d=8$. One of them has three fully separable bases (every eigenvector of these three bases is a tensor product of states embedded in the Hilbert space of each single qubit) and six GHZ bases [48]. The other structure has nine sets of bases with eigenvectors where one qubit can be factorized and the other two qubits are in a maximally entangled state. If we follow again the algorithm in Ref. [47] we get Table II.

Table II is equivalent to the first MUB construction demonstrated in this space by Fields and Wootters [10] or to the example 2 of Sec. V in Ref. [44], in that it has two fully (that is, tri-) separable bases (marked with a 3 in the rightmost column), three biseparable bases (marked with a 2), and four nonseparable bases (marked with a 1). We will denote such a set of MUBs as a (2,3,4) structure.

Lawrence, Brukner, and Zeilinger [38] have pointed out that two other constructions are possible, namely a (3,0,6) set where three of the bases are fully separable, and the remaining six bases are nonseparable, and a (0,9,0) one, in which all the bases are biseparable. The corresponding operators are given in Figs. 2 and 4 of Ref. [38]. We would like to derive these bases from the ones in Table II. To this end we use the controlled-Z operator

TABLE II. Nine sets of operators defining a (2,3,4) MUB.

1	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}$	3
2	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_x$	3
3	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_x$	2
4	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_x$	2
5	$\hat{\sigma}_x\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_y$	1
6	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_x$	1
7	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_y$	1
8	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_y$	1
9	$\hat{\sigma}_y\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_y$	2

$$\hat{Z}_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.1}$$

This operator is unitary, nonseparable and, moreover, has the property that it is its own inverse and its own conjugate. It commutes with $\hat{\mathbb{1}}\otimes\hat{\sigma}_z$, $\hat{\sigma}_z\otimes\hat{\mathbb{1}}$, and $\hat{\sigma}_z\otimes\hat{\sigma}_z$. Let us first convert the (2,3,4) into an equivalent basis set by applying the local unitary (permutation) transformation $x\to y\to z\to x$ to the leftmost qubit. The operator performing this transformation (up to an overall phase factor) is

$$\hat{U}_p = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\pi/4} & e^{i3\pi/4} \\ e^{i\pi/4} & e^{-i\pi/4} \end{pmatrix}. \tag{3.2}$$

We also apply the permutation $y\leftrightarrow z$ to the middle and rightmost qubits. The corresponding operator is

$$\hat{U}_c = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \tag{3.3}$$

Applying the operator $\hat{U}_p\otimes\hat{U}_c\otimes\hat{U}_c$ to Table II above, we are left with an equivalent operator table, Table III, still defining a (2,3,4) MUB.

Now we apply the nonlocal unitary operator $\hat{\mathbb{1}}\otimes\hat{Z}_c$ to the three qubits (evidently, the operator is only nonlocal in the four-dimensional subsystem constituting the rightmost two qubits). The local transformation we performed above on the first qubit will of course not change the structure of the MUB, not even after subsequently applying the operator $\hat{\mathbb{1}}\otimes\hat{Z}_c$. The reason we made this local transformation is only to facilitate a comparison with the construction in Ref. [38]. We note that $\hat{Z}_c(\hat{\mathbb{1}}\otimes\hat{\sigma}_x)\hat{Z}_c^{\dagger}=\hat{Z}_c(\hat{\mathbb{1}}\otimes\hat{\sigma}_x)\hat{Z}_c=\hat{\sigma}_z\otimes\hat{\sigma}_x$, and therefore $\hat{Z}_c(\hat{\sigma}_z\otimes\hat{\sigma}_x)\hat{Z}_c^{\dagger}=\hat{\mathbb{1}}\otimes\hat{\sigma}_x$. The corresponding transformations for the other products are

$$\hat{\sigma}_x\otimes\hat{\mathbb{1}}\leftrightarrow\hat{\sigma}_x\otimes\hat{\sigma}_z,$$

$$\hat{\mathbb{1}}\otimes\hat{\sigma}_y\leftrightarrow\hat{\sigma}_z\otimes\hat{\sigma}_y,$$

$$\hat{\sigma}_y\otimes\hat{\mathbb{1}}\leftrightarrow\hat{\sigma}_y\otimes\hat{\sigma}_z. \tag{3.4}$$

From these, the remaining relations

$$\hat{\sigma}_x\otimes\hat{\sigma}_x\leftrightarrow\hat{\sigma}_y\otimes\hat{\sigma}_y,$$

$$\hat{\sigma}_x\otimes\hat{\sigma}_y\leftrightarrow\hat{\sigma}_y\otimes\hat{\sigma}_x \tag{3.5}$$

follow. Hence, applying this transformation to Table III, will result in Table IV. From the unitarity of $\hat{Z}_c$ it follows that all

TABLE III. A local unitary rotation of Table II defining a "different," but isomorphic (2,3,4) MUB.

1	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\mathbb{1}}$	3
2	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_x$	3
3	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_x$	2
4	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_x$	2
5	$\hat{\sigma}_y\hat{\sigma}_y\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_z$	1
6	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_x$	1
7	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_z$	1
8	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_z$	1
9	$\hat{\sigma}_z\hat{\sigma}_y\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_z$	2

TABLE IV. Nine sets of operators defining a (0,9,0) MUB.

1	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z$	2
2	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_x$	2
3	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_y$	2
4	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_x$	2
5	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_z$	2
6	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_y$	2
7	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_z$	2
8	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\mathbb{1}}$	2
9	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\mathbb{1}}$	2

TABLE V. Nine sets of operators defining a (1,6,2) MUB.

1	$\hat{\sigma}_x\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_y$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\mathbb{1}}$	2
2	$\hat{\sigma}_y\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_x$	2
3	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_x$	2
4	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_x$	2
5	$\hat{\sigma}_x\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_z$	1
6	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x$	1
7	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_z$	2
8	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_x$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_z$	2
9	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_z$	3

TABLE VI. A table, isomorphic to Table V, defining a (1,6,2) MUB.

1	$\hat{\sigma}_x\hat{\sigma}_y\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\mathbb{1}}$	2
2	$\hat{\sigma}_z\hat{\sigma}_y\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_y$	2
3	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_y$	2
4	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_y$	2
5	$\hat{\sigma}_x\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_z$	1
6	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_y$	1
7	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_z$	2
8	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_x$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_z$	2
9	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_z$	3

TABLE VII. Nine sets of operators defining a (3,0,6) MUB.

1	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_z$	1
2	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_y$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_x$	1
3	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_y$	3
4	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_y$	1
5	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\mathbb{1}}$	1
6	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_x$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_x$	1
7	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_z$	1
8	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\mathbb{1}}$	3
9	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_z$	3

inner products between the eigenstates of the simultaneous eigenvectors of the operators in the same or in different rows of the two tables will be identical. We can therefore be confident that Table IV corresponds to a set of MUBs.

However, this set of MUBs represents a different entanglement structure, because here every basis is biseparable. In our nomenclature it is a (0,9,0) set. In fact it is the same table (with some rows interchanged) as Fig. 4 in Ref. [38].

We can now continue and apply the operator $\hat{Z}_c \otimes \hat{\mathbb{1}}$ to Table III, above. Again the set of simultaneous eigenstates of the operators in any row will define a complete basis, and the set of bases will form a MUB. The result can be seen in Table V.

This yields a (1,6,2) MUB. That is, only one of the bases is fully separable. This construction is neither a structure of the Fields' and Wootters' type, nor is it one of Lawrence *et al.*'s two structures. We note from the tables above, that there are nine operators containing two identity operators and 27 containing a single identity operator. In each one of the s sets of operators defining a fully separable basis (i.e., in each one of the s rows), there are three entries with two identity operators and three entries with a single identity operator. Each one of the b operator sets defining a biseparable basis contains one operator with two identities and three operators with a single identity. Finally, the n sets of operators defining nonseparable bases contain no operators with two identities, and three operators with a single identity. In consequence, we have the equations

$$3s + b = 9,$$

$$3(s + b + n) = 27, \tag{3.6}$$

for all non-negative integers (smaller or equal to nine), which yields the four solutions [(2,3,4), (0,9,0), (1,6,2), (3,0,6)]. We conclude that, so far, we have derived explicit constructions for the first three structures, and have one more left to construct.

Before doing that, we make a small digression and note that for three qutrits [49], similar considerations lead to the conclusion that in this 27-dimensional Hilbert space, with 28 MUBs, there exist five MUB structures, namely [(0,12,16), (1,9,18), (2,6,20), (3,3,22), (4,0,24)].

Now we return to the three qubit space. The last possible (3,0,6) structure can be built up in the following way: Take Table V and perform the transformation $y \leftrightarrow z$ on the leftmost two qubits and the transformation $x \leftrightarrow y$ on the rightmost qubit. As we are transforming only the two rightmost qubits in a nonlocal fashion, the transformation on the leftmost qubit gives a table identical to one of the constructions in Ref. [38]. The operator performing the transformation $x \leftrightarrow y$ is

$$\hat{U}_r = \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}. \tag{3.7}$$

The result of the local transformations $\hat{U}_c \otimes \hat{U}_c \otimes \hat{U}_r$ is shown in Table VI.

If we subsequently transform Table VI with $\hat{\mathbb{1}} \otimes \hat{Z}_c$, we get the last of the possible MUB constructions in this eight-dimensional space, which is reproduced in Table VII.

This table is a (3,0,6) MUB. It is, in fact, exactly the same (with some rows permuted) as in Fig. 2 of Ref. [38]. The possibilities in the eight-dimensional space are now exhausted. No other MUBs with different entanglement structures within the considered context can be constructed. We summarize the operational relationship between the different MUB structures one can construct in Fig. 1.

IV. MUB STRUCTURES FOR FOUR QUBITS

With four qubits, the MUBs can take five different forms with respect to their separability. We have fully sepa-

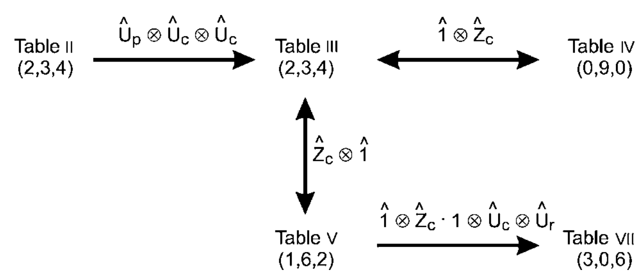

FIG. 1. The operational relationship between the different MUB constructions.

TABLE VIII. Seventeen sets of four operators defining a (2,0,4,2,9) MUB.

1	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\mathbb{1}}$	4
2	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_x$	4
3	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_x$	1
4	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_y$	2
5	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_x$	2
6	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_x$	1
7	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_x$	2
8	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_y$	2B
9	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_y$	1
10	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_y$	1
11	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_y$	2
12	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_x$	1
13	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_y$	2B
14	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_x$	1
15	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_y$	1
16	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_y$	1
17	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_x$	1

rable bases, triseparable bases ($2\times2\times4$), two kinds of biseparable bases (one that factors 2×8 and the other that factors 4×4), and finally nonseparable bases. If we follow once more Ref. [47], but write explicitly only the first four columns of each basis to save space, we get Table VIII.

The remaining 11 columns of the table can be generated through the relation $O_{r,c}=O_{r,c-4}O_{r,c-1}$. The 2 in the last column indicates a basis biseparable in a 2×8 space, while 2B indicate a basis biseparable in a 4×4 space. This basis will be denoted as a (2,0,4,2,9) MUB, referring how many of the bases that are fully separable, triseparable, biseparable (in a

TABLE IX. Seventeen sets of four operators defining a (0,4,4,2,7) MUB.

1	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\mathbb{1}}$	3
2	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_x$	3
3	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_x$	2B
4	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y\hat{\mathbb{1}}$	2
5	$\hat{\sigma}_y\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_y$	3
6	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_x$	2B
7	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_x$	2
8	$\hat{\sigma}_y\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_y\hat{\mathbb{1}}$	3
9	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_z$	1
10	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_z$	1
11	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_z$	2
12	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_y$	2
13	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_y\hat{\mathbb{1}}$	1
14	$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_y$	1
15	$\hat{\sigma}_x\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\hat{\mathbb{1}}$	1
16	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_z$	1
17	$\hat{\sigma}_x\hat{\sigma}_y\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_y$	1

TABLE X. The separability of the bases (left column) and the number of triplets, pairs, and single identity operators contained in the basis defining operator set.

Basis separability	$\hat{\mathbb{1}}\otimes\hat{\mathbb{1}}\otimes\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\otimes\hat{\mathbb{1}}$	$\hat{\mathbb{1}}$
4	4	6	4
3	2	4	6
2	1	3	7
2B	0	6	0
1	0	2	8
Available entries	12	54	108

2×8 and in a 4×4 space, respectively), and nonseparable. We now apply the operator

$$\hat{\mathbb{1}}\otimes\hat{\mathbb{1}}\otimes\hat{Z}_c\cdot\hat{\mathbb{1}}\otimes\hat{\mathbb{1}}\otimes\hat{U}_c\otimes\hat{U}_c, \tag{4.1}$$

which first locally rotates the two rightmost qubits so that $\hat{\sigma}_y\leftrightarrow\hat{\sigma}_z$ and then entangles (or disentangles) the same two qubits. The result is Table IX.

Before venturing further, it is instructive to see how many different MUB structures there are in the four-qubit space. Again we use the fact that identity operators play a special role in defining the separability of the bases. Table X shows how many products of three, two, and single identity operators define a basis of a certain kind. In the table we have disregarded the ordering of the multiple identities. We shall briefly come back to this issue, below.

Solving the three equations for the different number of identity operators, we find 16 different MUB structures: [(3,0,0,2,12), (2,0,4,2,9), (2,1,2,2,10), (2,2,0,2,11), (1,0,8,2,6), (1,1,6,2,7), (1,2,4,2,8), (1,3,2,2,9), (1,4,0,2,10), (0,0,12,2,3), (0,1,10,2,4), (0,2,8,2,5), (0,3,6,2,6), (0,4,4,2,7), (0,5,2,2,8), (0,6,0,2,9)]. Of these, we have explicitly given the tables for the (2,0,4,2,9) and the (0,4,4,2,7) structures. Deriving the transformations between any two of the 16 structures goes beyond the scope of this paper. However, note that applying the operator $\hat{\mathbb{1}}\otimes\hat{Z}_c\otimes\hat{\mathbb{1}}$ to the entries of Table IX will yield a (0,3,6,2,6) structure. Moreover, using instead the operator

$$\hat{Z}_c\otimes\hat{\mathbb{1}}\otimes\hat{\mathbb{1}}\cdot\hat{U}_c\otimes\hat{U}_c\otimes\hat{\mathbb{1}}\otimes\hat{\mathbb{1}} \tag{4.2}$$

in each entry of Table IX will yield a (1,2,4,2,8) structure. Since we know that a sequence of controlled-NOT or controlled-*Z* operations, together with local unitary rotations, suffice to make any entanglement transformation on qubits, it is clear that similar transformations will yield the whole set of different MUB structures, starting from Table VIII.

An interesting observation, which is neither obvious from Table VIII nor from Table IX, is that the eigenstates of the nonseparable bases are not generalized GHZ states. Instead, they are more akin to four-qubit graph states [50]. In particular, they have a high "persistency of entanglement" [51] in that tracing over any one of the qubits will leave the ensuing state entangled.

V. CONCLUSIONS

For one and two qubits, there exist only one MUB structure (in which for two qubits the bases are either separable or maximally entangled). For more qubits, the situation is more involved, four different MUB structures appear for three qubits and 16 for four qubits. The difference between these structures lies in how the bases are entangled. For both three and four qubits, MUBS exist that have 3, 2, 1, and no fully separable basis set(s). For three qubits it is possible to find one MUB that have no fully nonseparable bases. This is no longer possible in the four-qubit case.

In a quantum protocol relying on MUBs, the entanglement structure of the MUB is usually inconsequential. What counts is usually only the mutual unbiasedness, not the separability of the bases. Experimentally, however, it may be easier to generate one set of bases rather than another. Some of the bases can be generated locally, accessing each qubit separately. However, as we have shown, when several qubits are involved, most bases are entangled in one way or another, requiring joint operations on the qubits. In this paper we have tried to delineate the possible MUB structures for up to four qubits. The method we have employed can of course be extended to any number of qubits, although the complexity and variety of bases grows very rapidly with the number of qubits.

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