Two-fluid hydrodynamic modes in a trapped superfluid gas

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In the collisional region at finite temperatures, the collective modes of superfluids are described by the Landau two-fluid hydrodynamic equations. This region can now be probed over the entire BCS–Bose-Einstein-condensate crossover in trapped Fermi superfluids with a Feshbach resonance, including the unitarity region. Building on the approach initiated by Zaremba, Nikuni, and Griffin in 1999 for trapped atomic Bose gases, we present a variational formulation of two-fluid hydrodynamic collective modes based on the work of Zilsel in 1950 developed for superfluid helium. Assuming a simple variational Ansatz for the superfluid and normal fluid velocities, the frequencies of the hydrodynamic modes are given by solutions of coupled algebraic equations, with constants only involving spatial integrals over various equilibrium thermodynamic derivatives. This variational approach is both simpler and more physical than a direct attempt to solve the Landau two-fluid differential equations. Our two-fluid results are shown to reduce to those of Pitaevskii and Stringari for a pure superfluid at $T=0$.

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I. INTRODUCTION

In 1938, Tisza [1] first suggested that the classic features of superfluid helium were a manifestation of a two-fluid hydrodynamics, originating from a Bose-Einstein condensate (BEC). The correct two-fluid equations in the nondissipative limit were formulated and solved by Landau a few years later [2,3]. Two-fluid hydrodynamics describes the coupled dynamics of the superfluid and normal-fluid components when collisions produce a state of local thermodynamic equilibrium. This collision-induced state is described by the usual thermodynamic variables, which are now dependent on position and time. The two-fluid equations are essentially the same as the fluid dynamics of a normal fluid, but with additional equations describing the new superfluid component. In simple terms they describe the collective oscillations of frequency ω such that $\omega\tau_R \ll 1$, where τ_R describes the time it takes for a nonequilibrium state to reach local equilibrium (see, for example, Ref. [4]). For brief accounts of two-fluid hydrodynamics in the context of uniform quantum gases, see Refs. [5,6]. The two-fluid equations for a trapped Bose-condensed gas have been derived starting from a microscopic model in Ref. [7].

The two-fluid regime was difficult to reach in the first wave of experiments on trapped Bose-condensed gases after 1995. The typical collisional cross section and the achievable densities were not sufficient to reach the region described by Landau's two-fluid hydrodynamics. An exception was the pioneering experiment by Stamper-Kurn *et al.* [8], which later theoretical work showed was well within the two-fluid region [4]. However, there is renewed interest in the study of the two-fluid collisional dynamics in trapped gases. One reason is that with atom chips, one can now produce tightly localized high-density atomic Bose condensates. In addition, in a trapped two-component Fermi gas, one can use a Feshbach resonance to adjust the magnitude and sign of the s-wave scattering length a between Fermi atoms prepared in two different hyperfine states. This allows one to study the collective modes across the BCS-BEC crossover region in great detail (see, for example, Refs. [9,10]), with extremely large values of $|a|$. One also expects that even at the unitarity limit ($|a|\to\infty$) of trapped Fermi superfluids, one is dealing with a region where the Landau two-fluid equations are now the appropriate description of the dynamics at finite temperatures. Several recent papers [11,12] have considered first and second sound in a uniform two-component Fermi gas in the unitarity region.

So far, discussions of collective modes in a trapped Fermi gas with strong interactions have been limited to either $T=0$, where one is dealing with a pure superfluid [13–19], or to $T>T_c$, where one is dealing with the hydrodynamics of a normal Fermi liquid [20–22]. The much richer collective-mode spectrum at intermediate temperatures described by Landau two-fluid hydrodynamics awaits exploration. In particular, the BEC side of the BCS-BEC crossover is well described as a strongly interacting Bose gas of very stable molecules [23]. This should be an ideal system to study the analog of first and second sound in a uniform Bose superfluid predicted by the Landau two-fluid equations.

Evidence that recent collective mode experiments in trapped Fermi superfluids are described by collisional hydrodynamics is given by Massignan *et al.* [22]. In this paper, the relaxation time τ_R in a normal Fermi gas is calculated as a function of temperature and scattering length. Using the experimental parameters of Ref. [9], close to unitarity, it was estimated that $(\omega_\perp\tau_R)^{-1}\sim 3$ over the experimental temperature range $0.1 \leq T/T_F \leq 0.3$. Here $\omega_\perp$ is the radial trapping frequency and is typically much larger than the axial trapping frequency ω_z. This means that the axial mode is definitely in the collisionally hydrodynamic domain. We also note that the analysis in Ref. [4] shows that for Bose-condensed gases, the value of τ_R is mainly determined by collisions between the (high-density) condensate and (low-density) noncondensate particles. As a result, the value of τ_R is strongly suppressed as one goes into the Bose-condensed phase, allowing one to easily reach the collisional domain $\omega\tau_R \ll 1$. This result is also relevant to the molecular BEC regime of the BCS-BEC crossover where the condensate fraction can be quite large, even close to unitarity [10].

The frequencies of the in-phase (first-sound) and out-of-phase (second-sound) oscillations of the two fluids were first calculated by Zaremba, Griffin, and Nikuni (ZGN) [24] as well as Shenoy and Ho [25] for trapped, weakly interacting, dilute Bose-condensed gases. Their results demonstrated that the frequencies of the in-phase modes studied in recent experiments are largely independent of temperature. This behavior is found in superfluid Fermi gases close to unitarity [26]. The frequencies of the out-of-phase modes, on the other hand, are predicted to be strongly temperature dependent, vanishing above T_c. Thus, in addition to giving direct evidence for Fermi superfluidity, the experimental observation of the second-sound modes close to unitarity will provide an important test for microscopic theories of this strong interaction region.

For a trapped superfluid, it is very difficult to directly solve the Landau equations for the normal modes of the gas [7,25]. In this paper, we develop a variational approach of the kind first used by ZGN to solve for these modes (for a slightly modified version of the two-fluid equations reviewed in Appendix B). Our discussion is based on an action given in terms of the velocity and density fields of the superfluid and normal fluid, first proposed by Zilsel in 1950 [27]. The stationary point of the action corresponds to the solutions of the Landau two-fluid differential equations. Considering fluctuations of this action about an equilibrium state, we introduce simple variational *Ansätze* for the velocity fields. The end result is that the two-fluid hydrodynamic modes can be described in terms of two coupled harmonic oscillators. The effective masses and spring constants of the two oscillators are given explicitly in terms of spatial integrals over various static equilibrium thermodynamic functions and their derivatives. The latter can be computed in a straightforward manner, once we have a specific model for the thermal excitations of the trapped superfluid.

We illustrate our formalism in Sec. VI by using it to derive first- and second-sound modes in a uniform superfluid as well as the frequencies of the dipole and breathing modes of a trapped superfluid.

Our discussion of the two-fluid hydrodynamic equations using a variational approach is a natural extension of the approach recently used to deal with a pure superfluid in trapped gases at $T=0$ [15,17,19]. In Sec. VII, we consider the $T=0$ limit of our formalism to make contact with this "quantum hydrodynamic" theory, originally pioneered by Pitaevskii and Stringari [28] to extend the mean-field Gross-Pitaevskii theory of collective modes [29].

The discussion in Secs. II–IV is fairly technical. Readers primarily interested in the final results can immediately go to Sec. V and the examples in Sec. VI. Detailed numerical predictions for the hydrodynamic mode frequencies at finite T based on the formal expressions derived here will be presented in future papers.

II. LANDAU-KHALATNIKOV TWO-FLUID THERMODYNAMICS FOR A TRAPPED GAS

We note that since two-fluid hydrodynamics describes a system in local equilibrium, all thermodynamic quantities we discuss are functions of position and time. This dependence will usually be left implicit. Even in static equilibrium in the presence of a trapping potential, most thermodynamic quantities will still be position dependent.

Our variational principle will make use of a Lagrangian density of the form

$$\mathcal{L} = T - U, \tag{1}$$

where T is the kinetic energy density of the fluid and U is the internal energy density. Thus, we need to formulate the thermodynamics in terms of the internal energy density U, rather than the total energy $E=T+U$, as is normally done in two-fluid hydrodynamics (see, for example, p. 521 in Ref. [30]). Our approach will be to obtain U from the total energy density used by Landau and Khalatnikov (LK) to derive the Landau two-fluid (LTF) equations [3]. For a trapped two-fluid system, we define the total energy density E as the sum of the kinetic and potential energy densities,

$$E = \frac{1}{2}\rho_s \mathbf{v}_s^2 + \frac{1}{2}\rho_n \mathbf{v}_n^2 + U + \rho U_{\text{ext}}, \tag{2}$$

where ρ_n and ρ_s are the normal-fluid and superfluid densities, $\mathbf{v}_n$ and $\mathbf{v}_s$ are the normal-fluid and superfluid velocities, and $\rho=\rho_s+\rho_n$ is the total density. Also,

$$U_{\text{ext}}(\mathbf{r}) = \frac{1}{2}(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) \tag{3}$$

is the harmonic trapping potential *divided by the particle mass*, as follows from our use of the mass density ρ, rather than the number density n.

Following LK, the total energy density $E_0(\mathbf{r},t)$ as measured in a frame of reference moving with the local superfluid velocity is related to E by

$$E_0 = E - \frac{1}{2}\rho \mathbf{v}_s^2 - \mathbf{v}_s \cdot [\rho_n(\mathbf{v}_n - \mathbf{v}_s)]. \tag{4}$$

Using the LK identity for local equilibrium,

$$dE_0 = TdS + \mu d\rho + (\mathbf{v}_n - \mathbf{v}_s)\cdot d[\rho_n(\mathbf{v}_n - \mathbf{v}_s)], \tag{5}$$

with Eqs. (2) and (4), we obtain the following identity for the internal energy density [31]:

$$dU = (\mu - U_{\text{ext}})d\rho + TdS + \frac{1}{2}(\mathbf{v}_n - \mathbf{v}_s)^2 d\rho_n. \tag{6}$$

In a similar fashion, the LK expression for the pressure, $P=\partial(E_0 V)/\partial V$, including the effects of a trapping potential can be recast in terms of the internal energy density, giving

$$P = -U - \rho U_{\text{ext}} + TS + \mu\rho + \frac{1}{2}\rho_n(\mathbf{v}_n - \mathbf{v}_s)^2. \tag{7}$$

The two definitions given by Eqs. (6) and (7) can be combined to give

$$\rho d\mu = dP - SdT - \rho_n(\mathbf{v}_n - \mathbf{v}_s)\cdot d(\mathbf{v}_n - \mathbf{v}_s). \tag{8}$$

The three thermodynamic identities given by Eqs. (6)–(8) define all thermodynamic properties that we will require in developing our variational approach. We note that Eq. (6) implies that

$$\mu = \left(\frac{\partial U}{\partial \rho}\right)_{S,\rho_n} + U_{\rm ext}. \tag{9}$$

III. ZILSEL's HYDRODYNAMICS

In both classical and quantum mechanics, one can obtain dynamical equations of motion by equating to zero the variation of some action, usually expressed in the form $A = \int dt(T-U)$, where T and U are the kinetic and potential energies, respectively. To derive the LTF equations, we take an action of the form [27]

$$A = \int d^4x\left[\frac{1}{2}(\rho-\rho_n)\mathbf{v}_s^2 + \frac{1}{2}\rho_n\mathbf{v}_n^2 - U(\rho,\rho_n,S) - \rho U_{\rm ext}\right]. \tag{10}$$

Here $d^4x = d^3r dt$, while U is the internal energy defined by Eq. (6) and $U_{\rm ext}$ is the harmonic trapping potential given by Eq. (3). An action similar to Eq. (10) was originally used by Zilsel in 1950 to derive the LTF equations. In this section, we largely follow Zilsel's approach, with some differences noted in Appendix A.

In taking the variation of the action in Eq. (10), the variables $\mathbf{v}_n$, $\mathbf{v}_s$, ρ, ρ_n, and S will be treated as independent so that, for instance,

$$\frac{\delta U}{\delta \rho} \equiv \left(\frac{\partial U}{\partial \rho}\right)_{S,\rho_n}. \tag{11}$$

Two important conservation laws not incorporated into the action given by Eq. (10) are the conservation of mass and entropy, described by

$$\frac{\partial \rho}{\partial t} + \nabla\cdot[(\rho-\rho_n)\mathbf{v}_s + \rho_n\mathbf{v}_n] = 0, \tag{12}$$

$$\frac{\partial S}{\partial t} + \nabla\cdot(S\mathbf{v}_n) = 0. \tag{13}$$

The variation of the action in Eq. (10) must be taken subject to the constraints given by Eqs. (12) and (13). An additional constraint not employed by Zilsel describes conservation of circulation in the normal fluid and allows for the possibility of vorticity. This constraint takes the form [32–34]

$$\frac{\partial S\eta}{\partial t} + \nabla\cdot(S\eta\mathbf{v}_n) = 0, \tag{14}$$

where η depends on $\mathbf{r}$ and t. Even though the inclusion of this constraint has no effect on the final form of the LTF equations, it eliminates the restrictions on the normal-fluid velocity field that were present in Zilsel's original work. A more complete discussion of this point is given in Appendix A. Following the approach pioneered by Eckart [35], this constraint, along with entropy and mass conservation, can be incorporated into the variational principle by introducing Lagrange multipliers ϕ, α, and γ (all dependent on $\mathbf{r}$ and t) so that the action becomes

$$\begin{aligned} A = \int d^4x\Big[&\frac{1}{2}(\rho-\rho_n)\mathbf{v}_s^2 + \frac{1}{2}\rho_n\mathbf{v}_n^2 - U(\rho,\rho_n,S) - \rho U_{\rm ext} \\ &+ \phi\left\{\frac{\partial\rho}{\partial t} + \nabla\cdot[(\rho-\rho_n)\mathbf{v}_s + \rho_n\mathbf{v}_n]\right\} \\ &+ \alpha\left\{\frac{\partial S}{\partial t} + \nabla\cdot(S\mathbf{v}_n)\right\} + \gamma\left\{\frac{\partial S\eta}{\partial t} + \nabla\cdot(S\eta\mathbf{v}_n)\right\}\Big]. \end{aligned} \tag{15}$$

It is understood that the time and spatial integration in Eq. (15) is done between two fixed points with the fluctuations of the variables appearing in the action vanishing at both.

Setting the variation of the action given by Eq. (15) with respect to ρ, S, η, $\mathbf{v}_s$, and $\mathbf{v}_n$ equal to zero and making use of the thermodynamic identities implied by Eq. (6), we obtain the following relations:

$$\frac{\delta A}{\delta\rho} = \frac{1}{2}\mathbf{v}_s^2 - \frac{\partial\phi}{\partial t} - \mathbf{v}_s\cdot\nabla\phi - \mu = 0, \tag{16}$$

$$\frac{\delta A}{\delta S} = -\frac{\partial\alpha}{\partial t} - \mathbf{v}_n\cdot\nabla\alpha - T - \eta\left(\frac{\partial\gamma}{\partial t} + \mathbf{v}_n\cdot\nabla\gamma\right) = 0, \tag{17}$$

$$\frac{\delta A}{\delta\eta} = -S\left(\frac{\partial\gamma}{\partial t} + \mathbf{v}_n\cdot\nabla\gamma\right) = 0, \tag{18}$$

$$\frac{\delta A}{\delta\mathbf{v}_s} = \rho_s(\mathbf{v}_s - \nabla\phi) = \mathbf{0}, \tag{19}$$

and

$$\frac{\delta A}{\delta\mathbf{v}_n} = \rho_n(\mathbf{v}_n - \nabla\phi) - S(\nabla\alpha + \eta\nabla\gamma) = \mathbf{0}. \tag{20}$$

Taking the variation of the action with respect to ρ_n and using Eq. (19), one recovers a thermodynamic identity already known from Eq. (6):

$$\left(\frac{\partial U}{\partial\rho_n}\right)_{S,\rho} = \frac{1}{2}(\mathbf{v}_n - \mathbf{v}_s)^2. \tag{21}$$

Additionally, taking the variation of the action with respect to ϕ, α, and γ recovers the continuity equation given by Eq. (12), as well as the entropy and circulation conservation laws given by Eqs. (13) and (14).

Equations (16)–(20) can be rearranged to yield the useful expressions

$$\mathbf{v}_s = \nabla\phi, \tag{22}$$

$$\frac{\rho_n}{S}(\mathbf{v}_n - \mathbf{v}_s) = \nabla\alpha + \eta\nabla\gamma, \tag{23}$$

$$\frac{\partial\alpha}{\partial t} + \eta\frac{\partial\gamma}{\partial t} = -T - \frac{\rho_n}{S}\mathbf{v}_n\cdot(\mathbf{v}_n - \mathbf{v}_s), \tag{24}$$

$$\frac{\partial \gamma}{\partial t} + \mathbf{v}_n \cdot \nabla \gamma = 0, \tag{25}$$

and

$$\frac{\partial \phi}{\partial t} = -\left(\mu + \frac{1}{2}\mathbf{v}_s^2\right). \tag{26}$$

Equation (22) shows that the superfluid velocity is irrotational. Furthermore, comparing Eqs. (13) and (14), we find

$$\frac{\partial \eta}{\partial t} + \mathbf{v}_n \cdot \nabla \eta = 0. \tag{27}$$

Another relationship that will be helpful in deriving the equation of motion for the normal fluid, found from Eqs. (23), (25), and (27), is

$$\frac{\partial \eta}{\partial t} \nabla \gamma - \frac{\partial \gamma}{\partial t} \nabla \eta = \mathbf{v}_n \times \left(\nabla \times \left[\frac{\rho_n}{S}(\mathbf{v}_n - \mathbf{v}_s)\right]\right). \tag{28}$$

Taking the time derivative of Eq. (22) and the gradient of Eq. (26), we find the following equation for the superfluid velocity:

$$\frac{\partial \mathbf{v}_s}{\partial t} = -\nabla\left(\mu + \frac{1}{2}\mathbf{v}_s^2\right). \tag{29}$$

Rearranging Eq. (7) to obtain an equation for μ and making use of the result

$$\nabla U = \left(\frac{\partial U}{\partial \rho}\right)_{\rho_n,S} \nabla \rho + \left(\frac{\partial U}{\partial \rho_n}\right)_{\rho,S} \nabla \rho_n + \left(\frac{\partial U}{\partial S}\right)_{\rho,\rho_n} \nabla S,$$

$$= (\mu - U_{\text{ext}}) \nabla \rho + \frac{1}{2}(\mathbf{v}_n - \mathbf{v}_s)^2 \nabla \rho_n + T \nabla S, \tag{30}$$

we obtain the expression

$$\nabla \mu = \frac{1}{\rho} \nabla P - \frac{S}{\rho} \nabla T + \nabla U_{\text{ext}} - \frac{1}{2}\frac{\rho_n}{\rho} \nabla [(\mathbf{v}_n - \mathbf{v}_s)^2]. \tag{31}$$

Thus, Eq. (29) for the superfluid velocity can be rewritten as

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_s \cdot \nabla\right)\mathbf{v}_s = \frac{S}{\rho} \nabla T - \frac{1}{\rho} \nabla P - \nabla U_{\text{ext}} + \frac{1}{2}\frac{\rho_n}{\rho} \nabla [(\mathbf{v}_n - \mathbf{v}_s)^2]. \tag{32}$$

To determine the analogous equation for the velocity of the normal fluid, we take the time derivative of Eq. (23) and the gradient of Eq. (24). Then, using Eqs. (12)–(14), (23), and (28), as well as the identity $\mathbf{v} \cdot \nabla \mathbf{v} = (1/2)\nabla(\mathbf{v}^2) - \mathbf{v} \times (\nabla \times \mathbf{v})$, after some laborious algebra, we find

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_n \cdot \nabla\right)\mathbf{v}_n = -\frac{\rho_s}{\rho_n}\frac{S}{\rho} \nabla T - \frac{1}{\rho} \nabla P - \nabla U_{\text{ext}} - \frac{1}{2}\frac{\rho_s}{\rho} \nabla [(\mathbf{v}_n - \mathbf{v}_s)^2] - \frac{\Gamma}{\rho_n}(\mathbf{v}_n - \mathbf{v}_s), \tag{33}$$

where the "source function" Γ is defined by

$$\Gamma \equiv \frac{\partial \rho_n}{\partial t} + \nabla \cdot (\rho_n \mathbf{v}_n). \tag{34}$$

From the continuity equation given by Eq. (12), Eq. (34) implies that

$$\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s \mathbf{v}_s) = -\Gamma. \tag{35}$$

Combining Eqs. (32)–(35) to obtain an equation of motion for the current, $\mathbf{j} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n$, the source terms of the two components cancel and we find

$$\frac{\partial \mathbf{j}}{\partial t} = -\nabla P - \rho \nabla U_{\text{ext}} - \rho_s \mathbf{v}_s \cdot \nabla \mathbf{v}_s - \rho_n \mathbf{v}_n \cdot \nabla \mathbf{v}_n - \mathbf{v}_s \nabla \cdot (\rho_s \mathbf{v}_s) - \mathbf{v}_n \nabla \cdot (\rho_n \mathbf{v}_n). \tag{36}$$

More familiarly, in component form, this equation can be written as [3]

$$\frac{\partial j_i}{\partial t} = -\frac{\partial}{\partial x_j}[P\delta_{ij} + \rho_s v_{si} v_{sj} + \rho_n v_{ni} v_{nj}] - \rho \frac{\partial U_{\text{ext}}}{\partial x_i}, \tag{37}$$

where the index j is summed over.

Taken together, Eqs. (12), (13), (29), and (37) constitute the Landau two-fluid equations in the nondissipative limit, generalized to include a static external potential.

In Ref. 27, Zilsel claims that the source term appearing in Eq. (33) does not appear in the LTF equations of Landau [2]. However, Landau only derived equations of motion for the superfluid velocity, as well as the local current density. These are identical to Eqs. (29) and (37), apart from the external potential we are including. If one works backwards from these two equations to derive an equation for the normal fluid velocity, *the source term in Eq. (33) does appear*. Thus, the LTF equations do include a term that contains Γ. We also note that the appearance of Γ in the equation of motion for the normal-fluid velocity is consistent with the results derived by ZNG from a microscopic model (for further discussion, see Appendix B).

Even though our variational method has been developed so that we do not have to solve the linearized LTF equations, to facilitate comparison with that approach we derive those equations here. For the case where $\mathbf{v}_{s0} = \mathbf{v}_{n0} = \mathbf{0}$, the linearized LTF equations for the velocity fields are

$$\frac{\partial \mathbf{v}_s}{\partial t} = \frac{S_0}{\rho_0} \nabla \delta T - \frac{1}{\rho_0} \nabla \delta P + \frac{\nabla P_0}{\rho_0^2}\delta\rho = -\nabla \delta\mu \tag{38}$$

and

$$\frac{\partial \mathbf{v}_n}{\partial t} = -\frac{\rho_{s0}}{\rho_{n0}}\frac{S_0}{\rho_0}\nabla\,\delta T - \frac{1}{\rho_0}\nabla\,\delta P + \frac{\nabla P_0}{\rho_0^2}\delta\rho$$

$$= -\nabla\delta\mu - \frac{S_0}{\rho_{n0}}\nabla\,\delta T. \qquad (39)$$

Here we have made use of the fact that Γ is zero in equilibrium [7]. The last term in Eq. (33) is second order in the fluctuations and thus vanishes in a linearized theory. We have also made use of the fact that in equilibrium $\nabla T_0 = \mathbf{0}$. Furthermore, in equilibrium we also have $\nabla\mu_0 = \mathbf{0}$ so that for $\mathbf{v}_{n0} = \mathbf{v}_{s0} = \mathbf{0}$, Eq. (31) yields the well-known relation

$$\nabla U_{\rm ext} = -\frac{1}{\rho_0}\nabla P_0. \qquad (40)$$

Using this, the linearized hydrodynamic velocity equations reduce to

$$\frac{\partial \mathbf{v}_s}{\partial t} = \frac{S_0}{\rho_0}\nabla\,\delta T - \frac{1}{\rho_0}\nabla\,\delta P - \frac{\nabla U_{\rm ext}}{\rho_0}\delta\rho, \qquad (41)$$

$$\frac{\partial \mathbf{v}_n}{\partial t} = -\frac{\rho_{s0}}{\rho_{n0}}\frac{S_0}{\rho_0}\nabla\,\delta T - \frac{1}{\rho_0}\nabla\,\delta P - \frac{\nabla U_{\rm ext}}{\rho_0}\delta\rho. \qquad (42)$$

Combining these, we obtain

$$\frac{\partial \delta\mathbf{j}}{\partial t} = -\nabla\delta P - \delta\rho\,\nabla\,U_{\rm ext}. \qquad (43)$$

IV. ACTION FOR LINEARIZED LANDAU TWO-FLUID HYDRODYNAMICS

As was shown in the previous section, variation of the action given by Eq. (15) with respect to ρ, S, η, $\mathbf{v}_s$, $\mathbf{v}_n$, ϕ, α, and γ leads to the nondissipative LTF hydrodynamic equations [27]. To determine the low-energy collective modes of this system given by the solutions of the linearized hydrodynamic equations, we Taylor-expand the action about the equilibrium values of these variables. In principle, one could then take the variation of the resulting action with respect to the fluctuations of its variables—i.e., $\delta\rho$, δS, $\delta\eta$, $\delta\mathbf{v}_s$, $\delta\mathbf{v}_n$, $\delta\phi$, $\delta\alpha$, and $\delta\gamma$—to obtain the linearized hydrodynamic equations. These could then be solved to find the collective-mode frequencies. In practice, rather than dealing with the Lagrange multipliers ϕ, α, and γ in Eq. (15), it is more convenient to introduce displacement fields for the two velocities. This allows one to incorporate the conservation laws directly into expressions for $\delta\rho$ and δS, thereby eliminating the need for Lagrange multipliers. We will employ a simplified Rayleigh-Ritz method in conjunction with our variational approach to obtain estimates of the collective-mode frequencies by using physically reasonable *Ansätze* for the displacement fields of the superfluid and normal fluid.

Recalling that the conservation of circulation constraint does not affect the form of the equations resulting from the variational principle, we will omit this term in the action for the sake of clarity. However, if we were interested in collective oscillations of the normal fluid with nonzero circulation, we would need to incorporate this term and allow for fluctuations of η and γ.

Considering fluctuations of the action about equilibrium we set $\rho = \rho_0 + \delta\rho$, $\rho_n = \rho_{n0} + \delta\rho_n$, $S = S_0 + \delta S$, $\phi = \phi_0 + \delta\phi$, $\mathbf{v}_n = \mathbf{v}_{n0} + \delta\mathbf{v}_n$, and $\mathbf{v}_s = \mathbf{v}_{s0} + \delta\mathbf{v}_s$ and expand the action given by Eq. (15) up to quadratic order in these fluctuations. As in the above, we assume that the two fluid components are stationary in equilibrium, so that $\delta\mathbf{v}_n = \mathbf{v}_n$ and $\delta\mathbf{v}_s = \mathbf{v}_s$. Writing $A = A^{(0)} + A^{(1)} + A^{(2)} + \cdots$, the contribution to the action from terms which are quadratic in the fluctuations is

$$\begin{aligned} A^{(2)} = \int d^4x \Bigg[& \frac{1}{2}\rho_{s0}\mathbf{v}_s^2 + \frac{1}{2}\rho_{n0}\mathbf{v}_n^2 - \frac{1}{2}\left(\frac{\partial\mu}{\partial\rho}\right)_{S,\rho_n}(\delta\rho)^2 \\ & - \left(\frac{\partial T}{\partial\rho}\right)_{S,\rho_n}\delta S\,\delta\rho - \frac{1}{2}\left(\frac{\partial T}{\partial S}\right)_{\rho,\rho_n}(\delta S)^2 + \delta\phi\Bigg\{\frac{\partial\delta\rho}{\partial t} \\ & + \nabla\cdot(\rho_{s0}\mathbf{v}_s + \rho_{n0}\mathbf{v}_n)\Bigg\} + \phi_0\{\nabla\cdot[(\delta\rho - \delta\rho_n)\mathbf{v}_s + \delta\rho_n\mathbf{v}_n]\} \\ & + \delta\alpha\left\{\frac{\partial\delta S}{\partial t} + \nabla\cdot(S_0\mathbf{v}_n)\right\} + \alpha_0\{\nabla\cdot(\delta S\mathbf{v}_n)\}\Bigg], \end{aligned} \qquad (44)$$

where we have made use of Eq. (6) to rewrite the coefficients in terms of thermodynamic derivatives. Note that there is an alternative expression for the coefficient multiplying $\delta S\,\delta\rho$, given by the Maxwell relation

$$(\partial T/\partial\rho)_S = (\partial\mu/\partial S)_\rho. \qquad (45)$$

Contributions to the fluctuations of the internal energy arising from fluctuations of ρ_n have vanished since, as can be seen from Eq. (21), the equilibrium value $(\partial U/\partial\rho_n)_{S,\rho}$ is zero when $\mathbf{v}_{n0} = \mathbf{v}_{s0}$. We do not consider the contributions which are linear in fluctuations since we know from the stationary action principle that $A^{(1)} = 0$.

Since $\mathbf{v}_{s0} = \mathbf{0}$, Eq. (22) gives us $\nabla\phi_0 = \mathbf{0}$. Also, with $\mathbf{v}_{n0} = \mathbf{v}_{s0} = \mathbf{0}$, from Eq. (23) we find $\nabla\alpha_0 = \mathbf{0}$ since η is an independent and arbitrary variable. With these relations, having integrated by parts the terms in Eq. (44) which contain ϕ_0 and α_0, the quadratic action simplifies to

$$\begin{aligned} A^{(2)} = \int d^4x \Bigg\{ & \frac{1}{2}\rho_{s0}\mathbf{v}_s^2 + \frac{1}{2}\rho_{n0}\mathbf{v}_n^2 - \frac{1}{2}\left(\frac{\partial\mu}{\partial\rho}\right)_{S,\rho_n}(\delta\rho)^2 \\ & - \left(\frac{\partial T}{\partial\rho}\right)_{S,\rho_n}\delta S\,\delta\rho - \frac{1}{2}\left(\frac{\partial T}{\partial S}\right)_{\rho,\rho_n}(\delta S)^2 + \delta\phi\Bigg[\frac{\partial\delta\rho}{\partial t} \\ & + \nabla\cdot(\rho_{s0}\mathbf{v}_s + \rho_{n0}\mathbf{v}_n)\Bigg] + \delta\alpha\left[\frac{\partial\delta S}{\partial t} + \nabla\cdot(S_0\mathbf{v}_n)\right]\Bigg\}. \end{aligned} \qquad (46)$$

In the present form, the variational principle has been reduced to equating to zero the variation of the action given by the first two lines in Eq. (46), subject to the constraints given by the linearized continuity and entropy conservation equations

$$\frac{\partial \delta\rho}{\partial t} + \nabla \cdot (\rho_{s0}\mathbf{v}_s + \rho_{n0}\mathbf{v}_n) = 0 \quad (47)$$

and

$$\frac{\partial \delta S}{\partial t} + \nabla \cdot (S_0\mathbf{v}_n) = 0. \quad (48)$$

For our purposes, it is more convenient to incorporate these two constraints into expressions for $\delta\rho$ and δS than to use the Lagrange multipliers $\delta\phi$ and $\delta\alpha$. Introducing the displacement fields

$$\mathbf{v}_s(\mathbf{r},t) \equiv \frac{\partial \mathbf{u}_s(\mathbf{r},t)}{\partial t}, \quad \mathbf{v}_n(\mathbf{r},t) \equiv \frac{\partial \mathbf{u}_n(\mathbf{r},t)}{\partial t}, \quad (49)$$

the linearized continuity and entropy conservation equations can be expressed in terms of these fields as

$$\delta\rho(\mathbf{r},t) = -\nabla \cdot [\rho_{s0}(\mathbf{r})\mathbf{u}_s(\mathbf{r},t) + \rho_{n0}(\mathbf{r})\mathbf{u}_n(\mathbf{r},t)] \quad (50)$$

and

$$\delta S(\mathbf{r},t) = -\nabla \cdot [S_0(\mathbf{r})\mathbf{u}_n(\mathbf{r},t)]. \quad (51)$$

Removing the terms that contain the Lagrange multipliers from Eq. (46) and substituting these expressions for $\delta\rho$ and δS, we find our central result for the action that is second order in the fluctuations $\mathbf{u}_s$ and $\mathbf{u}_n$,

$$A^{(2)} = \int d^4x \Bigg\{ \frac{1}{2}\rho_{s0}\dot{\mathbf{u}}_s^2 + \frac{1}{2}\rho_{n0}\dot{\mathbf{u}}_n^2 - \frac{1}{2}\left(\frac{\partial \mu}{\partial \rho}\right)_{S,\rho_n} [\nabla \cdot (\rho_{s0}\mathbf{u}_s + \rho_{n0}\mathbf{u}_n)]^2 - \left(\frac{\partial T}{\partial \rho}\right)_{S,\rho_n} [\nabla \cdot (S_0\mathbf{u}_n)][\nabla \cdot (\rho_{s0}\mathbf{u}_s + \rho_{n0}\mathbf{u}_n)] - \frac{1}{2}\left(\frac{\partial T}{\partial S}\right)_{\rho,\rho_n} [\nabla \cdot (S_0\mathbf{u}_n)]^2 \Bigg\}. \quad (52)$$

To make our expressions as compact as possible, in what follows we will drop the ρ_n term constraining the thermodynamic derivatives in Eq. (52). However, it should be remembered that in evaluating the various local thermodynamic derivatives, ρ_n is always fixed.

V. HYDRODYNAMIC MODES DESCRIBING FLUCTUATIONS

The fluctuating quantities $\delta\phi$, $\delta\alpha$, δS, $\delta\rho$, $\mathbf{v}_s$, and $\mathbf{v}_n$, appearing in Eq. (46), have been replaced in Eq. (52) by the two displacement fields $\mathbf{u}_n$ and $\mathbf{u}_s$. Thus, the variational principle has been reduced to taking the variation of the quadratic action given by Eq. (52) with respect to only $\mathbf{u}_n$ and $\mathbf{u}_s$. The linearized LTF hydrodynamic equations and, hence, low-energy collective modes of the system are completely determined by the variational equations

$$\frac{\delta A^{(2)}}{\delta \mathbf{u}_s(\mathbf{r},t)} = \mathbf{0}, \quad \frac{\delta A^{(2)}}{\delta \mathbf{u}_n(\mathbf{r},t)} = \mathbf{0}. \quad (53)$$

Using Eq. (52) as well as the identities given by Eqs. (50) and (51), Eq. (53) gives

$$\ddot{\mathbf{u}}_s = -\nabla\left[\delta\rho\left(\frac{\partial \mu}{\partial \rho}\right)_S + \delta S\left(\frac{\partial \mu}{\partial S}\right)_\rho\right] \quad (54)$$

and

$$\ddot{\mathbf{u}}_n = -\nabla\left[\delta\rho\left(\frac{\partial \mu}{\partial \rho}\right)_S + \delta S\left(\frac{\partial \mu}{\partial S}\right)_\rho\right] - \frac{S_0}{\rho_{n0}}\nabla\left[\delta\rho\left(\frac{\partial T}{\partial \rho}\right)_S + \delta S\left(\frac{\partial T}{\partial S}\right)_\rho\right], \quad (55)$$

where we have used the Maxwell relation given in Eq. (45). Taking $\delta F = \delta S(\partial F/\partial S)_\rho + \delta\rho(\partial F/\partial \rho)_S$, where F denotes either one of μ or T, with the displacement fields defined by Eqs. (49), it is apparent that Eqs. (54) and (55) are equivalent to the linearized two-fluid hydrodynamic equations given by Eqs. (38) and (39). This verifies that the variational principle given by Eqs. (52) and (53) is the correct one.

Solutions of Eq. (53) corresponding to collective modes will have harmonic time-dependence of the form

$$\mathbf{u}_s(\mathbf{r},t) = \mathbf{u}_s(\mathbf{r})\cos(\omega t), \quad \mathbf{u}_n(\mathbf{r},t) = \mathbf{u}_n(\mathbf{r})\cos(\omega t). \quad (56)$$

Because these are exact solutions of the variational equations, we can insert these expressions into the action directly. Substituting Eq. (56) into Eq. (52) and performing the time integration, we obtain, apart from an irrelevant constant factor, the following Lagrangian:

$$L^{(2)} = K[\mathbf{u}_s,\mathbf{u}_n]\omega^2 - U[\mathbf{u}_s,\mathbf{u}_n], \quad (57)$$

where

$$K[\mathbf{u}_s,\mathbf{u}_n] = \frac{1}{2}\int d^3r\{\rho_{s0}\mathbf{u}_s^2 + \rho_{n0}\mathbf{u}_n^2\} \quad (58)$$

and

$$U[\mathbf{u}_s,\mathbf{u}_n] = \frac{1}{2}\int d^3r\Bigg\{\left(\frac{\partial \mu}{\partial \rho}\right)_S [\nabla \cdot (\rho_{s0}\mathbf{u}_s + \rho_{n0}\mathbf{u}_n)]^2 + 2\left(\frac{\partial T}{\partial \rho}\right)_S [\nabla \cdot (S_0\mathbf{u}_n)][\nabla \cdot (\rho_{s0}\mathbf{u}_s + \rho_{n0}\mathbf{u}_n)] + \left(\frac{\partial T}{\partial S}\right)_\rho [\nabla \cdot (S_0\mathbf{u}_n)]^2\Bigg\}. \quad (59)$$

Since the fields $\mathbf{u}_s(\mathbf{r})$ and $\mathbf{u}_n(\mathbf{r})$ are no longer time dependent, it suffices to consider variations of the Lagrangian given by Eq. (57), and the variational equations now become

$$\frac{\delta L^{(2)}}{\delta \mathbf{u}_s(\mathbf{r})} = \mathbf{0}, \quad \frac{\delta L^{(2)}}{\delta \mathbf{u}_n(\mathbf{r})} = \mathbf{0}. \quad (60)$$

Solving Eq. (60) is still equivalent to solving the linearized hydrodynamic equations, with the only simplification being that we have assumed a harmonic time dependence for the solutions. Fortunately, one can easily obtain approximate expressions for the collective-mode frequencies within a variational approach. Following Ref. [7], we use a simplified Rayleigh-Ritz method and make an *Ansatz* for each Cartesian component of the displacement fields of the form

$$u_{si}(\mathbf{r}) = a_{si} f_i(\mathbf{r}), \quad u_{ni}(\mathbf{r}) = a_{ni} g_i(\mathbf{r}), \tag{61}$$

where the constant coefficients a_{si} and a_{ni} are variational parameters. Substituting this *Ansatz* into Eq. (57) and equating to zero the variation of the resulting expression with respect to these parameters, we have the six variational equations

$$\frac{\delta L^{(2)}}{\delta a_{si}} = 0, \quad \frac{\delta L^{(2)}}{\delta a_{ni}} = 0. \tag{62}$$

In practice, the symmetry of the problem usually allows us to reduce the number of equations. From these equations, one obtains a rigorous *upper bound* for the collective-mode frequencies ω [36].

A more precise Rayleigh-Ritz method would require us to expand each component of the displacement fields in terms of $n(>1)$ elements of some complete set of functions. As n is increased, solving the resulting $6n$ variational equations, one iteratively generates successively better approximations for the collective-mode frequencies. For harmonically confined gases, however, there exist simple trial functions for $f_i(\mathbf{r})$ and $g_i(\mathbf{r})$ which are sufficiently close to the exact solutions that excellent results are obtained by considering only a single expansion term as in Eq. (61) [7]. Our choices of *Ansätze* for the displacement fields at finite temperatures will be guided by the known exact solutions at $T=0$ [29] and $T > T_c$ (for bosons, see Ref. [37]; for fermions, see Ref. [20]) for modes of experimental interest, such as the dipole and breathing modes.

Using the *Ansatz* given by Eq. (61), the Lagrangian given by Eqs. (57)–(59) describes the dynamics of a pair of coupled harmonic oscillators, with a_{si} and a_{ni} representing the displacements of the two oscillators from equilibrium. The effective spring constants are determined by the equilibrium thermodynamic quantities of the system. This is a useful picture to have when envisioning the low-energy dynamics of the two fluids. It immediately implies, for instance, the existence of in-phase as well as out-of-phase oscillation modes of the two fluids.

VI. EXAMPLES OF COLLECTIVE MODES

A. Uniform gas

As a simple example of our formalism, we consider first a uniform gas. In this case, ρ_{s0}, ρ_{n0}, and the equilibrium thermodynamic derivatives appearing in Eqs. (58) and (59) are all independent of position. Anticipating sound modes with a dispersion relation of the form $\omega = uk$, where u is the sound speed, we use the plane-wave *Ansätze*

$$\mathbf{u}_s(\mathbf{r}) = \hat{\mathbf{x}} \mathcal{N} a_s \cos(\omega x/u),$$

$$\mathbf{u}_n(\mathbf{r}) = \hat{\mathbf{x}} \mathcal{N} a_n \cos(\omega x/u), \tag{63}$$

where a_s and a_n are the variational parameters. The normalization constant $\mathcal{N}$ is chosen so that $\int d^3r \mathbf{u}_s^2 = a_s^2$. Inserting Eq. (63) into Eqs. (58) and (59), we find

$$K[a_s, a_n] = \frac{1}{2}\rho_{s0} a_s^2 + \frac{1}{2}\rho_{n0} a_n^2 \tag{64}$$

and

$$U[a_s,a_n] = \frac{\omega^2}{u^2}\Bigg\{ \frac{a_s^2}{2}\left[(\rho_{s0})^2 \left(\frac{\partial \mu}{\partial \rho}\right)_S \right] + a_s a_n \left[\rho_{s0}\rho_{n0}\left(\frac{\partial \mu}{\partial \rho}\right)_S + S_0 \rho_{s0}\left(\frac{\partial T}{\partial \rho}\right)_S \right] + \frac{a_n^2}{2}\left[(\rho_{n0})^2 \left(\frac{\partial \mu}{\partial \rho}\right)_S + 2S_0\rho_{n0}\left(\frac{\partial T}{\partial \rho}\right)_S + (S_0)^2\left(\frac{\partial T}{\partial S}\right)_\rho \right]\Bigg\}. \tag{65}$$

The trial functions given by Eq. (63) are, in fact, exact. Thus, the variational equations given by Eq. (62), which in the present case take the form

$$\frac{\delta L^{(2)}}{\delta a_s} = 0, \quad \frac{\delta L^{(2)}}{\delta a_n} = 0, \tag{66}$$

will give the exact eigenvalues ω or, equivalently, the exact sound speeds u. Using the above expressions for K and U, Eqs. (57) and (66) give the following quadratic equation for u^2:

$$u^4 - u^2\left[\rho_0 \left(\frac{\partial \mu}{\partial \rho}\right)_S + 2S_0\left(\frac{\partial T}{\partial \rho}\right)_S + \frac{(S_0)^2}{\rho_{n0}}\left(\frac{\partial T}{\partial S}\right)_\rho \right] + \frac{\rho_{s0}}{\rho_{n0}}(S_0)^2 \frac{\partial(\mu,T)}{\partial(\rho,S)} = 0. \tag{67}$$

Here $\partial(\mu,T)/\partial(\rho,S)$ denotes the Jacobian of the transformation between μ, T and ρ, S.

In the usual textbook discussions, one works with the entropy density $\sigma = S/\rho$, rather than the entropy S [3,5,6]. To express Eq. (67) in terms of σ, we use the following identities which are found from the transformation properties of thermodynamic derivatives (see, for example, Chap. 16 in Ref. [38]):

$$\left(\frac{\partial \mu}{\partial \rho}\right)_S = -2\sigma\left(\frac{\partial T}{\partial \rho}\right)_\sigma + \frac{\sigma^2}{\rho}\left(\frac{\partial T}{\partial \sigma}\right)_\rho + \frac{1}{\rho}\left(\frac{\partial P}{\partial \rho}\right)_\sigma, \tag{68}$$

$$\left(\frac{\partial T}{\partial \rho}\right)_S = \left(\frac{\partial T}{\partial \rho}\right)_\sigma - \frac{\sigma}{\rho}\left(\frac{\partial T}{\partial \sigma}\right)_\rho, \tag{69}$$

$$\frac{\partial(\mu,T)}{\partial(\rho,S)} = \frac{1}{\rho^2}\left(\frac{\partial P}{\partial \rho}\right)_T\left(\frac{\partial T}{\partial \sigma}\right)_\rho, \tag{70}$$

and

$$\left(\frac{\partial T}{\partial S}\right)_\rho = \frac{1}{\rho}\left(\frac{\partial T}{\partial \sigma}\right)_\rho. \tag{71}$$

With these relations, Eq. (67) for the sound speed can be rewritten as

$$u^4 - u^2\left[\left(\frac{\partial P}{\partial \rho}\right)_\sigma + \frac{\rho_{s0}}{\rho_{n0}}(\sigma_0)^2\left(\frac{\partial T}{\partial \sigma}\right)_\rho \right] + \frac{\rho_{s0}}{\rho_{n0}}\sigma_0^2\left(\frac{\partial P}{\partial \rho}\right)_T\left(\frac{\partial T}{\partial \sigma}\right)_\rho = 0. \tag{72}$$

This is the standard equation giving the first- and second-sound speeds in a uniform superfluid [2,3,5,6].

B. Dipole mode

We next consider a dipole mode in a harmonically confined gas. This mode is characterized by displacements of the center of masses of the two fluids along one of the axes of the harmonic trap, x_i. In this case, we use the following *Ansätze* for the displacement fields:

$$\mathbf{u}_s = \hat{\mathbf{x}}_i a_s, \quad \mathbf{u}_n = \hat{\mathbf{x}}_i a_n, \tag{73}$$

where a_s and a_n describe the displacements of the center of masses of the two fluids from the trap center. Substituting these *Ansätze* into Eqs. (58) and (59), we find

$$K[a_s,a_n] = \frac{1}{2}M_s a_s^2 + \frac{1}{2}M_n a_n^2 \tag{74}$$

and

$$U[a_s,a_n] = \frac{1}{2}k_s a_s^2 + \frac{1}{2}k_n a_n^2 + \frac{1}{2}k_{sn}(a_s - a_n)^2, \tag{75}$$

where M_s and M_n are the the masses of the superfluid and normal fluids, respectively, given by

$$M_s = \int d^3r \rho_{s0}, \quad M_n = \int d^3r \rho_{n0}. \tag{76}$$

The spring constants k_s, k_n, and k_{sn} are given by

$$k_s = \int d^3r\left\{\left[\left(\frac{\partial\mu}{\partial\rho}\right)_S\frac{\partial\rho_0}{\partial x_i} + \left(\frac{\partial\mu}{\partial S}\right)_\rho\frac{\partial S_0}{\partial x_i}\right]\frac{\partial\rho_{s0}}{\partial x_i}\right\}, \tag{77}$$

$$k_n = \int d^3r\left\{\left[\left(\frac{\partial\mu}{\partial\rho}\right)_S\frac{\partial\rho_0}{\partial x_i} + \left(\frac{\partial\mu}{\partial S}\right)_\rho\frac{\partial S_0}{\partial x_i}\right]\frac{\partial\rho_{n0}}{\partial x_i} + \left[\left(\frac{\partial T}{\partial\rho}\right)_S\frac{\partial\rho_0}{\partial x_i} + \left(\frac{\partial T}{\partial S}\right)_\rho\frac{\partial S_0}{\partial x_i}\right]\frac{\partial S_0}{\partial x_i}\right\}, \tag{78}$$

and

$$k_{sn} = -\int d^3r\left\{\left(\frac{\partial\mu}{\partial\rho}\right)_S\frac{\partial\rho_{n0}}{\partial x_i}\frac{\partial\rho_{s0}}{\partial x_i} + \left(\frac{\partial T}{\partial\rho}\right)_S\frac{\partial S_0}{\partial x_i}\frac{\partial\rho_{s0}}{\partial x_i}\right\}. \tag{79}$$

The variational equation, equivalent to the expression for the homogeneous gas given by Eq. (66) with K and U given by Eqs. (74) and (75), gives the following secular equation for the dipole frequencies:

$$\begin{pmatrix} M_s\omega^2 - k_s - k_{sn} & k_{sn} \\ k_{sn} & M_n\omega^2 - k_n - k_{sn}\end{pmatrix}\begin{pmatrix} a_s \\ a_n\end{pmatrix} = 0. \tag{80}$$

Applying some thermodynamic identities, the spring constants k_s and k_n can be simplified considerably. Specifically, with $\mathbf{v}_{n0}=\mathbf{v}_{s0}=\mathbf{0}$, Eq. (6) can be used to find useful expressions for the gradient of various equilibrium thermodynamic quantities. One finds, for instance, from the definition $T=(\partial U/\partial S)_\rho$,

$$\nabla T_0 = \left(\frac{\partial^2 U}{\partial S\partial\rho}\right)\nabla\rho_0 + \left(\frac{\partial^2 U}{\partial S^2}\right)_\rho\nabla S_0 = \left(\frac{\partial T}{\partial\rho}\right)_S\nabla\rho_0 + \left(\frac{\partial T}{\partial S}\right)_\rho\nabla S_0. \tag{81}$$

Similarly, using Eq. (9),

$$\nabla\mu_0 = \left(\frac{\partial\mu}{\partial\rho}\right)_S\nabla\rho_0 + \left(\frac{\partial\mu}{\partial S}\right)_\rho\nabla S_0 + \nabla U_{\text{ext}}, \tag{82}$$

which, it can be shown, is equivalent to Eq. (31) evaluated at equilibrium, when $\mathbf{v}_{n0}=\mathbf{v}_{s0}=\mathbf{0}$. For a harmonic trapping potential given by Eq. (3), noting that $\nabla\mu_0=\mathbf{0}$, Eq. (82) provides us with the useful expression

$$\left(\frac{\partial\mu}{\partial\rho}\right)_S\frac{\partial\rho_0}{\partial x_i} + \left(\frac{\partial\mu}{\partial S}\right)_\rho\frac{\partial S_0}{\partial x_i} = -\frac{\partial}{\partial x_i}U_{\text{ext}} = -\omega_i^2 x_i, \tag{83}$$

where ω_i is the trap frequency along the x_i axis. Also, since $\nabla T_0=\mathbf{0}$, Eq. (81) gives us

$$\left(\frac{\partial T}{\partial\rho}\right)_S\frac{\partial\rho_0}{\partial x_i} + \left(\frac{\partial T}{\partial S}\right)_\rho\frac{\partial S_0}{\partial x_i} = 0. \tag{84}$$

Substituting Eqs. (83) and (84) into Eqs. (77) and (78) and integrating by parts, the expressions for k_s and k_n readily simplify, reducing to $k_s=\omega_i^2 M_s$ and $k_n=\omega_i^2 M_n$. With these values, Eq. (80) becomes

$$[M_s M_n(\omega^2-\omega_i^2) - k_{sn}(M_s+M_n)](\omega^2-\omega_i^2) = 0. \tag{85}$$

This gives an in-phase dipole mode with frequency

$$\omega = \omega_i \tag{86}$$

and an out-of-phase dipole mode with frequency

$$\omega^2 = \omega_i^2 + \frac{M_s+M_n}{M_s M_n}k_{sn}. \tag{87}$$

The result given by Eq. (86) is the expected Kohn mode in a harmonic trap and corresponds to the solution $a_s=a_n$. This mode is a rigid oscillation of the superfluid and normal-fluid static distributions and, as a result, the interactions have no effect on the collective mode frequency. The frequency of the second mode given by Eq. (87) does depend on interactions as these determine the thermodynamic functions appearing in the k_{sn} spring constant given by Eq. (79). This mode corresponds to the solution $M_s a_s + M_n a_n = 0$. This is precisely the analog in a trapped system of second sound, since the displacements of the superfluid and normal fluid have opposite signs, producing an out-of-phase oscillation of the two components.

The expression for the frequency of the out-of-phase dipole mode given by Eq. (87) agrees with that found in Refs. [7,24], but with a slightly different value of k_{sn} in Eq. (79) as a result of working with a modified form of two-fluid hydrodynamics (see Appendix B for further discussion).

C. Breathing modes

For the breathing modes, we consider *Ansätze* of the form

$$\mathbf{u}_s = (a_{s1}x, a_{s2}y, a_{s3}z), \quad \mathbf{u}_n = (a_{n1}x, a_{n2}y, a_{n3}z). \tag{88}$$

Substituting Eq. (88) into Eq. (57), we find

$$K[a_s,a_n] = \frac{1}{2}\sum_i [M_{si}a_{si}^2 + M_{ni}a_{ni}^2] \tag{89}$$

and

$$U[a_s,a_n] = \frac{1}{2}\sum_{ij} [k_{s,ij}a_{si}a_{sj} + k_{n,ij}a_{ni}a_{nj} + 2k_{sn,ij}a_{si}a_{nj}], \tag{90}$$

where the weighted masses M_i are defined by

$$M_{si} \equiv \int d^3r \rho_{s0} x_i^2, \quad M_{ni} \equiv \int d^3r \rho_{n0} x_i^2, \tag{91}$$

and the constants $k_{s,ij}$, $k_{n,ij}$, and $k_{sn,ij}$ (which can be related to spring constants) are

$$k_{s,ij} = \int d^3r \left\{ \left(\frac{\partial \mu}{\partial \rho}\right)_S \frac{\partial(\rho_{s0}x_i)}{\partial x_i}\frac{\partial(\rho_{s0}x_j)}{\partial x_j} \right\}, \tag{92}$$

$$k_{n,ij} = \int d^3r \left\{ \left(\frac{\partial \mu}{\partial \rho}\right)_S \frac{\partial(\rho_{n0}x_i)}{\partial x_i}\frac{\partial(\rho_{n0}x_j)}{\partial x_j} + 2\left(\frac{\partial T}{\partial \rho}\right)_S \frac{\partial(\rho_{n0}x_i)}{\partial x_i}\frac{\partial(S_0x_j)}{\partial x_j} + \left(\frac{\partial T}{\partial S}\right)_\rho \frac{\partial(S_0x_i)}{\partial x_i}\frac{\partial(S_0x_j)}{\partial x_j} \right\}, \tag{93}$$

and

$$k_{sn,ij} = \int d^3r \left\{ \left(\frac{\partial \mu}{\partial \rho}\right)_S \frac{\partial(\rho_{s0}x_i)}{\partial x_i}\frac{\partial(\rho_{n0}x_j)}{\partial x_j} + \left(\frac{\partial T}{\partial \rho}\right)_S \frac{\partial(\rho_{s0}x_i)}{\partial x_i}\frac{\partial(S_0x_j)}{\partial x_j} \right\}. \tag{94}$$

Since we generally have six variational parameters (one for each Cartesian component of the two displacement fields), the collective-mode frequencies are found from the six coupled algebraic equations $\delta L^{(2)}/\delta a_{si}=0$ and $\delta L^{(2)}/\delta a_{ni}=0$. We find

$$M_{si}\omega^2 a_{si} = \frac{1}{2}\sum_j [(k_{s,ij}+k_{s,ji})a_{sj} + 2k_{sn,ij}a_{nj}], \tag{95}$$

$$M_{ni}\omega^2 a_{ni} = \frac{1}{2}\sum_j [(k_{n,ij}+k_{n,ji})a_{nj} + 2k_{sn,ji}a_{sj}]. \tag{96}$$

In experiments performed on trapped superfluid gases, one usually has an axisymmetric trap such that, for instance, $\omega_1 = \omega_2 \equiv \omega_\perp$ and $\omega_3 = \omega_z$ and the modes of interest are the radial and axial breathing modes (see, for example, Refs. [9,10] which deal with Fermi gases close to unitarity). These correspond to $a_{s1}=a_{s2}$ and $a_{n1}=a_{n2}$.

VII. HYDRODYNAMIC THEORY AT $T=0$

Since the variational principle given by Eqs. (57), (61), and (62) involves some fairly complex expressions, it is useful to consider the case of zero temperature where the Lagrangian simplifies tremendously. At $T=0$, the Landau two-fluid equations reduce to a single hydrodynamic differential equation that was first used by Pitaevskii and Stringari [28] to derive corrections to the mean-field Gross-Pitaevskii results [29] for the collective-mode frequencies in a dilute Bose gas.

A hydrodynamic description means that a few local variables are sufficient to describe the dynamics. While such a description requires rapid collisions and local equilibrium to be valid in the case of a normal fluid, a hydrodynamic description of a superfluid is always correct. Thus it is no suprise that the $T\to 0$ limit of two-fluid hydrodynamics gives the correct quantum description of the pure superfluid (see discussion on p. 170 of Ref. [6]).

At $T=0$, only the superfluid component persists and we have $S=0$, $\rho_{n0}=0$, and $\rho_{s0}=\rho_0$. Equations (57)–(59) then reduce to

$$L^{(2)}[\mathbf{u}_s] = \frac{1}{2}\int d^3r \left\{ \rho_0 \mathbf{u}_s^2 \omega^2 - \left(\frac{\partial \mu}{\partial \rho}\right)[\nabla \cdot (\rho_0 \mathbf{u}_s)]^2 \right\}. \tag{97}$$

From $\delta L^{(2)}/\delta \mathbf{u}_s = \mathbf{0}$, one obtains

$$\omega^2 \mathbf{u}_s = -\nabla\left[\left(\frac{\partial \mu}{\partial \rho}\right) \nabla \cdot (\rho_0 \mathbf{u}_s) \right]. \tag{98}$$

Using the $T=0$ linearized continuity equation $\delta\rho = -\nabla\cdot(\rho_0\mathbf{u}_s)$, Eq. (98) can be rewritten as

$$\omega^2 \mathbf{u}_s = \nabla\left[\left(\frac{\partial \mu}{\partial \rho}\right) \delta\rho \right]. \tag{99}$$

Multiplying both sides of this expression by ρ_0 and taking the divergence, we obtain a closed equation for the density fluctuations,

$$\omega^2 \delta\rho = -\nabla \cdot \left[\rho_0 \nabla \left\{ \left(\frac{\partial \mu}{\partial \rho}\right) \delta\rho \right\} \right], \tag{100}$$

which is the basis of the $T=0$ hydrodynamic theory developed by Pitaevskii and Stringari [28]. Equation (100) describes the low-energy collective modes of both atomic Bose and two-component Fermi superfluid gases at $T=0$. The only difference between the two quantum gases lies in the equation of state $\mu(\rho)$.

Our variational approach should give expressions for the collective-mode frequencies at $T=0$ that agree with results derived from solving Eq. (100) directly (see, for example, Ref. [28] for Bose gases and Refs. [13,14,16] for Fermi gases close to unitarity). However, it is still useful to show explictly how our formalism reproduces the $T=0$ results obtained in the recent literature. As a specific application, we consider the breathing modes which were discussed at finite T in the previous section. The breathing modes of trapped Fermi gases close to unitarity have been studied extensively at $T=0$ [13–19], where a simple *Ansatz* for the density dependence of the chemical potential $\mu(\rho)$ allows one to obtain simple expressions for the frequencies of these modes. We show that in the limit of zero temperature, our general results for the breathing-mode frequencies at finite T, given by Eqs. (95) and (96), reproduce these well-known expressions.

Since the normal fluid vanishes at zero temperature, we only have equations for the superfluid component, given by Eq. (95), which reduce to

$$M_{si}\omega^2 a_{si} = \sum_j k_{s,ij} a_{sj} \tag{101}$$

at $T=0$, where k_n and k_{sn} are zero. The expression for $k_{s,ij}$, given by Eq. (92), becomes

$$k_{s,ij} = \int d^3r \left(\frac{\partial \mu}{\partial \rho}\right) \frac{\partial(\rho_0 x_i)}{\partial x_i} \frac{\partial(\rho_0 x_j)}{\partial x_j}, \tag{102}$$

since $\rho_{s0}=\rho_0$ at $T=0$.

To obtain a simple analytic expression for the collective-mode frequencies, the following identity, found by integration by parts, will prove useful:

$$\int d^3r \left(\frac{\partial \mu}{\partial \rho}\right) \rho_0 x_j \frac{\partial \rho_0}{\partial x_j} = -\frac{1}{2} \int d^3r \rho_0^2 \left\{ \left(\frac{\partial \mu}{\partial \rho}\right) + x_j \frac{\partial}{\partial x_j} \left(\frac{\partial \mu}{\partial \rho}\right) \right\}. \tag{103}$$

Expanding out the derivatives in Eq. (102) and using Eq. (103) and (102) can be rewritten as

$$k_{s,ij} = \int d^3r \left\{ \left(\frac{\partial \mu}{\partial \rho}\right) \frac{\partial \rho_0}{\partial x_i} \frac{\partial \rho_0}{\partial x_j} x_i x_j - \frac{\rho_0^2}{2} \left(x_j \frac{\partial}{\partial x_j} + x_i \frac{\partial}{\partial x_i} \right) \left(\frac{\partial \mu}{\partial \rho}\right) \right\}. \tag{104}$$

At $T=0$, Eq. (83) reduces to

$$\left(\frac{\partial \mu}{\partial \rho}\right) \frac{\partial \rho_0}{\partial x_i} = -\omega_i^2 x_i. \tag{105}$$

Using this result in Eq. (104) and integrating by parts, we find

$$k_{s,ij} = \int d^3r \rho_0 \left\{ (2\delta_{ij}+1)\omega_i^2 x_i^2 - \frac{\rho_0}{2} x_j \frac{\partial}{\partial x_j}\left(\frac{\partial \mu}{\partial \rho}\right) - \frac{\rho_0}{2} x_i \frac{\partial}{\partial x_i}\left(\frac{\partial \mu}{\partial \rho}\right) \right\}. \tag{106}$$

Assuming a so-called *polytropic* equation of state (see, for example, Refs. [14,16,17,19,39,40]),

$$\mu[\rho] \propto \rho^{\gamma}, \tag{107}$$

we find

$$\rho_0 \frac{\partial}{\partial x_i}\left(\frac{\partial \mu}{\partial \rho}\right) = (\gamma-1)\left(\frac{\partial \mu}{\partial \rho}\right)\frac{\partial \rho_0}{\partial x_i} = -(\gamma-1)\omega_i^2 x_i. \tag{108}$$

Substituting this relation into Eq. (106) and making use of the definition of the weighted mass M_{si} given by Eq. (91), we obtain the following expression for the ratio of the spring constant and the weighted mass:

$$\frac{k_{s,ij}}{M_{si}} = 2\omega_i^2\delta_{ij} + \left[\frac{(\gamma+1)}{2}\omega_i^2 + \chi_{ji}\frac{(\gamma-1)}{2}\omega_j^2\right], \tag{109}$$

where

$$\chi_{ji} = \frac{\int d^3r \rho_0(\mathbf{r}) x_j^2}{\int d^3r \rho_0(\mathbf{r}) x_i^2}. \tag{110}$$

Using the Thomas-Fermi result $\mu_0=\mu[\rho_0(\mathbf{r})]+U_{\rm ext}(\mathbf{r})$, the equilibrium density ρ_0 can be expressed as a function of $U_{\rm ext}(\mathbf{r})=(1/2)(\omega_1^2x_1^2+\omega_2^2x_2^2+\omega_3^2x_3^2)$. Making a change of variables, $x_i'=\omega_i x_i$, χ_{ji} becomes

$$\chi_{ji} = \frac{\omega_i^2}{\omega_j^2} \frac{\int d^3r' \rho_0[x_1'^2+x_2'^2+x_3'^2]x_j'^2}{\int d^3r' \rho_0[x_1'^2+x_2'^2+x_3'^2]x_i'^2} = \frac{\omega_i^2}{\omega_j^2}. \tag{111}$$

With this result, Eq. (109) simplifies to

$$\frac{k_{s,ij}}{M_{si}} = 2\omega_i^2\delta_{ij} + \gamma\omega_i^2. \tag{112}$$

Combining this identity with Eq. (101), we find

$$\omega^2 a_{si} = \sum_j (2\omega_i^2\delta_{ij} + \gamma\omega_i^2) a_{sj} = 2\omega_i^2 a_{si} + \gamma\omega_i^2 \sum_j a_{sj}, \tag{113}$$

which is equivalent to Eq. (3) in Ref. [19] when a polytropic equation of state is used.

For an axisymmetric trap, $\omega_1=\omega_2=\omega_\perp$, $\omega_3=\omega_z$, and the axial and longitudinal breathing modes are characterized by solutions of the form $a_1=a_2=a$. In this case, Eq. (113) gives the normal-mode frequencies found by previous authors [14,16,19,40],

$$\omega^2 = \frac{1}{2}[2(\gamma+1)\omega_\perp^2 + (\gamma+2)\omega_z^2 \pm \sqrt{[2(\gamma+1)\omega_\perp^2 - (\gamma+2)\omega_z^2]^2 + 8\gamma^2\omega_z^2\omega_\perp^2}]. \tag{114}$$

VIII. CONCLUDING REMARKS

This paper has described a variational procedure for finding the solutions of the Landau two-fluid hydrodynamic equations for trapped quantum gases. These describe *all* superfluids with a two-component order parameter in terms of coupled differential equations for the local hydrodynamic variables and the velocity fields of the two fluids. The coefficients in these equations are given in terms of equilibrium thermodynamic functions. Thus, the only differences between the hydrodynamics of various superfluids are the values of these thermodynamic functions. These depend on the thermal excitations which are appropriate for different systems (superfluid helium, BCS superconductors, trapped Bose gases, etc.). The two-fluid equations for a BCS superfluid have been derived by Galasiewicz [41].

The usual derivation [3] of the Landau two-fluid hydrodynamic equations is based on conservation laws and macroscopic considerations, with key equations describing the irrotational superfluid velocity field $\mathbf{v}_s(\mathbf{r},t)$ being postulated. There is a considerable literature on this approach, which makes little contact with any microscopic theory of the origin of the superfluid component. Since the 1960s, however, it has generally been accepted that this originates from Bose-Einstein condensation (for a review, see Ref. [42]).

In the context of *trapped* atomic gas superfluids, an important development was the work of Zaremba, Nikuni, and Griffin (ZNG) in 1999 [7], who discussed a simple microscopic model of a condensate (described by a generalized Gross-Pitaevskii equation) coupled to a thermal gas of atoms moving in a self-consistent Hartree-Fock (HF) field (described by a kinetic Boltzmann equation generalized to include the Bose statistical factors). The ZNG coupled equations for the condensate (superfluid) and noncondensate (normal fluid) have been used to discuss the collisionless (see, for example, Ref. [43] and references therein) as well as the collision-dominated regions. In the latter region, ZNG showed that in the limit that collisions establish *complete* local equilibrium ($\omega\tau_R \to 0$), their equations can be reduced precisely to the Landau two-fluid equations, with all thermodynamic functions being given explicitly for a trapped gas of thermal atoms with a HF spectrum.

More than 60 years after Tisza [1] and Landau [2] proposed two-fluid hydrodynamics, their theory turns out to be very relevant in the study of the dynamics of trapped Fermi superfluid gases with strong interactions, close to unitarity. From linearized two-fluid hydrodynamics, one can obtain collective-mode frequencies in terms of equilibrium thermodynamic functions and thermodynamic derivatives. This is well known for uniform superfluids and still occurs in the case of trapped gases, apart from the fact that the equilibrium thermodynamic quantities are now dependent on position. This means that the two-fluid predictions are general, with a microscopic theory only being needed for the elementary excitations that determine the equilibrium thermodynamic functions.

Variational approaches have already been used extensively to obtain hydrodynamic collective-mode frequencies of superfluid Fermi gases at $T=0$ close to unitarity [15,17,19], as well as boson-fermion mixtures [44]. In this paper, we have extended this approach to finite temperatures to deal with the coupled hydrodynamics of a superfluid and normal fluid in a trapped gas.

We have presented a variational method to determine the collective-mode frequencies of a trapped superfluid, based on an action of the type first proposed by Zilsel in 1950 to derive Landau two-fluid hydrodynamics. By making an *Ansatz* for the superfluid and normal-fluid velocity fields, in this formulation, the problem of determing the collective-mode frequencies is reduced to solving a set of algebraic equations. This obviates the need to solve the Landau two-fluid differential equations numerically, a difficult task in a nonuniform trapped gas.

We illustrated our formalism by considering a uniform gas, as well as the dipole and breathing modes for a harmonically confined gas. To make contact with recent discussions in the literature on trapped superfluid Fermi gases, we also used our finite-T variational approach to discuss the collective modes in the limiting case of a pure superfluid at $T=0$ [13,28].

In this paper, we have only discussed how to determine the normal modes of the nondissipative Landau two-fluid equations for a trapped (nonuniform) superfluid. The damping of these modes due to various hydrodynamic relaxation processes can be calculated using the general expressions derived by Nikuni and Griffin [45].

In future work, we will apply the expressions derived in this paper and present detailed numerical predictions for the frequencies of various hydrodynamic modes in trapped superfluid gases at finite temperatures. In particular, we will discuss results in a Fermi superfluid for a strongly interacting molecular BEC at finite T (the BEC side of a Feshbach resonance).

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APPENDIX A: COMPARISON WITH ZILSEL'S VARIATIONAL PRINCIPLE

As discussed in the text, our major interest in this paper is to find an efficient way of calculating the collective-mode frequencies of a trapped superfluid in the region where Landau's two-fluid hydrodynamics are valid. Following Zilsel [27], we did this in Secs. II–IV by reformulating the two-fluid equations in terms of a variational theory involving the action. There is an extensive older literature on such a variational formulation in the context of superfluid helium. In this appendix, we review in more detail some of the subtle issues which arose in connection with Zilsel's seminal paper and their resolution.

One criticism concerned Zilsel's use of $x=\rho_n/\rho$ as an independent variable which, it was argued, conflicted with the Landau formalism [46]. Zilsel was ultimately vindicated by Jackson who demonstrated that Zilsel's approach was completely equivalent to those proposed by some of his critics [33].

In our approach to the two-fluid thermodynamics, using the internal energy density given by Eq. (6), we have deviated slightly from the variational method devised by Zilsel to derive the LTF equations. He considered the *specific* internal energy, which we denote by U_Z to avoid confusion. The two energies are related by $U=\rho U_Z$. Furthermore, while we have taken the internal energy density to be a function of the entropy S, the total density ($\rho=\rho_n+\rho_s$), and the normal-fluid density ρ_n, Zilsel took the specific internal energy to be a function of the entropy density $\sigma=S/\rho$, ρ, and the ratio $x=\rho_n/\rho$. However, one can show that the thermodynamics given in Sec. II, obtained from the Landau-Khalatnikov energy density, is identical to Zilsel's and our choice of independent variables is completely consistent with Zilsel's choice. To compare our thermodynamic identities with

Zilsel's, using Eqs. (6) and (7), with $U=\rho U_Z$ and, hence, $dU=\rho dU_Z+U_Z d\rho$, we find

$$dU_Z = Td\sigma + \frac{P}{\rho^2}d\rho + \frac{1}{2}(\mathbf{v}_n - \mathbf{v}_s)^2 dx, \quad \text{(A1)}$$

where $\sigma=S/\rho$ and $x=\rho_n/\rho$. This relation gives precisely the thermodynamic relations used by Zilsel in his derivation of the LTF equations. As well, Eq. (A1) was derived by Jackson using thermodynamic arguments [33]. Zilsel chose to use σ, ρ, and x as independent variables as these differentials appear in Eq. (A1). In contrast, with the internal energy density given by Eq. (6), the differentials of S, ρ, and ρ_n are the natural choice for independent variables.

A more valid criticism directed at Zilsel's work concerned the fact that, while he obtained the full LTF equations, his variational principle forced the normal-fluid velocity to be irrotational when S/ρ_n is a constant. Thus, Zilsel's variational principle only describes a subset of all possible solutions of the LTF equations. Following an argument originally due to Lin, this deficiency can be corrected by introducing an additional constraint, given by Eq. (14), into the variational principle that accounts for conservation of circulation (or vorticity) in the normal fluid [32–34,46–48]. To see this explicitly, consider Eq. (23). Without including the circulation constraint, equivalent to setting $\eta=0$, we recover Zilsel's expression for the normal-fluid velocity. It is apparent from Eq. (23) that for the case when the entropy divided by the normal-fluid density is uniform, the normal-fluid velocity is irrotational. By including the circulation constraint, the difference between the normal-fluid and superfluid velocities multiplied by ρ_n/S assumes the form of a Clebsch transformation [49], with the local functions η and γ acting as the Clebsch potentials. In this case, the normal-fluid velocity is no longer restricted.

APPENDIX B: ZNG vs ZGN HYDRODYNAMICS

We briefly summarize some key features of the derivation of two-fluid hydrodynamics for a trapped Bose gas. As discussed in detail in Refs. [7,50], the ZNG microscopic coupled equations involve in a crucial way a term Γ_{12} which is related to the collisionless transfer of atoms between the condensate and thermal cloud. In the linearized ZNG formalism, one can show that in local equilibrium,

$$\delta\Gamma_{12} = -\frac{n_{c0}}{k_B T\tau_{12}}\delta\mu_{\text{diff}}. \quad \text{(B1)}$$

This involves the difference in the chemical potentials of the condensate (μ_c) and the thermal cloud ($\tilde{\mu}$):

$$\mu_{\text{diff}} = \tilde{\mu} - \mu_c. \quad \text{(B2)}$$

In static equilibrium, Γ_{12} vanishes and $\tilde{\mu}_0=\mu_{c0}$, assuming that $\mathbf{v}_{c0}=\mathbf{v}_{n0}=\mathbf{0}$. In Eq. (B1), the relaxation time τ_{12} describes collisions between atoms in the condensate and thermal gas components. The relaxation time τ_μ introduced in Ref. [7], which determines the rate at which local diffusive equilibrium is achieved ($\tilde{\mu}=\mu_c$), is given by $\tau_\mu=\sigma_H\tau_{12}$, where σ_H is a dimensionless hydrodynamic normalization factor defined in Ref. [7] involving local equilibrium thermodynamic functions.

ZNG derive the nondissipative Landau two-fluid equations in the limit of $\omega\tau_\mu \ll 1$. In this limit, one finds $\delta\mu_{\text{diff}} \to 0$, but $\delta\Gamma_{12}$ in Eq. (B1) remains finite and is given by

$$\delta\Gamma_{12}(\mathbf{r},t) = \sigma_H\left\{\boldsymbol{\nabla}\cdot[n_c(\mathbf{v}_c-\mathbf{v}_n)] + \frac{1}{3}n_c\,\boldsymbol{\nabla}\cdot\mathbf{v}_n\right\}. \quad \text{(B3)}$$

As discussed in Ref. [7], this source term in the continuity equations (34) and (35) plays a crucial role in establishing the equivalence between the ZNG equations (in the limit that $\omega\tau_\mu\to 0$) and the standard Landau two-fluid equations considered in the text of this paper. Thus, Γ_{12} is a concrete example of the source term $\Gamma(=m\Gamma_{12})$ which arises in Eqs. (33) and (34), in the context of a dilute Bose gas.

This is a convenient place to note that in Refs. [7,24,51], another version of the two-fluid hydrodynamics was discussed in which the collisional transfer of atoms between the condensate and thermal cloud was ignored. This corresponds to setting $\Gamma_{12}=0$ or, equivalently, the limit $\omega\tau_\mu \gg 1$. This "ZGN" two-fluid hydrodynamics was used to calculate the temperature dependence of the frequencies of first and second sound in a weakly interacting uniform Bose gas [51]. In this case, the velocities were almost identical (see Fig. 1 of Ref. [7]) to those based on the Landau two-fluid hydrodynamics ($\omega\tau_\mu \ll 1$ limit). Expressions for the two sound velocities based on the LTF equations are given in Appendix C of Ref. [50], showing the small corrections which depend on the factor σ_H in Eq. (B3).

The variational theory developed in Ref. [7] for determining the normal-mode oscillations in a trapped gas was worked out originally for the ZGN two-fluid hydrodynamics [24]. The results for monopole, dipole, and quadrupole oscillations are summarized in Fig. 1 of Ref. [24] and Figs. 4 and 5 of Ref. [7]. For the dipole modes, the frequencies are identical to Eqs. (86) and (87), but with

$$k_{sn} = -2g\int d^3r\frac{\partial n_{c0}}{\partial x_i}\frac{\partial \tilde{n}_0}{\partial x_i}, \quad \text{(B4)}$$

where $g=4\pi\hbar^2 a/m$. It can be shown, however, making use of approximations consistent with those used to derive ZGN hydrodynamics, that Eq. (79) reduces to Eq. (B4). The details of this derivation will be given elsewhere.

The variational approach of Ref. [7] was formally extended to deal with the ZNG hydrodynamics in Ref. [45], but it was not used to calculate the collective-mode frequencies. Instead, the out-of-phase mode was approximated by extending the $T=0$ pure condensate mode to finite T by using a reduced condensate density. The in-phase solution was approximated at finite T by the thermal cloud oscillation above T_c, but with a reduced thermal cloud density. As discussed in Sec. VII of Ref. [7], these are often good first approximations.

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