# Landau damping of Bogoliubov excitations in two- and three-dimensional optical lattices at finite temperatures

Shunji Tsuchiya[1,2,*] and Allan Griffin[1,†]
[1]*Department of Physics, University of Toronto, Toronto, Ontario, Canada M5S 1A7*
[2]*Department of Physics, Waseda University, 3-4-1 Okubo, Tokyo 169-8555, Japan*
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We study the Landau damping of Bogoliubov excitations in two- and three-dimensional optical lattices at finite temperatures, extending our recent work on one-dimensional (1D) optical lattices. We use a Bose-Hubbard tight-binding model and the Popov approximation to calculate the temperature dependence of the number of condensate atoms $n^{c0}(T)$ in each lattice well. As with 1D optical lattices, damping only occurs if the Bogoliubov excitations exhibit anomalous dispersion (i.e., the excitation energy bends upward at low momentum), analogous to the case of phonons in superfluid $^4$He. This leads to the disappearance of all damping processes in a $D$-dimensional simple cubic optical lattice when $Un^{c0} \geqslant 6DJ$, where $U$ is the on-site interaction, and $J$ is the hopping matrix element.

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## I. INTRODUCTION

Bose condensates in periodic optical lattices have attracted much attention recently. However the effect of a thermal cloud on the condensate excitations in an optical lattice have not been studied very much. In a recent paper [1], we have given a detailed analysis of the Landau damping of the Bogoliubov excitations in a one-dimensional (1D) optical lattice. In the present paper, we extend these calculations to two-dimensional (2D) and three-dimensional (3D) optical lattices. As in Ref. [1], we use a Bose-Hubbard tight-binding model, calculating the temperature dependence of the condensate atom number $n^{c0}(T)$ in each lattice well using the static Popov approximation. As discussed in Ref. [1], for damping processes to occur, the dispersion relation of the Bloch-Bogoliubov excitations $E_q$ must initially bend upward as the quasimomentum $q$ increases. This is referred to as "anomalous dispersion" and is also the source of three-phonon damping of long wavelength phonons in superfluid $^4$He [2,3]. This condition leads to a dramatic disappearance of all damping processes of phonon modes in a $D$-dimensional optical lattice when $\alpha \equiv Un^{c0}/J > 6D$, where $U$ is the on-site interaction and $J$ is the hopping matrix element.

Recently, several experimental papers have reported results on the collective modes of Bose condensates in a one-dimensional optical lattice [4–7] and its damping at finite temperature [4]. In these experiments, a shift of the oscillation frequency in the presence of the optical lattice and a sharp change of the damping rate with increasing depth of the optical lattice have been observed. The measured frequency shift [5] of the collective modes is in good agreement with the renormalized mass theory of Krämer *et al.* [8]. The damping of condensate oscillations in a 1D optical lattice at $T=0$ has been measured [6]. However, damping of the Bogoliubov excitations at finite $T$ has not been studied in any detail.

In Sec. II we briefly recall the well-known Bose-Hubbard tight-binding model for Bose gases in an optical lattice, and give the dispersion relation for the excitations. We present results for the condensate fraction as a function of the optical lattice depth and temperature. In Sec. III, we discuss the Landau damping of Bogoliubov excitations in 2D and 3D optical lattices, and compare it with the results for a 1D optical lattice discussed in Ref. [1]. We also briefly discuss Beliaev damping in a 1D optical lattice.

## II. EXCITATIONS IN A TIGHT-BINDING MODEL AT FINITE $T$

We consider bosonic atoms in an optical lattice potential

$$V_{\mathrm{op}}(\mathbf{r}) = sE_R\sum_{i=1}^{D} \sin^2(kx_i), \tag{1}$$

where $s$ is the usual dimensionless parameter describing the optical lattice depth in units of the photon recoil energy $E_R \equiv \hbar^2k^2/2m$. $D$ is the dimension of the optical lattice and $d=\pi/k=\lambda/2$ is the lattice period. We only consider simple cubic lattices considered in recent experiments [9,10]. We call attention to the recent technique [6] of producing a two-dimensional array of long, tightly confined condensate tubes by loading a Bose condensate into a deep 2D optical lattice potential, which prevents atoms from hopping between different tubes. With an additional 1D optical lattice potential along a tube, an ideal 1D system can be experimentally realized [6]. One can also have an ideal 2D system by loading a condensate into a deep 1D optical lattice and a shallow 2D optical lattice. We assume this experimental setup for the realization of 1D and 2D optical lattices in the present paper. We also assume that the laser intensity determining the depth of the optical lattice wells is large enough to make the atomic

*Present address: Institute of Industrial Science, University of Tokyo, 4-6-1 Komaba, Meguro-ku, Tokyo 153-8505, Japan. Electronic address: stuchiya@iis.u-tokyo.ac.jp
†Electronic address: griffin@physics.utoronto.ca

 

wave functions well localized on the individual sites (i.e., where we can use the tight-binding approximation). The energy gap between the first and the second excitation bands is large compared to the thermal energy ($2k_{\mathrm{B}}T/E_R \ll s$), and thus only the first band is thermally occupied.

Within a tight-binding approximation, the Hamiltonian is effectively described by the Bose-Hubbard model [11,12] as

$$H = -J\sum_{\langle j,l\rangle}(a_j^\dagger a_l + a_l^\dagger a_j) + \frac{1}{2}U\sum_j a_j^\dagger a_j^\dagger a_j a_j, \tag{2}$$

where $a_j$ and $a_j^\dagger$ are destruction and creation operators of atoms on the $j$th lattice site. $\langle j,l\rangle$ represents nearest-neighbor pairs of lattice sites. The first term describes the kinetic energy due to the hopping of atoms between sites. The hopping matrix element $J$ is given by

$$J = -\int d\mathbf{r} w_j^*(\mathbf{r})\left(-\frac{\hbar^2\nabla^2}{2m} + V_{\mathrm{op}}(\mathbf{r})\right)w_l(\mathbf{r}), \tag{3}$$

where $w_j(\mathbf{r})$ is a wave function localized on the $j$th lattice site, and $m$ is the atomic mass. Expanding the optical lattice potential around the minima of the potential wells, the well trap frequency is $\omega_s \equiv s^{1/2}(\hbar k^2/m)$. Approximating the localized function as the ground state wave function of a harmonic oscillator with frequency $\omega_s$ at the potential minima of $j$th site

$$w_j(\mathbf{r}) = \left(\frac{m\omega_s}{\pi\hbar}\right)^{D/4}\exp\left(-\frac{m\omega_s}{2\hbar}(\mathbf{r}-\mathbf{r}_j)^2\right), \tag{4}$$

one obtains

$$\frac{J}{E_R} \sim \left[\left(\frac{\pi^2 s}{4} - \frac{s^{1/2}}{2}\right) - \frac{1}{2}s[1+\exp(-s^{-1/2})]\right]e^{-\pi^2 s^{1/2}/4}. \tag{5}$$

Here $s^{1/2} \sim [\sqrt{2m(sE_R)}/\hbar]d$ can be interpreted as a WKB factor for tunneling in an optical lattice potential which has height $sE_R$ and well width $d$.

The second term in Eq. (2) describes the interaction between atoms when they are at the same site. We assume that atoms can move along the $z$ direction in the 1D case and in $xy$ plane in the 2D case. The on-site interaction $U$ depends on the dimensionality of the optical lattice. $U$ is given by [12]

$$U = g\int d\mathbf{r}_\perp|\phi_\perp(\mathbf{r}_\perp)|^4\int dz|w_j(z)|^4, \quad (1\mathrm{D}) \tag{6}$$

$$= g\int dz|\phi_\parallel(z)|^4\int dxdy|w_j(x,y)|^4, \quad (2\mathrm{D}) \tag{7}$$

$$= g\int d\mathbf{r}|w_j(\mathbf{r})|^4, \quad (3\mathrm{D}) \tag{8}$$

where $g = 4\pi\hbar^2 a/m$ and $a$ is the $s$-wave scattering length. Here $\phi_\perp(\mathbf{r}_\perp) = (m\omega_\perp/\pi\hbar)^{1/2}\exp(-(m\omega_\perp/2\hbar)\mathbf{r}_\perp^2)$ and $\phi_\parallel(z) = (m\omega_\parallel/\pi\hbar)^{1/4}\exp(-(m\omega_\parallel/2\hbar)z^2)$ are the ground state wave functions in optical lattice well traps for confining atoms in 1D and 2D. Approximating the localized function $w_j$ as a simple Gaussian, one obtains [12]

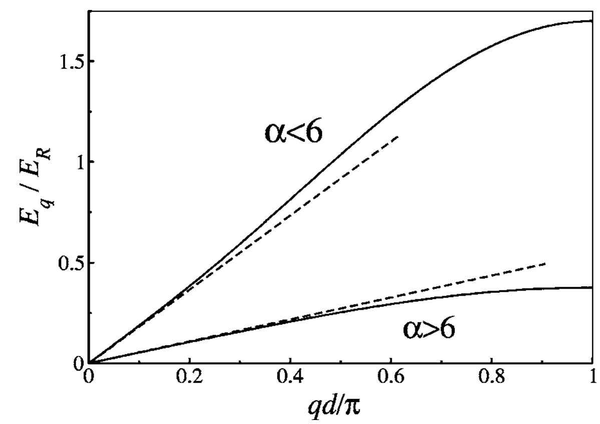


FIG. 1. The Bogoliubov excitation spectrum $E_q$ in a 1D optical lattice plotted as a function of the quasimomentum $q$ in the first Brillouin zone, for $\alpha < 6$ and $\alpha > 6$. In contrast with Fig. 1 of [1], $E_q$ is normalized to the recoil energy instead of $J$.

$$\frac{U}{E_R} \sim \frac{g}{(2\pi)^{3/2}a_\perp^2 a_s} = \frac{2^{3/2}ad}{\pi^{3/2}a_\perp^2}s^{1/4}, \quad (1\mathrm{D}) \tag{9}$$

$$\sim \frac{g}{(2\pi)^{3/2}a_\parallel a_s^2} = \frac{2^{3/2}a}{\pi^{1/2}a_\parallel}s^{1/2}, \quad (2\mathrm{D}) \tag{10}$$

$$\sim \frac{g}{(2\pi)^{3/2}a_s^3} = \frac{2^{3/2}\pi^{1/2}a}{d}s^{3/4}, \quad (3\mathrm{D}) \tag{11}$$

where $a_s = \sqrt{\hbar/m\omega_s}$, $a_\perp = \sqrt{\hbar/m\omega_\perp}$, and $a_\parallel = \sqrt{\hbar/m\omega_\parallel}$.

The Bogoliubov excitation spectrum for a uniform optical lattice is easily calculated for the model discribed by Eq. (2) [13–15], as summarized in Ref. [1] for a 1D optical potential. This discussion is easily extended to 2D and 3D optical lattices: The Bloch-Bogoliubov excitation energy in a $D$-dimensional tight-binding optical lattice is given by

$$E_{\mathbf{q}} = \sqrt{\epsilon_{\mathbf{q}}^0(\epsilon_{\mathbf{q}}^0 + 2Un^{c0})}, \tag{12}$$

where

$$\epsilon_{\mathbf{q}}^0 \equiv 4J\sum_{i=1}^{D}\sin^2\frac{q_i d}{2} \tag{13}$$

and $n^{c0}$ denotes the number of condensate atoms trapped in each well of the optical lattice.

We call attention to an important feature of the dispersion relation $E_q$ in Fig. 1, considered as a function of the dimensionless interaction parameter

$$\alpha \equiv \frac{Un^{c0}}{J}. \tag{14}$$

For $\alpha \leq 6$, the excitation energy $E_q$ bends up before bending over, as $q$ approaches the Brillouin zone (BZ) boundary. This behavior is analogous to the so-called "anomalous dispersion" of the phonon spectrum in superfluid $^4$He [2,3,16]. For $\alpha > 6$, in contrast, the spectrum simply bends over as one leaves the low $q$ (phonon) region. The excitation spectrum in 2D and 3D optical lattices also exhibit this kind of spectrum. However, the critical value of $\alpha$ then depends on the direction of $\mathbf{q}$, since simple cubic optical lattices in 2D and 3D do not have rotational symmetry. The crucial effect of this anomalous dispersion on damping processes is calculated in Sec. III.

As discussed in Ref. [1], one can easily extend the usual $T=0$ excitation spectrum to finite $T$ within the standard Popov approximation [17,18]. This assumes that the noncondensate in each lattice well is always in thermal equilibrium, i.e., the dynamics is ignored. Within this approximation, $E_{\mathbf{q}}$ is identical to the $T=0$ result in Eq. (12), except that now $n^{c0}(T)$ is temperature-dependent [1].

Expressing the number of noncondensate atoms in a lattice site $\tilde{n}^0(T)$ in terms of Bogoliubov-Popov excitations, we have [1,13,14,17]

$$n = n^{c0} + \frac{1}{I^D}\sum_{\mathbf{q}\neq 0}\langle a^\dagger_{\mathbf{q}} a_{\mathbf{q}}\rangle = n^{c0} + \frac{1}{I^D}\sum_{\mathbf{q}\neq 0}[(u^2_{\mathbf{q}} + v^2_{\mathbf{q}})f^0(E_{\mathbf{q}}) + v^2_{\mathbf{q}}], \tag{15}$$

where $I^D$ is the total number of lattice sites. $u_{\mathbf{q}}$ and $v_{\mathbf{q}}$ are the standard Bogoliubov transformation functions [1]. $f^0(E_{\mathbf{q}}) = [\exp(\beta E_{\mathbf{q}}) - 1]^{-1}$ is the Bose distribution function. Apart from the limiting case of $\alpha \ll 1$, we must always use the *full* Bogoliubov spectrum $E_{\mathbf{q}}$ to describe the thermal cloud composed of excitations in the first band of an optical lattice. This is different from Bose gases trapped in harmonic potentials, where one can always use the Hartree-Fock approximation [19] for the excitations describing the thermal cloud as long as the kinetic energy of the atoms ($\sim k_B T$) is much larger than the interaction energy ($Un^{c0}$).

The number of condensate atoms $n^{c0}(T)$ at a site is found by solving Eq. (15) self-consistently for a fixed value of the total site density $n$. The condensate fraction $n^{c0}(T)/n$ in a $D$-dimensional optical lattice is shown in Fig. 2. We take $n=2$ and use the parameters from Ref. [6]. These experimental parameters are quite different from those used in the calculations reported in Ref. [1], which were based on the experiments in Ref. [4]. A 3D optical lattice used to produce 1D systems in Ref. [6] decreases the number of atoms per site compared to the experiments in Ref. [4] with a combined potential of a 1D optical lattice and a harmonic trap. The spurious finite jump in the condensate atom number $n^{c0}$ at the transition temperature $T_c$ is an inherent problem of the Bogoliubov theory in a uniform gas [20].

In Fig. 3, we plot the parameter $\alpha \equiv Un^{c0}(T)/J$ as a function of the temperature for several values of the optical depth $s$. The dimensionless interaction parameter $\alpha$ and the results in Fig. 3 will be very important in our analysis in Sec. III. Since we limit our discussion to the first energy band of the optical lattice, our results only apply when $s \gg 2k_B T/E_R$. Higher excitation bands would be thermally populated if we consider lower values of $s$ (weak optical lattices).

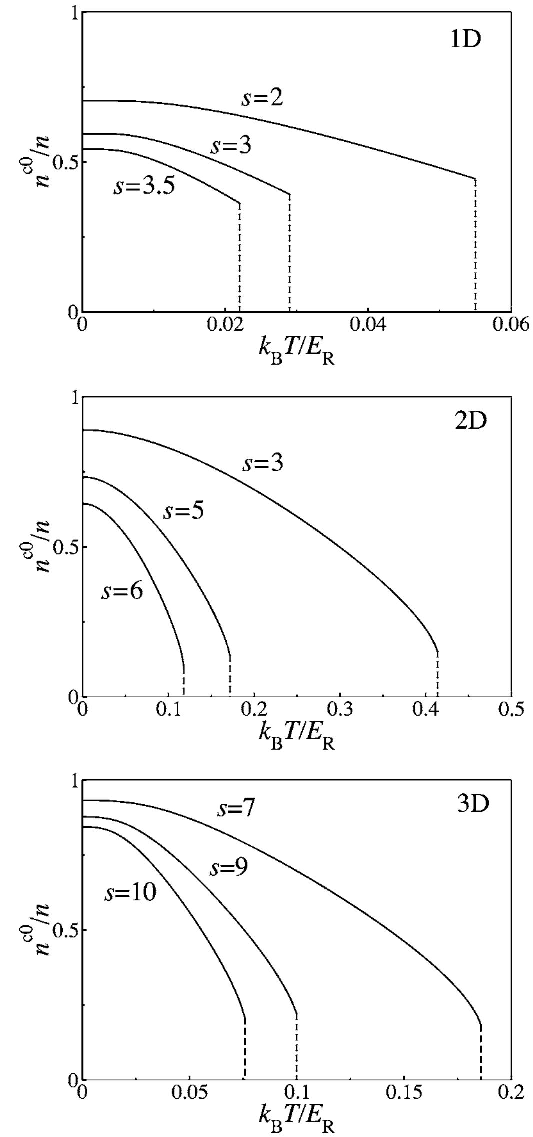


FIG. 2. The condensate fraction $n^{c0}/n$ in $D$-dimensional optical lattice as a function of temperature. The height of the optical lattice potential (in units of $E_R$) is always denoted by $s$. The experimental parameters used are from Ref. [6].

## III. DAMPING OF BOGOLIUBOV EXCITATIONS AT FINITE $T$

The damping of Bogoliubov excitations ($\omega_{\mathbf{q}} = E_{\mathbf{q}} - i\Gamma_{\mathbf{q}}$) in an optical lattice is given by

$$\Gamma_{\mathbf{q}} = \Gamma^L_{\mathbf{q}} + \Gamma^B_{\mathbf{q}}, \tag{16}$$

where

$$\Gamma^L_{\mathbf{q}} = \pi \sum_{\mathbf{p}_1,\mathbf{p}_2\neq 0} |M_{\mathbf{q},\mathbf{p}_2;\mathbf{p}_1}|^2 [f^0(E_{\mathbf{p}_2}) - f^0(E_{\mathbf{p}_1})]\delta(E_{\mathbf{q}} - E_{\mathbf{p}_1} + E_{\mathbf{p}_2}), \tag{17}$$

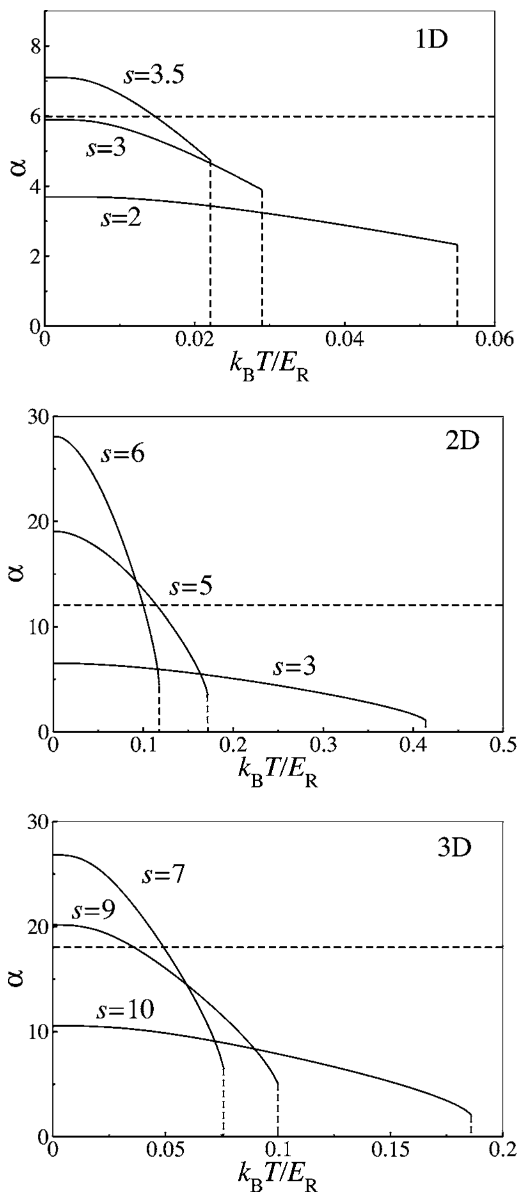


FIG. 3. The demensionless interaction parameter $\alpha = Un^{c0}(T)/J$, plotted as a function of temperature, for several values of the optical well depth $s$. The number of condensate atoms at a lattice site $n^{c0}(T)$ is given in Fig. 2. The dashed line shows the critical value $\alpha_c = 6D$, above which there is no damping.

$$\Gamma^B_{\mathbf{q}} = \frac{\pi}{2} \sum_{\mathbf{p}_1,\mathbf{p}_2 \neq 0} |M_{\mathbf{p}_1,\mathbf{p}_2;\mathbf{q}}|^2 [1 + f^0(E_{\mathbf{p}_1}) + f^0(E_{\mathbf{p}_2})] \times \delta(E_{\mathbf{q}} - E_{\mathbf{p}_1} - E_{\mathbf{p}_2}). \tag{18}$$

The matrix element is

$$M_{\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}_3} = 2U\sqrt{\frac{n^{c0}}{I^D}} \sum_{\mathbf{G}} [(u_{\mathbf{p}_1}u_{\mathbf{p}_3} + v_{\mathbf{p}_1}v_{\mathbf{p}_3} - v_{\mathbf{p}_1}u_{\mathbf{p}_3})u_{\mathbf{p}_2} - (u_{\mathbf{p}_1}u_{\mathbf{p}_3} + v_{\mathbf{p}_1}v_{\mathbf{p}_3} - u_{\mathbf{p}_1}v_{\mathbf{p}_3})v_{\mathbf{p}_2}]\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{G}}, \tag{19}$$

where $\mathbf{G}$ is a reciprocal lattice vector. The Kronecker delta $\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{G}}$ expresses the conservation of quasimomentum of the three excitation scattering processes (Umklapp processes are associated with $\mathbf{G} \neq 0$).

Landau damping $\Gamma^{\mathrm{L}}_{\mathbf{q}}$ in Eq. (17) is expected to be dominant at higher temperatures where there are thermally excited quasiparticles. In contrast, Beliaev damping $\Gamma^{\mathrm{B}}_{\mathbf{q}}$ in Eq. (18) is due to a decay process and can arise in the absence of thermally excited excitations. Beliaev damping is possible even at $T=0$ [i.e., $f^0(E_{\mathbf{p}})=0$] and is expected to be dominant at low temperatures.

### A. Landau damping in 2D and 3D optical lattices

The Landau damping has been calculated in Ref. [1] for a 1D optical lattice. This simple case is useful background to the analogous but more algebraically complicated calculations in the 2D and 3D optical lattices. The energy conservation condition $E_{\mathbf{q}} + E_{\mathbf{p}} = E_{\mathbf{q}+\mathbf{p}+\mathbf{G}} = E_{\mathbf{q}+\mathbf{p}}$ needs to be satisfied, where $\mathbf{G}$ is a reciprocal lattice vector. The solution of the energy conservation condition $E_q + E_p = E_{q+p}$ for a 1D optical lattice is illustrated in Fig. 4 of Ref. [1]. As discussed in Ref. [1], for $E_q + E_p = E_{q+p}$ to be satisfied, the dispersion relation $E_q$ must bend up as $q$ increases, before bending over. From Fig. 1, we see that the 1D optical lattice dispersion relation $E_q$ has this feature only for $\alpha < 6$ and thus Landau damping can occur only when $\alpha < 6$. If the solution $E_{q+p}$ is outside of the first Brillouin zone, it has to be reduced in the first Brillouin zone by subtracting a reciprocal lattice vector $G_m = 2\pi m/d$ ($m$ is an integer), corresponding to an Umklapp process.

We now discuss the analogous energy conservation condition in a 2D optical lattice. We imagine a surface in a three-dimensional space which satisfies the Bogoliubov dispersion relation $(q_x, q_y, E_{\mathbf{q}})$. Then, we draw a new dispersion $E_{\mathbf{q}}$ in the three-dimensional space, with a point on the surface $(q_{1x}, q_{1y}, E_{\mathbf{q}_1})$ as origin. That is, we draw $(q_x, q_y, E_{\mathbf{q}-\mathbf{q}_1} + E_{\mathbf{q}_1})$. If those two surfaces intersect, the energy conservation condition $E_{\mathbf{q}_1} + E_{\mathbf{q}_2} = E_{\mathbf{q}_1+\mathbf{q}_2}$ is satisfied, the intersection being given by $(q_{1x}+q_{2x}, q_{1x}+q_{2y}, E_{\mathbf{q}_1+\mathbf{q}_2})$. For this condition to be satisfied, the surface $(q_x, q_y, E_{\mathbf{q}})$ has to be above the other surface $(q_x, q_y, E_{\mathbf{q}-\mathbf{q}_1} + E_{\mathbf{q}})$ around $(q_{1x}, q_{1y}, E_{\mathbf{q}_1})$. Since the Bogoliubov spectrum is phononlike $E_{\mathbf{q}} \simeq cq$ for small $q$, the maximum gradient of $E_{\mathbf{q}}$ at $(q_{1x}, q_{1y}, E_{\mathbf{q}_1})$ must be greater than $c$. Since $|\nabla_{\mathbf{q}} E_{\mathbf{q}}|_{\mathbf{q}=\mathbf{q}_1}$ is the maximum gradient of $E_{\mathbf{q}}$ at $\mathbf{q}_1$, this condition is equivalent to the requirement:

$$|\nabla_{\mathbf{q}} E_{\mathbf{q}}|_{\mathbf{q}=\mathbf{q}_1} = 2Jd \frac{\tilde{E}_{\mathbf{q}_1}}{E_{\mathbf{q}_1}} \sqrt{\sin^2 q_{1x}d + \sin^2 q_{1y}d} > c. \tag{20}$$

$\tilde{E}_{\mathbf{q}} = \epsilon_{\mathbf{q}}^0 + Un^{c0}$ is the Hartree-Fock excitation spectrum. Equation (20) is the 2D version of the condition for the Bogoliubov spectrum of a 1D optical lattice to have anomalous dispersion. If Eq. (20) is satisfied, the energy conservation condition $E_{\mathbf{q}_1} + E_{\mathbf{q}_2} = E_{\mathbf{q}_1+\mathbf{q}_2}$ can be satisfied. An excitation $E_{\mathbf{q}_1}$ can then decay into $E_{\mathbf{q}_1+\mathbf{q}_2}$ by absorbing $E_{\mathbf{q}_2}$ (Landau), or an excitation $E_{\mathbf{q}_1+\mathbf{q}_2}$ can decay into two excitations $E_{\mathbf{q}_1}$ and $E_{\mathbf{q}_2}$ (Beliaev).

The condition for such a $\mathbf{q}_1$ to exist is that the maximum value of $|\nabla_{\mathbf{q}} E_{\mathbf{q}}|$ as a function of $(q_x, q_y)$ is greater than $c$. Due to $|\nabla_{\mathbf{q}} E_{\mathbf{q}}|_{\mathbf{q}=0}=c$, we only need to consider the condition that $|\nabla_{\mathbf{q}} E_{\mathbf{q}}|$ takes its maximum at $\mathbf{q}\neq 0$. When $|\nabla_{\mathbf{q}} E_{\mathbf{q}}|$ has its maximum value,

$$\nabla_{\mathbf{q}}|\nabla_{\mathbf{q}} E_{\mathbf{q}}| = \frac{2Jd^2 \sin q_x d}{\sqrt{\sin^2 q_x d + \sin^2 q_y d}} \times \left[ \frac{\tilde{E}_{\mathbf{q}}}{E_{\mathbf{q}}} \begin{pmatrix} \cos q_x d \\ \cos q_y d \end{pmatrix} - \frac{2JU^2 (n^{c0})^2}{E_{\mathbf{q}}^3} (\sin^2 q_x d + \sin^2 q_y d) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] = 0. \tag{21}$$

From Eq. (21), $|\nabla_{\mathbf{q}} E_{\mathbf{q}}|$ has its maximum value when $\cos q_x d = \cos q_y d$, i.e., at the two values $q_x = \pm q_y$. If we assume $q_x = q_y$ and define $u \equiv \sin^2 q_x d$, Eq. (21) reduces to

$$u = \sin^2 q_x d = \frac{-(3\alpha - 4) + \sqrt{5\alpha^2 + 24\alpha + 16}}{16}. \tag{22}$$

From Eq. (22), one sees that $u$ decreases as $\alpha$ increases, vanishing when $\alpha = 12$. Therefore we conclude that the Landau damping when $\alpha \lesssim 12$ is due to excitations with momentum $q_x = \pm q_y$, and all damping processes in a 2D optical lattice will vanish when $\alpha > 12$. We have confirmed this analytical result by numerically solving the energy conservation condition in Eq. (17).

The condition for the disappearance of Landau damping in a 3D optical lattice can be derived by generalizing the procedure described above for a 2D optical lattice. One can show that $|\nabla_{\mathbf{q}} E_{\mathbf{q}}|$ has its maximum when $\cos q_x d = \cos q_y d = \cos q_z d$, i.e., $q_x = \pm q_y = \pm q_z$ and

$$\sin^2 q_x d = \frac{-3(\alpha - 2) + \sqrt{5\alpha^2 + 36\alpha + 36}}{24}. \tag{23}$$

One finds that $u \to 0$ when $\alpha \to 18$, and damping in a 3D optical lattice vanishes when $\alpha \geqslant 18$. As in the 2D case, Landau damping when $\alpha \lesssim 18$ only occurs for an excitation with momentum $q_x = \pm q_y = \pm q_z$.

We next discuss the energy conservation condition in a 2D optical lattice in detail, restricting ourselves to the damping of a long wavelength phonon. Using the approximation for the long wavelength phonon $E_{\mathbf{q}} \sim cq$ and $E_{\mathbf{q}+\mathbf{p}} \sim E_{\mathbf{q}} + \nabla_{\mathbf{p}} E_{\mathbf{p}} \cdot \mathbf{q}$, the energy conservation condition $E_{\mathbf{q}} + E_{\mathbf{q}} = E_{\mathbf{q}+\mathbf{p}}$ can be written as

$$\frac{\alpha}{2}(q_x^2 + q_y^2) = \frac{\tilde{E}_{\mathbf{p}}^2}{E_{\mathbf{p}}^2} [\sin^2(p_x d) q_x^2 + 2 \sin(p_x d) \sin(p_y d) q_x q_y + \sin^2(p_y d) q_y^2]. \tag{24}$$

When $q_x > 0$ and $q_y = 0$, Eq. (24) can be solved easily. Defining $X \equiv \sin^2(p_x d/2)$ and $Y \equiv \sin^2(p_y d/2)$, the solution of Eq. (24) is

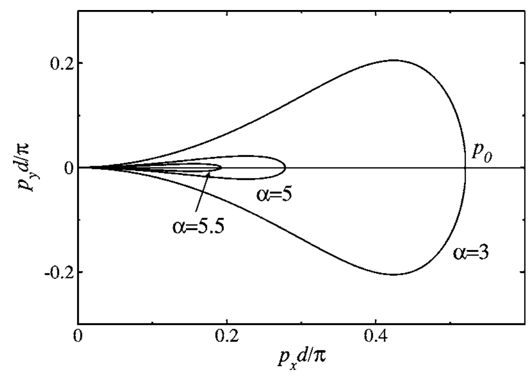


FIG. 4. The solution of the energy conservation condition $E_{\mathbf{q}} + E_{\mathbf{p}} = E_{\mathbf{q}+\mathbf{p}}$ for a 2D optical lattice, with $q_x > 0$ and $q_y = 0$. $p_0$ is defined in Eq. (26) and we note $p_0 \to 0$ as $\alpha \to 6$.

$$Y = -\left(X + \frac{\alpha}{4}\right) + \frac{1}{4}\sqrt{\frac{\alpha^3}{8X^2 - 8X + \alpha}}. \tag{25}$$

One can confirm that Eq. (25) reduces to the 1D result given in Ref. [1] when $Y = 0$, namely

$$\sin^2\left(\frac{p_0 d}{2}\right) = \frac{-(\alpha - 2) + \sqrt{2(\alpha + 2)}}{4}. \tag{26}$$

Equation (25) is plotted in Fig. 4 for several values of $\alpha$. We see that as $\alpha \to 6$ the line in the $(p_x, p_y)$ plane which satisfies the energy conservation condition shrinks and vanishes when $\alpha > 6$. Therefore the Landau damping of an excitation with $q_y = 0$ disappears when $\alpha > 6$.

For a long wavelength phonon $\mathbf{q}$ with $q_x = q_y > 0$, we solve the energy conservation condition numerically. The solution is shown in Fig. 5. There is no solution when $\alpha > 12$, as expected. Figures 4 and 5 clearly show that the threshold value of $\alpha$ for the disappearance of damping strongly depends on the direction of $\mathbf{q}$ due to the anisotropy of 2D square lattice. This result also holds for a 3D simple cubic optical lattice.

As in the 1D case, $\Gamma_{\mathbf{q}}^{L}$ becomes larger than $E_{\mathbf{q}}$ around the threshold value of $\alpha$ in 2D and 3D optical lattices. In this

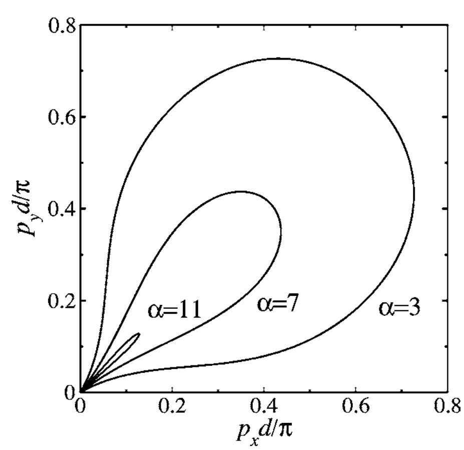


FIG. 5. The solution of the energy conservation condition $E_{\mathbf{q}} + E_{\mathbf{p}} = E_{\mathbf{q}+\mathbf{p}}$ in a 2D optical lattice, for $q_x = q_y = 0.1\pi/d$.

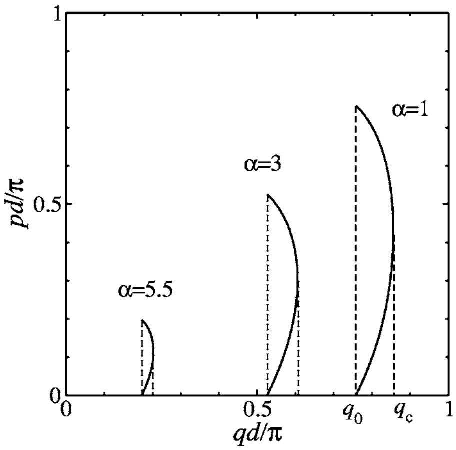


FIG. 6. The solution of the energy conservation condition $E_q = E_p + E_{q-p}$ for Beliaev damping in a 1D optical lattice.

case, the simple Golden Rule expression of the Landau damping given by Eq. (17) is no longer valid. One would have to extend it using higher order perturbation theory [21] in order to calculate the Landau damping when $\alpha$ is close to the threshold value [22].

### B. Beliaev damping

We briefly discuss the Beliaev damping of the Bogoliubov excitations, which can occur even at $T=0$. Beliaev damping is due to spontaneous decay of an excitation into two excitations, and thus we need to satisfy the energy conservation condition $E_{\mathbf{q}} = E_{\mathbf{q}-\mathbf{p}} + E_{\mathbf{p}}$. For simplicity, we only consider a 1D optical lattice.

In Fig. 6, the solution of the energy conservation condition for 1D optical lattice $E_q = E_{q-p} + E_p$ is shown in a $(q, p)$ plane. As predicted in the previous section (see Ref. [1] for more details), one finds that the curve of the solution shrinks as $\alpha$ increases and vanishes when $\alpha > 6$, which indicates the disappearance of the Beliaev damping for $\alpha > 6$.

Even for values of $\alpha < 6$, Beliaev damping of an excitation $E_q$ is only possible when $q$ is between the threshold momenta $q_0$ and $q_c$ shown in Fig. 6. The same phenomenon has been discussed for phonons in superfluid $^4$He [16,21]. At the threshold momenta $q_0$ and $q_c$, two excitations $E_p$ and $E_{q-p}$ created by the decay of an excitation $E_q$ have the same velocity [21]. At $q=q_0$, one of the generated excitations is a phonon having the sound velocity $c$. Therefore the other one also has the velocity equal to the sound velocity $c$, namely $q_0 = p_0$. It is clear from Fig. 6 that at the upper threshold momentum $q_c$ an excitation decays into two identical excitations $E_{q_c/2}$ [21]. Generally, the upper threshold momentum $q_c^n$ for the decay process of an excitation into $n$ excitations is given by Eq. (4) in Ref. [16]. For the decay into two excitations, $q_c = \sqrt{\frac{4}{3}} p_0$. As discussed in [16], due to the uncertainties in energy conservation, $q_c^{n=\infty} = \sqrt{\frac{5}{3}} p_0$ is a true upper threshold momentum for Beliaev damping.

In addition to Landau damping and Beliaev damping, one also has intercollisional damping arising from two body collisions which transfer atoms between the condensate and thermal cloud at finite temperatures [18,19]. Such processes also involve the energy conservation condition for three-excitation processes. Thus the intercollisional damping also disappears when $\alpha \geqslant 6D$ in a $D$-dimensional optical lattice.

## IV. CONCLUSIONS

In conclusion, we have given a detailed treatment of the damping of Bogoliubov excitations associated with Bose condensates in 2D and 3D optical lattices at finite temperature using the tight-binding Bose-Hubbard model. This extends our recent work on a 1D optical lattice in Ref [1].

We have used the Popov approximation in the Bose-Hubbard model to extend the usual $T=0$ theory to finite temperatures. As a by-product, we have calculated the number of condensate atoms per lattice site as a function of both the temperature and the lattice well depth $s$. These results may be of interest in other problems.

We have calculated and compared the Landau damping of Bogoliubov excitations in 1D, 2D, and 3D optical lattices. Previous work (see, for example, Refs. [23,24]) on this problem only considered dynamical instabilities and did not include the dissipation processes we have considered here. In optical lattices of any dimension, the Bogoliubov-Popov excitation spectrum $E_{\mathbf{q}}$ must exhibit "anomalous dispersion" for damping processes to occur. This is analogous to the case of phonon damping in superfluid $^4$He. In the absence of this "bending-up" of the low $\mathbf{q}$ spectrum, energy conservation cannot be satisfied. As a consequence, we find that the excitation damping is absent when $\alpha = Un^{c0}/J > 6D$, where $D$ is the dimension of the optical lattice.

The first studies [4] of damping of excitations were limited to 1D optical lattices along the axis of a cigar-shape magnetic trap, but one would need a much tighter magnetic trap (in the radial direction) for our 1D model results to apply. In the more recent experiments by Stöferle *et al.* [6], a 3D optical lattice is prepared first, and the lattice potential depths in two lattice axes are then made much larger than the third one to produce a 2D array of tightly bound 1D optical lattices. An analogous 2D optical lattice can be formed by choosing a much larger lattice potential depth along one lattice axis than that in the other two axes. This effectively 2D optical lattice might be better for checking our theoretical predictions than the 2D periodic array of 1D tubes used in Ref. [10], since excitations along the direction perpendicular to the 2D lattice potential can be neglected.

Due to our use of a tight-binding approximation, our results are not directly applicable to the damping of excitations found in very weak optical lattices ($s<1$) [7]. Extension of our calculations to such weak optical lattices would be clearly of interest [25].

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