Lamb shift of muonic deuterium

E. Borie
Forschungszentrum Karlsruhe (IHM), Hermann-von-Helmholtzplatz 1, 76344 Eggenstein-Leopoldshafen, Germany

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My previous calculations of the Lamb shift in muonic hydrogen are repeated for muonic deuterium.

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I. INTRODUCTION

The energy levels of muonic atoms are very sensitive to effects of quantum electrodynamics (QED), nuclear structure, and recoil, since the muon is about 206 times heavier than the electron [1]. A number of theoretical analyses of the Lamb shift (the $2p$-$2s$ transition) in light muonic atoms have been published, including [2–8], most recently in view of a proposed measurement of the Lamb shift im muonic hydrogen [9]. The present paper extends the independent recalculation of some of the most important effects [3] to the case of muonic deuterium, including effects that were not considered previously [10]. Muonic deuterium is in many ways similar to muonic hydrogen, but there are some differences. In addition to the different mass the deuteron has spin 1 and thus both magnetic and quadrupole moments.

In the numerical calculations the fundamental constants from CODATA 2002 [11] are used, i.e., α^{-1}, $\hbar c$, m_μ, m_e, m_u=137.035 999 1, 197.326 97 MeV·fm, 105.658 369 MeV, 0.510 998 9 MeV, and 931.5050 MeV, respectively.

Also, the following properties of deuteron were used: m_d=1875.613 MeV/c^2, R_d=2.139±0.003 fm and μ_d=0.85744μ_N=0.307 012μ_p [11]. The quadrupole moment of the deuteron is taken to be Q=0.2860(15) fm^2 [12–14].

Although the main purpose of this paper is to give numerical results for muonic deuterium, some results for hydrogen from [3] are repeated.

II. VACUUM POLARIZATION

The most important QED effect for muonic atoms is the virtual production and annihilation of a single e^+e^- pair. It has as a consequence an effective interaction of order $\alpha Z\alpha$ which is usually called the Uehling potential [15,16]. This interaction describes the most important modification of Coulomb's law. Numerically it is so important that it should not be treated using perturbation theory; instead the Uehling potential should be added to the nuclear electrostatic potential before solving the Dirac equation. However, a perturbative treatment is also useful in the case of very light atoms, such as hydrogen. However, unlike some other authors, we prefer to use relativistic (Dirac) wave functions to describe the muonic orbit. Since these contributions have been extensively discussed in the literature [1–4] (among others), there is no need to go into detail here. The results (in meV), calculated as the expectation value of the Uehling potential using point-Coulomb Dirac wave functions with reduced mass, are, for muonic deuterium:

	Point nucleus		R_d=2.139 fm	
	$2p_{1/2}-2s_{1/2}$	$2p_{3/2}-2s_{1/2}$	$2p_{1/2}-2s_{1/2}$	$2p_{3/2}-2s_{1/2}$
Uehling	227.6577	227.6635	227.5985	227.6043
Kaellen-Sabry	1.666 22	1.666 26	1.665 77	1.665 82

The effect of finite nuclear size calculated here can be parametrized as $-0.0129\langle r^2\rangle$ (here the correction is in meV and the nuclear radius in fm).

Corresponding numbers for muonic hydrogen, calculated as the expectation value of the Uehling potential using point-Coulomb Dirac wave functions with reduced mass, were given in [3]. The finite size contribution corresponding to the one calculated here can be parametrized as $-0.0109\langle r^2\rangle$.

However, higher iterations can change these results. Up to now, these have not been calculated well for muonic deterium, as far as I know. For hydrogen, the contributions due to two and three iterations have been calculated by [4,21], respectively, giving a total of 0.151 meV. An additional higher iteration including finite size and vacuum polarization is given in Ref. [4] [Eqs. (66) and (67)] and Ref. [2] [Eqs. (264) and (268)]. These amount to $-0.0164\langle r^2\rangle$.

The best way to calculate these effects would be an accurate numerical solution of the Dirac equation in the combined Coulomb-Uehling potential.

The mixed muon-electron vacuum polarization correction ([2,19]) is 0.000 08 meV for deuterium and 0.000 07 meV for hydrogen.

The Wichmann-Kroll contribution was calculated using the parametrization for the potential given in [1]. For deuterium, the contribution is −0.001 11 meV. The result obtained for hydrogen is −0.001 03 meV, consistent with that given in [2].

The equivalent potential for the virtual Delbrück effect was recomputed from the Fourier transform given in [1,20]. The resulting potential was checked by reproducing previously calculated results for the $2s$-$2p$ transition in muonic helium, and the $3d$-$2p$ transitions in muonic Mg and Silt for deuterium is +(0.001 47±0.000 16) meV, and for hydrogen it is +(0.001 35±0.000 15) meV. As in the case of muonic helium, this contribution very nearly cancels the Wichmann-Kroll contribution. The contribution corresponding to three photons to the muon and one to the proton should be analogous to the light by light contribution to the muon anomalous moment; to my knowledge, the corresponding contribution to the muon form factor has never been calculated. It will be comparable to the other light by light contributions. This graph was included in contributions to the muon's anoma-

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lous magnetic moment; the contribution to the muon form factor is one of the most significant unknown corrections.

The sixth-order vacuum polarization corrections to the Lamb shift in muonic hydrogen have been calculated by Kinoshita and Nio [21]. Their result for the $2p$-$2s$ transition (in hydrogen) is

$$\Delta E^{(6)} = 0.120\,045(\alpha Z)^2 m_r\left(\frac{\alpha}{\pi}\right)^3 \approx 0.007\,61 \text{ meV}$$

and 0.008 04 meV for muonic deuterium.

However, I should remark that the contributions from Figs. 1 and 2 of Ref. [21] were checked by direct integration. Although the results agreed perfectly for the case of hydrogen, there were small but significant discrepancies for the case of deuterium. (hydrogen: Fig. 1 contributes 0.000 396 meV and Fig. 2 contributes 0.002 931 meV; deuterium: direct integration gave 0.000 472 meV and 0.003 364 meV, respectively, while the work of Ref. [21]. indicates values 0.000 419 meV and 0.003 906 meV, respectively). This indicates that, at least for these two graphs, integration over momentum transfer involves more than a single reduced mass factor.

The hadronic vacuum polarization contribution has been estimated by a number of authors [2,22,23]. It amounts to about 0.013 meV in deuterium and 0.012 meV in hydrogen. One point that should not be forgotten about the hadronic VP correction is the fact that the sum rule or dispersion relation that everyone (including myself) used does not take into account the fact that the proton (nucleus) can in principle interact strongly with the hadrons in the virtual hadron loop. It is irrelevant for the anomalous magnetic moment but probably not for muonic atoms. An estimation of this effect appears to be extremely difficult and could easily change the correction by up to 50%. Eides *et al.* [2] point out that the graph related to hadronic vacuum polarization can also contribute to the measured value of the nuclear charge distribution (and polarizability). It is not easy to determine where the contribution should be assigned. This may also be true for the so-called "proton self-energy" [2,5], which involves some of the same graphs as are present in the calculation of radiative corrections to electron scattering.

III. FINITE NUCLEAR SIZE AND NUCLEAR POLARIZATION

The main contribution due to finite nuclear size has been given analytically to order $(\alpha Z)^6$ by Friar [24]. The main result is

$$\Delta E_{ns} = -\frac{2\alpha Z}{3}\left(\frac{\alpha Z m_r}{n}\right)^3\left[\langle r^2\rangle - \frac{\alpha Z m_r}{2}\langle r^3\rangle_{(2)} + (\alpha Z)^2(F_{REL} + m_r^2 F_{NR})\right] \quad (1)$$

where $\langle r^2\rangle$ is the mean square radius of the nucleus.

For muonic deuterium, the main contribution amounts to $-6.0732\langle r^2\rangle = -(27.787\pm 0.078)$ meV. Depending on the model, the term proportional to $\langle r^3\rangle_{(2)}$ gives a contribution of 0.382 or 0.417 meV, depending on the model used for the charge distribution. The terms of order $Z\alpha^6$ contribute 0.0045 meV. This estimate includes all of the terms given in [24].

For the case of muonic hydrogen, Pachucki [4] has estimated a correction similar to the second term (proportional to $\langle r^3\rangle_{(2)}$) in Eq. (1). Friar [24] used the Coulomb approximation to calculate this contribution of order $Z\alpha^5$ as the dominant contribution to the very complicated integral using Eq. (58) of Ref. [4]. A superficial examination of the integrand suggests that this is the most infrared-sensitive part of the long expression and is exact in the limit of large m_N.

This two-photon correction requires further investigation, especially since the logarithmic terms in the two-photon correction without finite size (see below) also seem to be suspect. In particular, the parametrization of the form factors used in any calculation should reproduce the correct proton radius. Also the relationship among the different contributions needs to be specified more clearly. A useful approach might be to use the external field approximation and reclassify the remaining contributions to the complete expression as "recoil terms." Since the derivation given in Ref. [4] is only applicable to spin-1/2 nuclei, it cannot be used for muonic deuterium anyway.

As mentioned previously, the finite-size contributions to vacuum polarization in muonic hydrogen can be parametrized as $-0.0109\langle r^2\rangle - 0.0164\langle r^2\rangle$, giving a total of $-0.0273\langle r^2\rangle$ or $-0.0209(6)$ meV if the proton radius is 0.875 fm. For deuterium. only the contribution corresponding to the first term of the previous sum ($-0.0129\langle r^2\rangle$) has been calculated.

The contribution due to nuclear polarization has been calculated by Leidemann and Rosenfelder [25] to be 1.50 ± 0.025 meV (see also [26]).

IV. RELATIVISTIC RECOIL

As is well known, the center-of-mass motion can be separated exactly from the relative motion only in the nonrelativistic limit. Relativistic corrections have been studied by many authors and will not be reviewed here. The relativistic recoil corrections summarized in [1] include the effect of finite nuclear size to leading order in m_μ/m_N properly.

Up to now this method has been used to treat recoil corrections to vacuum polarization only in the context of extensive numerical calculations that include the Uehling potential in the complete potential, as described in [1]. They can be included explicitly, as a perturbation correction to point-Coulomb values. Recall that (to leading order in $1/m_N$) the energy levels are given by

$$E = E_r - \frac{B_0^2}{2m_N} + \frac{1}{2m_N}\langle h(r) + 2B_0 P_1(r)\rangle, \quad (2)$$

where E_r is the energy level calculated using the reduced mass and B_0 is the unperturbed binding energy. Also

$$h(r) = -P_1(r)\left(P_1(r) + \frac{1}{r}Q_2(r)\right) - \frac{1}{3r}Q_2(r)[P_1(r) + Q_4(r)/r^3]. \tag{3}$$

Here

$$P_1(r) = 4\pi\alpha Z\int_r^\infty r'\rho(r')dr' = -V(r) - rV'(r),$$

$$Q_2(r) = 4\pi\alpha Z\int_0^r r'^2\rho(r')dr' = r^2V'(r),$$

$$Q_4(r) = 4\pi\alpha Z\int_0^r r'^4\rho(r')dr' \tag{4}$$

Following Ref. [3], only the Coulomb and Uehling potentials are kept. One finds

$$P_1(r) = -\alpha Z\frac{2\alpha}{3\pi}(2m_e)\chi_0(2m_e r),$$

$$Q_2(r) = \alpha Z\left(1 + \frac{2\alpha}{3\pi}[\chi_1(2m_e r) + (2m_e r)\chi_0(2m_e r)]\right),$$

$$Q_4(r) = \alpha Z\frac{2\alpha}{3\pi}\int_1^\infty dz\frac{(z^2-1)^{1/2}}{z^2}\left(1+\frac{1}{2z^2}\right)\left(\frac{2}{\pi}\right)\int_0^\infty \frac{1}{q^2+4m_e^2z^2} \times\frac{[6qr-(qr)^3]\cos(qr)+[3(qr)^2-6]\sin(qr)}{q}dq,$$

where $\chi_n(x)$ is defined in [1]. Details of the calculations for the case of vacuum polarization are given in Ref. [3]. Corrections due to finite nuclear size can be included when a model for the charge distribution is given. This was done by Friar [24] (and confirmed independently for two different model charge distributions); the contribution to the recoil correction for the binding energy of the $2s$ level due to finite nuclear size is −0.019 meV. The factor $1/m_N$ is replaced by $1/(m_\mu+m_N)$, also consistent with the calculations presented in [24].

As in Ref. [3], it will be sufficient to approximate $Q_2(r)/r$ by $\alpha Z/r$, since vacuum polarization is assumed to be a relatively small correction to the Coulomb potential.

For muonic deuterium,

$$\frac{-2B_0}{(m_\mu+m_N)}\langle P_1(r)\rangle$$

is −0.000 176 meV for the $2s$ state and −0.000 030 meV for the $2p$ state.

$$\frac{1}{(m_\mu+m_N)}\left\langle\frac{\alpha Z}{r}P_1(r)\right\rangle$$

gives 0.005 543 meV for the $2s$ state and 0.000 206 meV for the $2p$ state, and

$$\frac{1}{(m_\mu+m_N)}\left\langle\frac{\alpha Z}{3r}Q_4(r)\right\rangle$$

gives 0.002 753 meV for the $2s$ state and 0.000 281 meV for the $2p$ state.

Combining the expectation values according to Eqs. (2) and (3), as in Ref. [3], one finds a contribution to the $2p$-$2s$ transition of −0.004 79 meV for deuterium and −0.004 19 meV for hydrogen.

To obtain the full relativistic and recoil corrections, one must add the difference between the expectation values of the Uehling potential calculated with relativistic and nonrelativistic wave functions, giving a total correction of 0.0179 meV for muonic deuterium. The corresponding result for hydrogen (0.0166 meV) was in quite good agreement with the correction of 0.0169 meV calculated by Veitia and Pachucki [27]. The treatment presented here and in [3] has the advantage of avoiding second-order perturbation theory.

The review by Eides *et al.* [2] gives a better version of the two photon recoil [Eq. (136)] than was available for the review by Borie and G. Rinker [1]. Evaluating this expression for muonic hydrogen gives a contribution of −0.022 656 meV to the $2p$-$2s$ transition in deuterium and −0.044 97 meV in hydrogen.

However, some of the contributions to the expressions given in [2] involve logarithms of the mass ratio m_μ/m_N. Logarithms can only arise in integrations in the region from m_μ to m_N; in this region the effect of the nuclear form factor should not be neglected. As discussed above, it might be useful to split the two-photon correction into a Coulomb term and recoil corrections to it.

An additional recoil correction for states with $\ell \neq 0$ has been given by [28] (see also [2]). It is

$$\Delta E_{n,\ell,j} = \frac{(\alpha Z)^4 m_r^3}{2n^3 m_N^2}(1-\delta_{\ell 0})\left(\frac{1}{\kappa(2\ell+1)}\right) \tag{5}$$

where κ is equal to $-(\ell+1)$ if $j=\ell+\frac{1}{2}$ and $+\ell$ if $j=\ell-\frac{1}{2}$. When evaluated for the $2p$ states of muonic deuterium, one finds a contribution to the $2p$-$2s$ transition energy of 0.0168 meV for the $2p_{1/2}$ state and −0.0084 meV for the $2p_{3/2}$ state in deuterium.

A final point about recoil corrections is that in the case of light muonic atoms, the mass ratio m_μ/m_N is considerably larger than the usual perturbation expansion parameter αZ. Contributions of higher order in the mass ratio could be significant.

V. MUON LAMB SHIFT

For the calculation of muon self-energy and vacuum polarization, the lowest order (one-loop approximation) contribution is well known, at least in perturbation theory. Including also muon vacuum polarization (0.019 68 meV) and an

TABLE I. Contributions to the muon Lamb shift $[E(2p_{1/2}) - E(2s_{1/2})]$ in muonic hydrogen and deuterium, in meV.

Transition	$2p_{1/2}-2s_{1/2}$	$2p_{3/2}-2s_{1/2}$
Deuterium		
Second order	−0.774 616	−0.755 125
Higher orders	−0.002 001	−0.001 926
Total	−0.776 617	−0.757 051
Hydrogen		
Second order	−0.66788	−0.65031
Higher orders	−0.00172	−0.00165
Total	−0.66960	−0.65196

extra term of order $(Z\alpha)^5$ as given in [2], which contributes −0.005 18 meV, one finds a contribution of −0.774 62 meV for the $2s_{1/2}-2p_{1/2}$ transition and −0.755 12 meV for the $2s_{1/2}-2p_{3/2}$ transition.

These results, and the higher order corrections [1,19], are summarized in Table I.

For hydrogen, Pachuki [4] has estimated an additional contribution of −0.005 meV for a contribution corresponding to a vacuum polarization insert in the external photon.

VI. SUMMARY OF CONTRIBUTIONS FOR MUONIC DEUTERIUM

For deuterium, one finds the transition energies in meV in Table II. Here the main vacuum polarization contributions are given for a point nucleus, using the Dirac equation with reduced mass. The finite size corrections for deuterium up to order $(\alpha Z)^5$ can be parametrized as $6.0732\langle r^2\rangle + 0.0129\langle r^2\rangle + 0.0409\langle r^3\rangle_{(2)}$, although not all contributions to the effect of finite size on the vacuum polarization correction are included.

VII. FINE AND HYPERFINE STRUCTURE

The Breit equation [2,7,28] contributions to the fine and hyperfine interactions for general potentials and arbitrary spins were given by Metzner and Pilkuhn [29]. Here a version applicable to the case of muonic atoms ($Z_1=-1$, $s_1=1/2$, $m_1=m_\mu$, $\kappa_1=a_\mu$, $Z_2=Z$) is given:

$$V_{L,s_1} = \frac{1}{2m_\mu}\frac{1}{r}\frac{dV}{dr}\left(\frac{1+a_\mu}{s_1 m_r} - \frac{1}{m_\mu}\right)\vec{L}\cdot\vec{s}_1. \qquad (6)$$

This can be rearranged to give the well-known form for spin-1/2 particles with an anomalous magnetic moment, namely

$$-\frac{1}{r}\frac{dV}{dr}\frac{1+a_\mu+(a_\mu+1/2)m_N/m_\mu}{m_N m_\mu}\vec{L}\vec{\sigma}_\mu.$$

Also

$$V_{L,s_2} = \frac{1}{2m_2}\frac{1}{r}\frac{dV}{dr}\left(\frac{1+\kappa_2/Z}{s_2 m_r} - \frac{1}{m_2}\right)\vec{L}\cdot\vec{s}_2.$$

Usually one writes

TABLE II. Contributions to the muonic deuterium Lamb shift. The deuteron radius is taken from [11].

Contribution	Value (meV)	Uncertainty (meV)
Uehling	227.6577	
Källen-Sabry	1.6662	
Wichmann-Kroll	−0.00111	
virtual Delbrueck	0.00147	0.00016
mixed mu-e VP	0.00008	
hadronic VP	0.013	0.002
sixth order [21]	0.00804	
recoil [2] (Eq. 136)	−0.02656	
recoil, higher order [2]	−0.00377	
recoil, finite size [24]	0.019	0.003
recoil correction to VP [1]	−0.0048	
additional recoil [28]	0.0168	
muon Lamb shift		
second order	−0.77462	
fourth order	−0.00200	
nuclear size (R_d=2.139 fm)		0.003 fm
main correction [24]	−27.787	0.078
order $(\alpha Z)^5$ [24]	0.0400	0.018
order $(\alpha Z)^6$ [24]	−0.0045	
correction to VP	−0.0592	
polarization	1.50	0.03
Other (not checked)		
VP iterations [4]	?	
VP insertion in self energy [4]	?	
additional size for VP [2]	?	

$$\frac{Z+\kappa_2}{m_2} = \frac{\mu_2}{m_p},$$

where μ_2 is the magnetic moment of the nucleus in units of nuclear magnetons ($\mu_N=e/2m_p$). A value of $\mu_d=0.857\,44\mu_N=0.307\,012\mu_p$ corresponds to $\kappa_d=0.714$:

$$V_{s_1,s_2} = \frac{2(1+a_\mu)\mu_2}{2s_2 m_\mu m_2}\left(\frac{1}{r}\frac{dV}{dr}(3\vec{s}_1\cdot\hat{r}\vec{s}_2\cdot\hat{r} - \vec{s}_1\cdot\vec{s}_2) - \frac{2}{3}\nabla^2 V\vec{s}_1\cdot\vec{s}_2\right)$$

$$V_Q = -\alpha Q\frac{1}{r}\frac{dV}{dr}(3\vec{s}_2\cdot\hat{r}\vec{s}_2\cdot\hat{r} - \vec{s}_2\cdot\vec{s}_2)$$

with Q in units of $1/m_2^2$. The quadrupole moment of the deuteron is taken to be $Q=0.2860(15)$ fm^2 [12–14]. In other units, one finds $Q=25.84/m_d^2=7.345\times10^{-6}$ MeV^{-2}.

Note that V_{L,s_1} describes the fine structure, while the hyperfine structure is described (in perturbation theory) by the expectation values of V_{L,s_2}, V_{s_1,s_2}, and V_Q (where applicable).

The Uehling potential has to be included in the potential $V(r)$. For states with $\ell>0$ in light atoms, and neglecting the effect of finite nuclear size, we may take

TABLE III. Contributions to the fine structure $[E(2p_{3/2}) - E(2p_{1/2})]$ of the $2p$ state in muonic hydrogen and deuterium, in meV.

	Hydrogen	Deuterium
Dirac	8.415 64	8.864 30
Uehling (VP)	0.0050	0.005 75
Källen-Sabry	0.000 04	0.000 05
Anomalous moment a_μ		
Second order	0.017 57	0.014 91
Higher orders	0.000 07	0.000 07
Recoil [Eq. (5)]	−0.0862	−0.0252
Total fine structure	8.352	8.864

$$\frac{1}{r}\frac{dV}{dr} = \frac{\alpha Z}{r^3}\left[1 + \frac{2\alpha}{3\pi}\int_1^\infty \frac{(z^2-1)^{1/2}}{z^2}\left(1+\frac{1}{2z^2}\right)\cdot(1+2m_e rz)e^{-2m_e rz}dz\right], \tag{7}$$

which is obtained from the Uehling potential [15,16] by differentiation. Then, assuming that it is sufficient to use nonrelativistic point Coulomb wave functions for the $2p$ state, one finds

$$\left\langle \frac{1}{r^3}\right\rangle_{2p} \to \left\langle \frac{1}{r^3}\right\rangle_{2p}(1+\varepsilon_{2p})$$

where

$$\varepsilon_{2p} = \frac{2\alpha}{3\pi}\int_1^\infty \frac{(z^2-1)^{1/2}}{z^2}\left(1+\frac{1}{2z^2}\right)\cdot\left(\frac{1}{(1+az)^2}+\frac{2az}{(1+az)^3}\right)dz \tag{8}$$

with $a=2m_e/(\alpha Z m_r)$. For hydrogen, $\varepsilon_{2p}=0.000\,365$, and for deuterium $\varepsilon_{2p}=0.000\,391$.

A. Fine structure of the 2*p* state

The fine structure of the $2p$ states can be calculated by using the relativistic Dirac energies, computing the vacuum polarization contributions with Dirac wave functions, and including the effect of the anomalous magnetic moment in the muon Lamb shift. An additional recoil correction [Eq. (5)] also has to be includedlts are summarized in Table III. One should also include the $B^2/2M_N$-type correction to the fine structure (see [2]). This is tiny (5.7×10^{-6} meV in hydrogen) and is not included in the table. Friar [24] has given expressions for the energy shifts of the $2p$ states due to finite nuclear size. These were calculated and found to give a negligible contribution (3.1×10^{-6} meV) to the fine structure of the $2p$ state in hydrogen.

B. Hyperfine structure of the 2*p*-state in muonic deuterium

For the $2p$ state, the matrix elements of the magnetic hyperfine structure have been given by Brodsky and Parsons [30]. For hydrogen they are the same as those calculated in [3].

Let

$$\beta_D = \frac{16(1+\kappa_d)}{m_\mu m_d}\frac{\alpha}{(\alpha Z m_r/n)^3}\ell(\ell+1)(2\ell+1) = \frac{(1+\kappa_d)}{6m_\mu m_d}(\alpha Z m_r)^3 = 4.0906\text{ meV}$$

(for a point Coulomb potential).

The matrix elements for the magnetic hyperfine structure are then given by

j	j'	Energy
1/2	1/2	$(\beta_D/6)(2+x_d+a_\mu)[-\delta_{F,1/2}+1/2\,\delta_{F,3/2}]$
3/2	3/2	$\delta+(\beta_D/4)(4+5x_d-a_\mu)[-1/6\,\delta_{F,1/2}-1/15\,\delta_{F,3/2}+1/10\,\delta_{F,5/2}]$
3/2	1/2	$(\beta_D/48)(1+2x_d-a_\mu)[\sqrt{2}\,\delta_{F,1/2}-\sqrt{5}\,\delta_{F,3/2}]$

where $x_d=(m_\mu^2/m_d m_r)(\kappa_d/(1+\kappa_d))=0.0248$ represents a recoil correction due to Thomas precession [7,28,30].

For the evaluation of the contributions of the quadrupole HFS, let

$$\epsilon_Q = \alpha Q\left\langle \frac{1}{r}\frac{dV}{dr}\right\rangle.$$

For a point Coulomb potential, and the $2p$ state, $\epsilon_Q=\alpha Q(Z\alpha m_r)^3/24=0.432\,43$ meV. The quadrupole interaction results in energy shifts of

j	j'	Energy
1/2	1/2	0
3/2	3/2	$\epsilon_Q[\delta_{F,1/2}-4/5\,\delta_{F,3/2}+1/5\,\delta_{F,5/2}]$
3/2	1/2	$\epsilon_Q[\sqrt{2}\,\delta_{F,1/2}-1/\sqrt{5}\,\delta_{F,3/2}]$

As mentioned before, the Uehling potential has to be included in the potential $V(r)$. For states with $\ell>0$ in light atoms, this can be taken into account by multiplying β_D and ϵ_Q by $(1+\varepsilon_{2p})$ where ε_{2p} is given by Eq. (8). With a numerical value of $\varepsilon_{2p}=0.000\,391$ for muonic deuterium, the value of ϵ_Q is increased to 0.434 40 meV and the value of β_D is increased to $\beta_D'=4.0922$ meV.

Then for the $2p$ level with $j=j'=3/2$ and $F=5/2$, the energy shift is given by

$$\delta + \epsilon_Q/5 + (\beta_D'/40)(4+5x_d-a_\mu) = 9.373\text{ meV}.$$

For the $2p$ levels with $F=1/2$ and $F=3/2$, the corresponding matrices have to be diagonalized. The resulting numerical values for the eigenvalues are, for $F=1/2$, −1.3834 and 8.5974 meV, and, for $F=3/2$, they are 0.6856 and 8.2410 meV.

C. Hyperfine structure of the 2s state

The expectation value of $V_{s_1 s_2}$ in an ns state with $j=1/2$ is

$$\Delta E_{ns} = \frac{2\mu_2 \alpha(\alpha Z)^3 m_r^3}{3n^3 m_\mu m_2 s_2}(1+a_\mu)[F(F+1)-s_2(s_2+1)-3/4].$$

When $s_2=1/2$ and $\mu_2/m_p=(1+\kappa_2)/m_2$, this reproduces the well-known result for muonic hydrogen:

$$\Delta E_{ns} = \frac{8(\alpha Z)^4 m_r^3}{3n^3 m_\mu m_2}(1+\kappa_2)(1+a_\mu) = (8/n^3)\times 22.8332 \text{ meV}$$

(see, for example, [2]).

For deuterium, with $s_2=1$, the corresponding hyperfine splitting is

$$\Delta E_{ns} = \frac{2(\alpha Z)^4 m_r^3}{3n^3 m_\mu m_2}(1+\kappa_d)(1+a_\mu)[F(F+1)-11/4]$$
$$= (8/n^3)\times 2.04766 \text{ meV} \times [F(F+1)-11/4]$$

for a total splitting of 6.14298 meV in muonic deuterium. This is in reasonably good agreement with the result given by Carboni [10].

As was shown in [2,7], the energy shift of the ns state has to be multiplied by

$$1+\varepsilon_{VP}+\varepsilon_{vertex}+\varepsilon_{Breit}+\varepsilon_{FS,rec}.$$

The QED corrections have been discussed by Borie [3] (see also [31]). For the 2s state [31],

$$\varepsilon_{vertex} = \frac{2\alpha(\alpha Z)}{3}\left(\ln(2)-\frac{13}{4}\right) = -1.36\times 10^{-4}$$

and ([2], Eq. (277))

$$\varepsilon_{Breit} = \frac{17(\alpha Z)^2}{8} = 1.13\times 10^{-4}.$$

The vacuum polarization correction has two contributions [3,31,32]. The detailed formulas can be found in Ref. [3]. For muonic deuterium, one obtains $\varepsilon_{VP1}=0.002\,18$ and $\varepsilon_{VP2}=0.003\,37$ for a point nucleus. Including the effect of deuteron size (with a dipole form factor) reduces these numbers to 0.002 07 and 0.003 26, respectively. For the case of muonic hydrogen, the corresponding numbers are $\varepsilon_{VP1}=0.002\,11(0.002\,06)$ and $\varepsilon_{VP2}=0.003\,25(0.003\,21)$, respectively.

In the case of ordinary hydrogen, each of these two contributions is equal to $3\alpha^2/8=1.997\times 10^{-5}$. The accuracy of the numerical integration was checked by reproducing these results. One can thus expect that muonic vacuum polarization will contribute $3\alpha^2/4\simeq 4\times 10^{-5}$, as in the case of normal hydrogen or deuterium. This amounts to an energy shift of 0.0002 meV.

The contribution to the hyperfine structure from the two loop diagrams [17,18] can be calculated as in [3]. The resulting additional contributions to ε_{VP1} and ε_{VP2} for muonic deuterium are 1.69×10^{-5} and 2.54×10^{-5}, respectively, giving a total shift of 0.0002 meV in muonic deuterium.

The correction due to finite size is taken to be

$$\varepsilon_{Zem} = -2\alpha Z m_r \langle r\rangle_{(2)},$$

where $\langle r\rangle_{(2)}$ is given in [7,24,34]. This is known as the Zemach correction [33].

For muonic deuterium, the coefficient of $\langle r\rangle_{(2)}$ is $-0.007\,398$ fm^{-1}, giving, with $\langle r\rangle_{(2)}=2.593\pm 0.016$ fm from [34], $\varepsilon_{Zem}=-0.019\,18\pm 0.000\,12$.

Additional recoil corrections have been discussed for the case of hydrogen by Pachucki [4] and by Martynenko [35]. Their numerical results differ by more than the expected experimental precision. It would be very desirable to understand the reasons for the discrepancy between Refs. [4,35] in the calculations of this effect. Also, since the Zemach radius seems to be sensitive to details of the electric and magnetic charge distributions [34], evaluations performed with a dipole-type form factor may not be good enough. The Zemach-moment term seems to be a very good approximation to the more complete expression; one could reclassify the higher-order terms not kept in the Zemach term as "recoil terms" of order $(m_\mu/m_N)^2$. This point requires further invesigation, including a generalization to the case of nuclear spin not equal to 1/2, and the additional terms are not kept here.

The total hyperfine splitting of the 2s state of muonic deuterium, including all corrections, is

$$\Delta E_{2s} = \tfrac{3}{2}\beta_D(1+a_\mu)(1+\varepsilon_{VP}+\varepsilon_{vertex}+\varepsilon_{Breit}+\varepsilon_{Zem})$$
$$= 6.0582 \text{ meV}.$$

Table IV gives the contributions to the transition energies due to fine and hyperfine structure.

TABLE IV. Fine and hyperfine contributions to the Lamb shift in muonic deuterium.

Transition	Energy shift in meV
$^2p_{1/2}-^2s_{1/2}$	2.655
$^2p_{3/2}-^2s_{1/2}$	12.636
$^4p_{1/2}-^2s_{1/2}$	4.724
$^4p_{3/2}-^2s_{1/2}$	12.280
$^2p_{1/2}-^4s_{1/2}$	−3.403
$^2p_{3/2}-^4s_{1/2}$	6.578
$^4p_{1/2}-^4s_{1/2}$	−1.334
$^6p_{3/2}-^4s_{1/2}$	6.222
$^6p_{3/2}-^4s_{1/2}$	7.354

VIII. SUMMARY OF CONTRIBUTIONS AND CONCLUSIONS

The most important contributions to the Lamb shift in muonic deuterium, including hyperfine structure, have been independently recalculated. A new calculation of some terms that were omitted in the literature, such as the virtual Delbrück effect [20] and an alternative calculation of the relativistic recoil correction, have been presented.

Numerically the results given in Table II add up to a total correction of $[230.073(31)-6.086\langle r^2\rangle+0.0409\langle r^2\rangle^{3/2}]$ meV, or 202.263 ± 0.95 meV. The complete dependence on the deuteron radius is uncertain since contributions from iteration of the potential are not included. Also, some other contributions are not included, as indicated in Table II.

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