Concurrence versus purity: Influence of local channels on Bell states of two qubits

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We analyze how a maximally entangled state of two qubits (e.g., the singlet ψ_s) is affected by the action of local channels described by completely positive maps $\mathcal{E}$. We analyze the concurrence and the purity of states $\varrho_{\mathcal{E}}=\mathcal{E}\otimes\mathcal{I}[\psi_s]$. Using the concurrence-versus-purity phase diagram we characterize local channels $\mathcal{E}$ by their action on the singlet state ψ_s. We specify a region of the concurrence-versus-purity diagram that is achievable from the singlet state via the action of unital channels. We show that even the most general (including nonunital) local channels acting just on a single qubit of the original singlet state cannot generate the maximally entangled mixed states. We study in detail various time evolutions of the original singlet state induced by local Markovian semigroups. We show that the decoherence process is represented in the concurrence-versus-purity diagram by a line that forms the lower bound of the achievable region for unital maps. On the other hand, the depolarization process is represented by a line that forms the upper bound of the region of maps induced by unital maps.

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I. INTRODUCTION

From two well-established properties of the entanglement—namely, from the fact that (i) *interactions create entanglement* and (ii) *entanglement cannot be shared freely* (monogamy [1–4])—we can conclude that any nonunitary evolution of a single qubit that is entangled with another qubit is accompanied with a deterioration of the original entanglement between these two qubits.

The aim of this paper is to address the question how local actions (channels) affect properties of quantum states of bipartite systems. In particular, we will analyze in detail how the entanglement and the purity of a two-qubit system that has been originally prepared in a maximally entangled Bell state depend on the action of a single-qubit channel; i.e., we assume that one of the qubits of the original Bell pair is affected by an environment.

We have a twofold task in front of us: First, we will analyze how local channels affect the entanglement and purity of the original Bell state. Second, we study how the time evolution (i.e., a one-parametric subset $\mathcal{E}_t$ of the set of all completely positive maps) can be represented as a one-parametric curve in the concurrence-versus-purity "phase" diagram. We will focus our attention on Markovian evolutions—i.e., those one-parametric subsets of channels for which the semigroup property $\mathcal{E}_t\mathcal{E}_s=\mathcal{E}_{t+s}$ holds. We will analyze in detail physical processes such as decoherence, decay, quantum homogenization, etc., in terms of the concurrence-versus-purity phase diagram.

Let us first define those quantities that we shall use through the paper. The purity that measures the degree of "mixedness" of a state that is described by the density operator ϱ will be quantified by the function

$$P(\varrho)=\mathrm{Tr}[\varrho^2], \tag{1.1}$$

which equals to unity for pure states and achieves its minimum for maximally mixed state; i.e., for the total mixture $\varrho=1/dI$, the purity achieves the minimal value that is equal to $1/d$.

The entanglement between two quantum systems described by a density operator $\varrho_{AB}\equiv\varrho$ will be quantified by a function called the *tangle*:

$$\tau(\varrho)=\min_{\varrho=\sum_k q_k\psi_k}\sum_k q_k S_2(\psi_k), \tag{1.2}$$

where ψ_k denotes the projection onto a pure state $|\psi_k\rangle$. The minimum in Eq. (1.2) is taken over all pure-state decompositions of the state ϱ while the function $S_2(\psi_k) =2[1-P(\mathrm{Tr}_B\psi_k)]$ is the so-called *linear entropy*. The quantity $C(\varrho)=\sqrt{\tau(\varrho)}$ (the square root of the tangle) is known in the literature as the *concurrence*. Wootters [5] has derived a simple analytic formula for the concurrence of two qubits in a state ϱ:

$$C(\varrho)=2\max\{\mu_j\}-\sum_j\mu_j, \tag{1.3}$$

where μ_j are square roots of eigenvalues of the matrix $R =\varrho(\sigma_y\otimes\sigma_y)\varrho^*(\sigma_y\otimes\sigma_y)$ and ϱ^* denotes complex conjugation of the original two-qubit density operator ϱ. From these definitions it is obvious that the entanglement and purity are closely related quantities and that for two-qubit state ϱ they cannot take arbitrary values.

One of the questions one can ask at this point is, which two-qubit states are maximally entangled providing that their purity is fixed and vice versa? This problem has been addressed in several earlier papers [6–11]. In particular, Ishizaka and Hiroshima [6] have introduced the so-called *maximally entangled mixed states* (MEMS's). These are the states that for a given value of the purity achieve the maximal entanglement. In Ref. [7] a slightly more general problem has been solved. The authors have shown which unitary transformation has to be applied on a given state ϱ in order to maximize the entanglement—in other words, which state maximizes the entanglement for a given spectrum of the density operator (i.e., for a given value of the purity). After these introductory papers appeared, many different aspects of the relation between the entanglement and the mixedness were analyzed. In addition, the entanglement-based "ordering" (parametrization) of the state space of two qubits, originally introduced by Eisert and Plenio in [12], has been investigated in detail. It has been shown that different entanglement measures define different ordering of states [13]. In fact, this feature is not only characteristic for entanglement measures, but also for different measures of mixedness. This means that the choice of the measures affects the final entanglement-purity picture of the state space. In Ref. [8] the entanglement-purity dependence for various measures has been analyzed.

In this paper we will study the entanglement-purity relation from a perspective of *local* operations. In what follows we will assume that the initial state of two qubits is a maximally entangled pure state; i.e., the two qubits are prepared in a Bell state. Without loss of generality we can consider that the two qubits are prepared in the singlet state

$$\psi_s = \frac{1}{4}(I \otimes I - \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z). \quad (1.4)$$

This state is transformed under the action of a local completely positive trace-preserving linear map $\mathcal{E}$ [14,15] that describes the most general quantum process into the state

$$\varrho_{\mathcal{E}} = \mathcal{E} \otimes \mathcal{I}[\psi_s]. \quad (1.5)$$

In general, any local action $\mathcal{E}$ (except for unitary operations) *decreases* the purity of the singlet state. Our aim is to find how much the entanglement is changed under the action of the map $\mathcal{E}\otimes\mathcal{I}$.

In Secs. II–IV we will analyze in detail the action of unital channels (i.e., those channels that do not affect the total mixture—i.e., $\mathcal{E}[I]=I$). The unital channels are defined in Sec. II. In Sec. III we present a geometrical representation of the space of all unital maps. In Sec. IV we introduce the concurrence-versus-purity phase diagram and we determine the region that is covered by the states $\varrho_{\mathcal{E}}=\mathcal{E}\otimes\mathcal{I}[\psi_s]$ that are obtained via the action of local unital maps on the singlet state of two qubits.

Sec. V is devoted to an investigation of the action of nonunital channels. Finally, in Sec. VI we will discuss the time evolution in the concurrence-versus-purity (C-P) phase diagram for specific quantum processes. In the Conclusion we will summarize the main results and discuss some open problems.

II. UNITAL CHANNELS

Let us first consider unital channels. Thanks to the seminal work of Ruskai *et al.* [16] the investigation (parametrization) of single-qubit channels can be significantly simplified. A single-qubit channel $\mathcal{E}$ (not only unital ones) can be written as a sequence of two unitary rotations and one specific completely positive map $\Phi_{\mathcal{E}}$ which belongs to a six-parametric family of maps. In particular, $\mathcal{E}[\varrho] = U\Phi_{\mathcal{E}}[V\varrho V^\dagger]U^\dagger$. Due to the fact that unitary operations preserve essentially all interesting properties of the original channel $\mathcal{E}$, one can reduce the analysis of single-qubit channels into the investigation of properties of the channel $\Phi_{\mathcal{E}}$. In other words, up to a unitary equivalence the original 15-parametric set of single-qubit channels can be reduced into a 6-parametric set of channels $\Phi_{\mathcal{E}}$. This reduction significantly simplifies the analysis of single-qubit channels. In the Bloch-sphere picture the general channel $\mathcal{E}$ transforms the Bloch vector $\vec{r}$ in an *affine* way—i.e., $\vec{r}\rightarrow\vec{r}\,'=T\vec{r}+\vec{t}$. The corresponding map $\Phi_{\mathcal{E}}$ acts as follows:

$$\vec{r} \rightarrow \vec{r}\,' = D\vec{r} + \vec{\tau}, \quad (2.1)$$

where $D=\mathrm{diag}\{\lambda_1,\lambda_2,\lambda_3\}$ is a diagonal matrix of singular values of the matrix T and $\vec{\tau}=R_U\vec{t}$, with R_U being a three-dimensional rotation associated with the unitary transformation U. The vector $\vec{\tau}$ represents the shift of the total mixture; i.e., it is associated with the nonunitality of the channel under consideration.

The set of unital channels $\Phi_{\mathcal{E}}$ forms a three-parametric family of completely positive (CP) maps of the form $\Phi_{\mathcal{E}} =\mathrm{diag}\{1,\lambda_1,\lambda_2,\lambda_3\}$ and the inequalities

$$\begin{aligned} 1 + \lambda_x - \lambda_y - \lambda_z &\geq 0, \\ 1 - \lambda_x + \lambda_y - \lambda_z &\geq 0, \\ 1 - \lambda_x - \lambda_y + \lambda_z &\geq 0, \\ 1 + \lambda_x + \lambda_y + \lambda_z &\geq 0, \end{aligned} \quad (2.2)$$

are equivalent to complete positivity.

Our task is to evaluate the purity and concurrence of states $\Omega_{\mathcal{E}}=\Phi_{\mathcal{E}}\otimes\mathcal{I}[\psi_s]$ [with $\psi_s=\frac{1}{4}(I\otimes I-\sigma_x\otimes\sigma_x-\sigma_y\otimes\sigma_y -\sigma_z\otimes\sigma_z)$] as functions of these three parameters. The state $\Omega_{\mathcal{E}}$ takes a simple form $\Omega_{\mathcal{E}}=\frac{1}{4}(I\otimes I-\lambda_x\sigma_x\otimes\sigma_x-\lambda_y\sigma_y\otimes\sigma_y -\lambda_z\sigma_z\otimes\sigma_z)$. The corresponding density matrix of this state reads, in the computational basis,

$$\Omega_{\mathcal{E}} = \begin{pmatrix} A & 0 & 0 & D \\ 0 & B & C & 0 \\ 0 & C & B & 0 \\ D & 0 & 0 & A \end{pmatrix}, \quad (2.3)$$

with $A=\frac{1}{4}(1-\lambda_z)$, $B=\frac{1}{4}(1+\lambda_z)$, $C=-\frac{1}{4}(\lambda_y+\lambda_x)$, and $D=\frac{1}{4}(\lambda_y -\lambda_x)$.

In order to evaluate the purity of the state $\Omega_{\mathcal{E}}$ we have to find eigenvalues of the matrix (2.3). These eigenvalues are given by the expression $\kappa_{1,2}=A\pm D$ and $\kappa_{3,4}=B\pm C$. Thus, for the purity of the state $\Omega_{\mathcal{E}}$ we obtain the expression

$$P(\Omega_{\mathcal{E}}) = \mathrm{Tr}[\Omega_{\mathcal{E}}^2] = \frac{1}{4}(1 + \lambda_x^2 + \lambda_y^2 + \lambda_z^2). \quad (2.4)$$

In order to find the concurrence of the state $\Omega_{\mathcal{E}}$ we have to evaluate the eigenvalues of the matrix

$$R = \Omega_{\mathcal{E}}\sigma_y \otimes \sigma_y \Omega_{\mathcal{E}}\sigma_y \otimes \sigma_y = \begin{pmatrix} X & 0 & 0 & Y \\ 0 & P & Q & 0 \\ 0 & Q & P & 0 \\ Y & 0 & 0 & X \end{pmatrix}, \quad (2.5)$$

with $X=A^2+D^2$, $Y=-2AD$, $P=C^2+B^2$, and $Q=2CB$. The square roots of these eigenvalues are $\mu_{1,2}=|B\pm C|$ and $\mu_{3,4}=|A\pm D|$. Since the eigenvalues of $\Omega_{\mathcal{E}}$ are positive, the square roots of eigenvalues of R and the eigenvalues of $\Omega_{\mathcal{E}}$ coincide; i.e., the absolute values can be removed and the concurrence is given by the formula

$$C(\Omega_{\mathcal{E}}) = \frac{1}{2}\max\begin{Bmatrix} \lambda_x + \lambda_y + \lambda_z - 1, \\ \lambda_x - \lambda_y - \lambda_z - 1, \\ -\lambda_x + \lambda_y - \lambda_z - 1, \\ -\lambda_x - \lambda_y + \lambda_z - 1, \\ 0 \end{Bmatrix}. \quad (2.6)$$

III. PARAMETRIZATION OF LOCAL CP MAPS: GEOMETRIC PICTURE

We have derived explicit relations for the purity and concurrence of a two-qubit density operator $\Omega_{\mathcal{E}}$ that is obtained via the action of the unital channel $\mathcal{E}$ on the singlet state (1.4). Unfortunately, the concurrence is not so easy to deal with. Let us illustrate the whole situation in the space of the parameters λ_x, λ_y, and λ_z. It is a well-known result of the analysis of qubit channels that unital channels $\Phi_{\mathcal{E}}$ form a tetrahedron with unitary Pauli operators in its vertices. The same geometrical picture holds for states of the form $\Omega_{\mathcal{E}}$. The extremal points of this tetrahedron are mutually orthogonal maximally entangled states (Bell basis). The convex combinations of these states form the tetrahedron—i.e., a classical probability simplex.

The states of the same purity correspond to a sphere of the radius proportional to a specific value of the purity centered at the point $\lambda_x=\lambda_y=\lambda_z=0$. In particular, $|\vec{\lambda}|^2=4P-1$. There are only four pure states represented by the intersection of the tetrahedron with the sphere of the radius $|\vec{\lambda}|^2=3$—i.e., the sphere in which the tetrahedron is embedded. These points are exactly the four maximally entangled states that form the Bell basis.

The equally entangled states specify planes (that form a polytope inside the tetrahedron). Because of the discrete symmetry represented by the four unitary transformations (rotations) described by the operators I, σ_x, σ_y, and σ_z, the analysis of possible values of the concurrence and the purity in terms of $\vec{\lambda}$ can be focused onto only two cases: (i) all λ_j are positive or (ii) all λ_j are negative. The symmetry relating these two options is the space inversion ($\vec{\lambda}\to-\vec{\lambda}$), which is not physical (i.e., this inversion cannot be realized by a CP map). In fact, the tetrahedron does not possess such symmetry. The tetrahedron is symmetric under rotations by an angle $\phi=\pi/2$ along each axis (σ matrices). Consequently, the analysis of the maximum of the concurrence reduces to an investigation of two cases (see Fig. 1): $\lambda_j\geq 0$ (for all j) and $\lambda_j\leq 0$ (for all j).

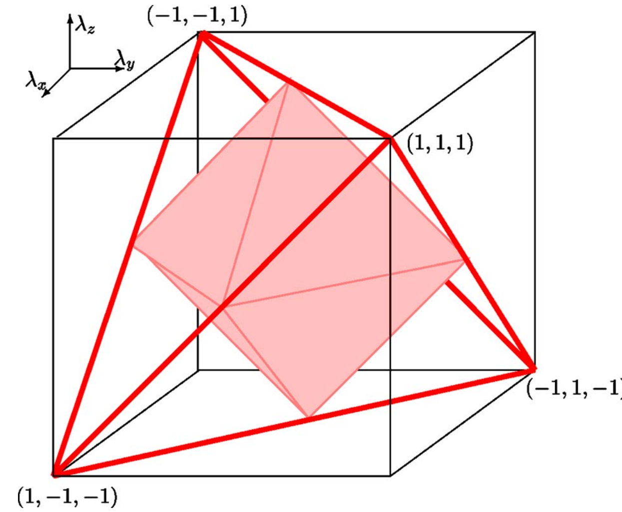

FIG. 1. (Color online) This figure presents the space of unital channels $\Phi_{\mathcal{E}}$ parametrized by $\vec{\lambda}=(\lambda_1,\lambda_2,\lambda_3)$ (with $-1\leq\lambda_j\leq 1$). Up to unitary rotations the set of states $\Omega_{\mathcal{E}}$ that are obtained by the action of unital channels $\mathcal{E}$ on the singlet state (1.4) form a tetrahedron with vertices corresponding to maximally entangled states that form the Bell basis. The octahedron inside the tetrahedron represents the set of all separable states. The remaining four tetrahedra in each corner of the original tetrahedron represent sets of entangled states. The closer the point is to the vertex, the more entangled the corresponding state is. The planes parallel to faces of the separable region contain states with the same degree of entanglement ("isoconcurrence" planes). States with the same purity form spheres ("isopurity" spheres) centered at the center of the cube—i.e., at the point $\vec{\lambda}=(0,0,0)$.

Let us start with the case of positive values of λ's. In this case the maximum of the concurrence (2.6) is achieved for $C(\Omega_{\mathcal{E}})=\frac{1}{2}(\lambda_x+\lambda_y+\lambda_z-1)$. Providing that this number is larger than zero, then states of the same concurrence C form the plane $\lambda_x+\lambda_y+\lambda_z=d$ with $d=2C+1$. These "isoconcurrence" planes intersecting the tetrahedron (its positive octant) are defined by normal vectors $\vec{n}=(1,1,1)$—i.e., $\vec{n}\cdot\vec{\lambda}=2C+1$. The plane with $C=0$ ($d=1$) forms a boundary (face) of separable states in this part of the tetrahedron. Due to the symmetry mentioned above, the same picture holds for the other four vertices of the tetrahedron—i.e., tetrahedrons "under" four maximally entangled states (that correspond to vertices of the tetrahedron).

The negative region of the tetrahedron ($\lambda_j\leq 0$) contains no entangled states; i.e., it consists of only separable states. One can prove property this by analyzing all possibilities or also by exploiting the geometrical picture. The set of allowable (negative) $\vec{\lambda}$ is bounded by the plane $1+\lambda_x+\lambda_y+\lambda_z=0$ that potentially contains entangled states in this "negative region." If not, then due to the convexity of separable states, the whole "negative region" is separable. Using the expressions for λ_x, λ_y, and λ_z determined by the equation of the plane and by calculating the concurrence we obtain $C=\frac{1}{2}\max\{-2,2\lambda_x,2\lambda_y,2\lambda_z,0\}$. Because of $\lambda_j\leq 0$, we obtain that the concurrence always equals zero.

So far, we have shown that entangled states belong to regions close to the vertices of the tetrahedron of all possible states $\Omega_{\mathcal{E}}$. Separable states form the octahedron embedded in the tetrahedron of all states (see Fig. 1). Our next task is to evaluate the concurrence and the purity for all entangled states. In particular, we are interested in the shape of the region formed by the states $\Omega_{\mathcal{E}}$ in the *C-P* phase diagram.

IV. CONCURRENCE-vs-PURITY DIAGRAM

One possible way to characterize bipartite states is to specify their mixedness and the value of their entanglement [6]. Such classification contains highly nontrivial information about the state itself. One can choose different measures for both the mixedness and the entanglement. The resulting characterization strongly depends on the particular choice of measures of entanglement and the mixedness [7,8]. In what follows we will use two most common measures: the purity for the mixedness and the concurrence for the entanglement.

As we have already mentioned the boundaries of the *C-P* phase diagram have been analyzed and the results are known. The states maximizing the concurrence for the given value of purity are known [6] as *maximally entangled mixed states*. This concept generalizes the notion of Bell states—i.e., maximally entangled pure states. MEMS have the following form (up to local unitary transformations):

$$\varrho_{\mathrm{MEMS}} = p|\phi_+\rangle\langle\phi_+| + (1-p)|01\rangle\langle 01| \tag{4.1}$$

for $p \in [2/3, 1]$ and

$$\varrho_{\mathrm{MEMS}} = p|\phi_+\rangle\langle\phi_+| + \frac{1}{3}|01\rangle\langle 01| + \left(\frac{1}{3} - p/2\right)(|00\rangle\langle 00| + |11\rangle\langle 11|) \tag{4.2}$$

for $p \in [0, 2/3]$ and $|\phi_+\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle)$. These states specify the region in the *C-P* phase diagram which is physical; i.e. any point in this part of the *C-P* phase diagram corresponds to a state of quantum-mechanical system (see Fig. 2). Our aim is to analyze this picture for the states of the form $\Omega_{\mathcal{E}}$, i.e., those states that are obtained by the action of local unital operations on the maximally entangled state ψ_s. In particular, our task is to find the maximum-minimum value of the concurrence for the given value of the purity.

As we have already shown, states of equal purity correspond to the sphere (parametrized by $\lambda_x^2+\lambda_y^2+\lambda_z^2=4P-1$) in the tetrahedron of all states $\Omega_{\mathcal{E}}$ (see Fig. 1). States with the same amount of entanglement determine a plane $\lambda_x+\lambda_y+\lambda_z=2C+1$. We have argued that it is sufficient to consider only the positive region ($\lambda_j \geqslant 0$) of the tetrahedron. Given the isopurity sphere, the isoentanglement plane with the maximal value of entanglement is the one that "intersects" the sphere only in a single point. One can exploit the geometric picture to see that this point (state) fulfills the condition $\lambda_x=\lambda_y=\lambda_z\equiv\lambda$. After we insert this condition into the equations for the purity and the concurrence we obtain

$$\left.\begin{aligned} 3\lambda^2 &= 4P-1 \\ 3\lambda &= 2C+1 \end{aligned}\right\} \Rightarrow C_{\max} = \frac{1}{2}[\sqrt{3(4P-1)} - 1]. \tag{4.3}$$

These states form the upper bound of the available region in the *C-P* phase diagram (see Fig. 2). In particular, this bound contains (is formed by) Werner states—i.e., $\varrho_W = q\psi_s + (1-q)\frac{1}{4}I$ and $C_{\max} = C_W$.

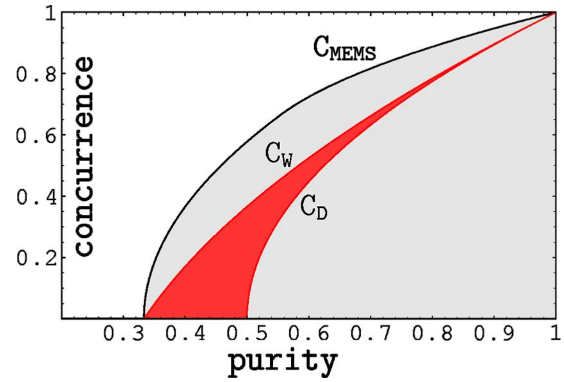

FIG. 2. (Color online) The concurrence-vs-purity phase diagram of two qubits. The region of states $\Omega_{\mathcal{E}}=\mathcal{E}\otimes\mathcal{I}[\psi_s]$ (region colored in dark red in the *C-P* phase diagram) that are obtained from the maximally entangled two-qubit states by the action of unital channels $\mathcal{E}$ is bounded by two lines corresponding to C_W and C_D for a given value of the purity. The upper bound of the region of physically realizable states (shaded sector of the *C-P* diagram) corresponds to maximally entangled mixed states (MEMS's). We note that the MEMS's cannot be achieved from Bell states by applying local channels on a *single* subsystem only. Both the concurrence and purity are presented in dimensionless units.

The next step is to analyze the minimum of the concurrence for a given value of purity. In general this minimum is trivial, because there always exist separable states for a given value of purity. However, in our case we investigate states that are generated by local unital channels from the maximally entangled Bell states. In this case a nontrivial lower bound exists. Exploiting the geometry of the states $\Omega_{\mathcal{E}}$ we can conclude that this minimum is zero for all purity spheres for which the intersection with the tetrahedron contains the plane $\lambda_1+\lambda_2+\lambda_3=1$ ($C=0$)—i.e., the boundary of the separable states. The states of the maximal purity belonging to this plane (i.e., the maximally mixed separable states) are the points with the two same components and the remaining one equals to unity. Without loss of generality let us consider the case $\lambda_z=1$ and $\lambda_x=\lambda_y\equiv\lambda$. The equation of the plane implies that $\lambda=0$ and, consequently, the purity $P=\frac{1}{4}(1+|\vec{\lambda}|^2)=\frac{1}{2}$. That is, for states with a purity larger than $1/2$ there are no separable states $\Omega_{\mathcal{E}}$.

The question about the minimum value of entanglement for a given value of the purity is equivalent to the question about the maximum of the purity for a given value of the concurrence. Using the same arguments as before we find that the state of the maximal purity satisfies the same conditions—i.e., $\lambda_z=1$ and $\lambda_x=\lambda_y\equiv\lambda$. Consequently, the equation of plane $\lambda_x+\lambda_y+\lambda_z=1+2C$ implies $\lambda=C$. As a result we obtain that $P=\frac{1}{2}(1+C^2)$. Inverting this formula we obtain the functional dependence of the concurrence as a function of the purity that specifies the lower bound of the allowable region in the *C-P* phase diagram for states $\Omega_{\mathcal{E}}$:

$$C_{\min} = \sqrt{2P-1}. \tag{4.4}$$

In conclusion, states induced by local unital channels from the initial maximally entangled state are represented in the concurrence-versus-purity diagram by points in the region that is bounded from above by the "line" $C_{\max}=C_W$ and from below by the line $C_{\min}$ (see the dark red region in Fig. 2). The whole region of physically relevant states is determined by the line C_{MEMS} corresponding to the MEMS. Unital channels do not allow us to achieve states that are within the lines C_W and C_{MEMS}. Simultaneously, we can conclude that the states $\Omega_{\mathcal{E}}$ with the purity larger than 1/2 remain entangled (this property is given by the lower bound $C_{\min}=C_D$).

V. NONUNITAL CHANNELS

In the previous section we have shown that unital channels (up to unitary rotations) are characterized by three parameters. The nonunital channels are characterized by six parameters. The triple $\vec{\tau}=(\tau_x,\tau_y,\tau_z)$ describes how the total mixture is transformed under the action of nonunital channels. In particular, under the action of a general nonunital map $\mathcal{E}$, the maximally entangled state (e.g., the singlet) is transformed into the state

$$\Omega_{\mathcal{E}}=\frac{1}{4}[(I+\vec{\tau}\cdot\vec{\sigma})\otimes I-\vec{\lambda}\cdot(\vec{\sigma}\otimes\vec{\sigma})], \tag{5.1}$$

where $\vec{\lambda}\cdot(\vec{\sigma}\otimes\vec{\sigma})=\lambda_x\sigma_x\otimes\sigma_x+\lambda_y\sigma_y\otimes\sigma_y+\lambda_z\sigma_z\otimes\sigma_z$. In other words,

$$\Omega_{\mathcal{E}}=\begin{pmatrix} A+\tau & F & 0 & D \\ F^* & B-\tau & C & 0 \\ 0 & C & B+\tau & F \\ D & 0 & F^* & A-\tau \end{pmatrix}, \tag{5.2}$$

where A, B, C, and D are defined as before and $F=(\tau_x-i\tau_y)/4$ and $\tau=\tau_z/4$. First, we address the question whether by using nonunital channels the upper bound on concurrence can be increased, in particular whether $\Omega_{\mathcal{E}}$ can be a MEMS state (i.e., the state on the C_{MEMS} line). In order to answer this question we use the identity $\mathrm{Tr}_A\Omega_{\mathcal{E}}=\mathrm{Tr}_A\psi_s=\frac{1}{2}I$ which holds for any *local* action $\mathcal{E}\otimes\mathcal{I}$. For MEMS states we find that $\mathrm{Tr}_A\varrho_{\mathrm{MEMS}}\neq\frac{1}{2}I$ and $\mathrm{Tr}_B\varrho_{\mathrm{MEMS}}\neq\frac{1}{2}I$ as well; i.e., $\Omega_{\mathcal{E}}$ cannot be a MEMS state. Therefore, we conclude that the MEMS states cannot be achieved from the maximally entangled state via local operations; i.e., MEMS are not the states of the form $\Omega_{\mathcal{E}}=\Phi_{\mathcal{E}}\otimes\mathcal{I}[\psi_s]$.

However, the action of nonunital channels on maximally entangled states can be different than the action of unital channels. That is, the states $\Omega_{\mathcal{E}}=\Phi_{\mathcal{E}}\otimes\mathcal{I}[\psi_s]$ that are obtained by the action of nonunital channels might lie in the region of the concurrence-versus-purity phase diagram that is bounded by the lines C_{MEMS} and C_W. The purity of states $\Omega_{\mathcal{E}}$ can be calculated by finding the trace of $\Omega_{\mathcal{E}}^2$. A direct calculation gives that $P=\frac{1}{4}(1+|\vec{\lambda}|^2+|\vec{\tau}|^2)$. Thus in the picture of $\vec{\lambda}$ parameters (for a fixed vector $\vec{\tau}$) the states of the same purity belong to the sphere $|\vec{\lambda}|^2=4P-1-|\vec{\tau}|^2$. Note that for the fixed $\vec{\tau}$ the set of all possible $\vec{\lambda}$ do not form a tetrahedron anymore.

Let us consider a specific case $\vec{\tau}=(0,0,\tau_z)$—i.e., $F=0$. In this case the eigenvalues of $\Omega_{\mathcal{E}}$ read

$$\mu_1=\frac{1}{4}(1-\lambda_z+\sqrt{(\lambda_x-\lambda_y)^2+\tau_z^2}),$$

$$\mu_2=\frac{1}{4}(1-\lambda_z-\sqrt{(\lambda_x-\lambda_y)^2+\tau_z^2}),$$

$$\mu_3=\frac{1}{4}(1+\lambda_z+\sqrt{(\lambda_x+\lambda_y)^2+\tau_z^2}),$$

$$\mu_4=\frac{1}{4}(1+\lambda_z-\sqrt{(\lambda_x+\lambda_y)^2+\tau_z^2}).$$

The positivity of these eigenvalues determines the set of all allowed values of parameters $\vec{\lambda}$, τ_z. Fixing τ_z we obtain four surfaces (setting $\mu_j=0$) in the three-dimensional space of parameters $\vec{\lambda}$ that form boundaries of the set of all possible states $\Omega_{\mathcal{E}}$. The identities $\mu_j=0$ can be rewritten as the equations

$$\lambda_z=1+\sqrt{(\lambda_x-\lambda_y)^2+\tau_z^2},$$

$$\lambda_z=1-\sqrt{(\lambda_x-\lambda_y)^2+\tau_z^2},$$

$$\lambda_z=-1-\sqrt{(\lambda_x+\lambda_y)^2+\tau_z^2},$$

$$\lambda_z=-1+\sqrt{(\lambda_x+\lambda_y)^2+\tau_z^2}. \tag{5.3}$$

Let us note that only the second and fourth of these equalities fulfills the (cube) constraints $|\lambda_j|\leqslant 1$, and therefore these two equalities completely specify the shape of the set in the space of parameters λ_x, λ_y, and λ_z. For nonunital channels the vertices of the tetrahedron are smoothed ("rounded") depending on the value of the shift $\vec{\tau}$.

In order to compute the concurrence C of state $\Omega_{\mathcal{E}}=\Phi_{\mathcal{E}}\otimes\mathcal{I}[\psi_s]$ we have to find eigenvalues of the matrix

$$R=\Omega_{\mathcal{E}}(\sigma_y\otimes\sigma_y)\Omega_{\mathcal{E}}^*(\sigma_y\otimes\sigma_y)=\begin{pmatrix} \alpha & 0 & 0 & \delta_+ \\ 0 & \beta & \gamma_- & 0 \\ 0 & \gamma_+ & \beta & 0 \\ \delta_- & 0 & 0 & \alpha \end{pmatrix},$$

where $\alpha=A^2-\tau^2+D^2$, $\beta=B^2-\tau^2+C^2$, $\gamma_\pm=2C(B\pm\tau)$, and $\delta_\pm=2D(A\pm\tau)$ $(\tau=\tau_z/4)$. Square roots of the eigenvalues of the matrix R read

$$\{\sqrt{A^2-\tau^2}\pm|D|,\sqrt{B^2-\tau^2}\pm|C|\}, \tag{5.4}$$

and for the concurrence we obtain the expression

$$C=\max\{0,2(|D|-\sqrt{B^2-\tau^2}),2(|C|-\sqrt{A^2-\tau^2})\}. \tag{5.5}$$

Using the parameters $\vec{\lambda}$, τ_z the concurrence can be rewritten in the form

$$C=\frac{1}{2}\max\{0,\ |\lambda_y-\lambda_x|-\sqrt{(1+\lambda_z)^2-\tau_z^2},\ |\lambda_x+\lambda_y| -\sqrt{(1-\lambda_z)^2-\tau_z^2}\}. \tag{5.6}$$

As we will show in the next section the states $\Omega_{\mathcal{E}}=\Phi_{\mathcal{E}}$

$\otimes\mathcal{I}[\psi_s]$ that are generated by nonunital channels can lie above the line $C_{\max}$ in the concurrence-versus-purity diagram. In order to have more physical insight into the action of the local channels on maximally entangled states let us consider a one-parametric set of local maps that correspond to specific time evolutions of two-qubit systems.

VI. EVOLUTION IN THE *C-P* DIAGRAM

In this section we will analyze how the evolution of a maximally entangled state under the action of a local channel is reflected in the *C-P* diagram. The case of unitary evolution is trivial: Under the action of local unitary transformations neither the concurrence nor the purity is changed. Therefore, the state $\Omega_t=(U_t\otimes I)\psi_s(U_{-t}\otimes I)$ is still a maximally entangled pure state. In what follows we will analyze several models of nonunitary dynamics. We will focus our attention on Markovian semigroup dynamics. In particular, we will consider processes of the decoherence, the depolarization, and the homogenization.

A. Decoherence

The decoherence of a qubit is induced by the master equation [17] $\dot{\varrho}=i[H,\varrho]+(T/2)[H,[H,\varrho]]$. The solution of this equation forms a semigroup of unital quantum channels:

$$\mathcal{E}_t=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & e^{-t/T}\cos\omega t & e^{-t/T}\sin\omega t & 0\\ 0 & -e^{-t/T}\sin\omega t & e^{-t/T}\cos\omega t & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \qquad (6.1)$$

The singular values of these channels are

$$\lambda_1(t)=\lambda_2(t)=e^{-t/T},$$

$$\lambda_3(t)=1. \qquad (6.2)$$

Due to the evolution of one qubit, the singlet is transformed into a state Ω_t with the purity

$$P_t=\frac{1}{4}(1+|\vec{\lambda}_t|^2)=\frac{1}{2}(1+e^{-2t/T}) \qquad (6.3)$$

and with the concurrence

$$C_t=\frac{1}{2}(\lambda_1(t)+\lambda_2(t)+\lambda_3(t)-1)=e^{-t/T}. \qquad (6.4)$$

Comparing these two equations we find that the purity and concurrence of the state $\Omega_t=\Phi_{\mathcal{E}}\otimes\mathcal{I}[\psi_s]$ induced by the decoherence channel acting on one qubit are related as

$$P_t=\frac{1}{2}(1+C_t^2) \qquad (6.5)$$

or, equivalently,

$$C_t=\sqrt{2P_t-1}. \qquad (6.6)$$

Since $P_t\in[1/2,1]$, we have obtained that the process of single-qubit decoherence in the *C-P* diagram corresponds to the lower bound of the allowed region for unital channels—i.e., $C_t:=C_D=C_{\min}$.

B. Depolarization

The process of a single-qubit depolarization in a specific basis is represented by the semigroup [14]

$$\mathcal{E}_t=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & e^{-t/T} & 0 & 0\\ 0 & 0 & e^{-t/T} & 0\\ 0 & 0 & 0 & e^{-t/T}\end{pmatrix}. \qquad (6.7)$$

In this case the states Ω_t are Werner states; i.e., this type of dynamics in the *C-P* diagram is represented by the line C_W for unital channels. In particular,

$$\Omega_t=e^{-t/T}\psi_s+(1-e^{-t/T})\frac{1}{4}I. \qquad (6.8)$$

Thus, we have found two processes that saturate the upper and lower bounds of the region in the *C-P* diagram that is allowed for unital channels. In particular, the decoherence saturates the lower bound while the depolarization process defines the upper bound.

C. Homogenization

The process of quantum homogenization is described by the semigroup of nonunital channels [18,19]

$$\mathcal{E}_t=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \multicolumn{2}{c}{e^{-t/T_2}\ R_{\omega t}} & 0\\ 0 & & & 0\\ w(1-e^{-t/T_1}) & 0 & 0 & e^{-t/T_1}\end{pmatrix}, \qquad (6.9)$$

where

$$R_{\omega t}=\begin{pmatrix}\cos\omega t & \sin\omega t\\ -\sin\omega t & \cos\omega t\end{pmatrix} \qquad (6.10)$$

is a rotation matrix. This process describes an evolution that transforms the whole Bloch sphere into a single point—i.e., a generalization of an exponential decay. That is, quantum homogenization is described by a contractive map with the fixed point that is the stationary state of the dynamics. The parameters in the description of the map (6.9) have the following meaning: w is the purity of the final state, T_1 is the decay time, T_2 is the decoherence time, and ω describes the unitary part of the evolution. The singular values are similar to those in the decoherence—i.e.,

$$\lambda_1(t)=\lambda_2(t)=e^{-t/T_2},$$

$$\lambda_3(t)=e^{-t/T_1}. \qquad (6.11)$$

The homogenization belongs to a class of nonunital channels that we have analyzed in the previous section. Therefore we can easily find expressions for the purity,

$$P_t=\frac{1}{4}[1+2e^{-2t/T_2}+e^{-2t/T_1}+w^2(1-e^{-t/T_1})^2], \qquad (6.12)$$

and the concurrence,

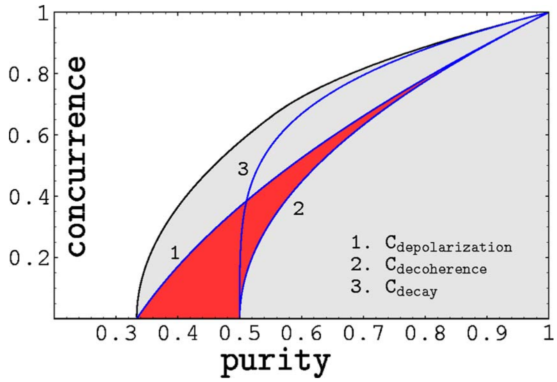

FIG. 3. (Color online) Parametric plots of the time evolution of the concurrence and purity in the processes of decoherence, homogenization, and depolarization. The lines are parametrized in such a way that the initial moment of the time evolution $t=0$ corresponds to the point $C=P=1$ (upper right corner of the C-P diagram). The decoherence line (2) represents the lower bound (C_D) of the unital region; the depolarization line (1) forms the upper bound (C_W) of the unital region. The homogenization (a nonunital process) is characterized by line (3), which is outside the unital region. In the present case the homogenization describes an exponential decay; i.e., the fixed point of the evolution is the state $|0\rangle$. Both the concurrence and purity are presented in dimensionless units.

$$C_t = \max\left\{0, e^{-t/T_2} - \frac{1}{2}(1 - e^{-t/T_1})\sqrt{1 - w^2}\right\}. \quad (6.13)$$

The resulting lines for all considered evolutions (decoherence, depolarization, and homogenization for the values $T_1/T_2=1/2$ and $w=1$) are depicted in Fig. 3.

In a special case, when $w=0$ (the final state is the total mixture), the homogenization process is unital. In this case the purity and the concurrence are

$$P_t = \frac{1}{4}(1 + e^{-2t/T_1} + 2e^{-2t/T_2}), \quad (6.14)$$

$$C_t = e^{-t/T_2} + \frac{1}{2}(e^{-t/T_1} - 1). \quad (6.15)$$

In Fig. 5, we can see the corresponding line for different values of the decay and decoherence times—i.e., T_1 and T_2, respectively. The interesting point is when these two rates coincide ($T_1=T_2$) when the homogenization saturates the Werner states line (i.e., C_W). In the limit of $T_1/T_2\to\infty$ the homogenization approaches the decoherence line (i.e., C_D).

For $w=1$ the homogenization describes the exponential decay to a pure state and

$$P_t = \frac{1}{2}(1 + e^{-2t/T_1} + e^{-2t/T_2} - e^{-t/T_1}), \quad (6.16)$$

$$C_t = e^{-t/T_2}. \quad (6.17)$$

In this case one can express the purity as a function of the concurrence—i.e., $P=\frac{1}{2}(1+C^2+C^{2T_2/T_1}-C^{T_2/T_1})$. In the special case $T_2=2T_1$ (i.e., the decoherence time is twice as fast as the decay time) the homogenization follows the line

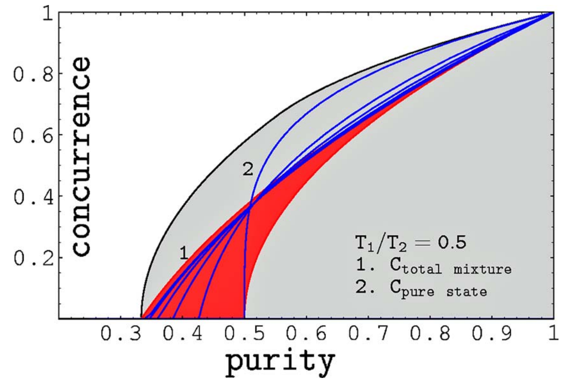

FIG. 4. (Color online) We present evolutions of the concurrence and purity for the homogenization process when the ratio of the characteristic times T_1 and T_2 is constant and $T_1/T_2=1/2$ while the fixed point of the evolution varies from the total mixture (a unital process) to a pure state (nonunital process). In particular, when the fixed point of the homogenization is the total mixture, then the homogenization dynamics in the C-P diagram is described by line (1). This line lies below the line C_W with the final point $C=0$ having the smallest value of the purity. On the other hand, when the fixed point of the evolution is a pure state, then the corresponding C-P line (2) ends in the point $C=0$ with the largest value of the purity $P=1/2$. Both the concurrence and purity are presented in dimensionless units.

$$C = \sqrt[4]{2P - 1}. \quad (6.18)$$

This line lies above the region determined by unital channels (see Fig. 3).

In Figs. 4–6 we analyze various regimes of the homogenization process: We consider processes with a fixed value of the ratio $T_1/T_2=1/2$, but we change fixed points of the evolution (Fig. 4). In Fig. 5 we consider the homogenization process with the fixed point being equal to the total mixture (i.e., in this case the homogenization is a unital process) and

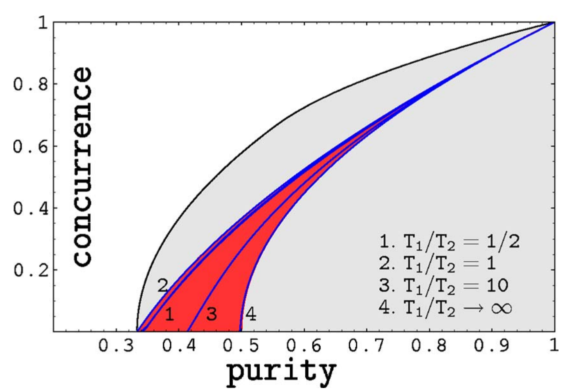

FIG. 5. (Color online) The evolution of concurrence and purity for the homogenization with the fixed point equal to the total mixture. In this case the homogenization is a unital process. We vary the ratio T_1/T_2 in the interval $[1/2,\infty]$. For $T_1=T_2$ the homogenization exactly covers line (2) of Werner states (C_W). In the limit of $T_1/T_2\to\infty$ this evolution coincides with the line of decoherence (C_D). Both the concurrence and purity are presented in dimensionless units.

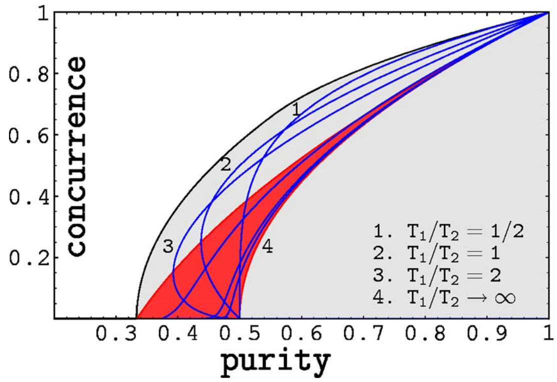

FIG. 6. (Color online) The parametric plot of the concurrence and purity for the homogenization with the fixed point that corresponds to a pure state. That is, we consider an exponential decay into a pure state. We vary the T_1/T_2 in the interval $[1/2,\infty]$. In this case, the evolution ends in the point $C=0$ and $P=1/2$ irrespective of the particular value of the ratio T_1/T_2. When $T_1/T_2=1/2$ the evolution is represented by the line which is "deep" in the nonunital region of the C-P diagram. On the contrary, for $T_1/T_2\rightarrow\infty$ the corresponding line coincides with the bound C_D (the decoherence line). Both the concurrence and purity are presented in dimensionless units.

we consider different values of the ratio T_1/T_2. In Fig. 6 we consider the homogenization process with the fixed point equal to a pure state (i.e., in this case the homogenization is a nonunital process) and we consider different values of the ratio T_1/T_2.

VII. CONCLUSION

In this paper we have investigated how a local transformation (quantum channel described by a CP map) of a subsystem affects the entanglement and global purity of the initial singlet state (or, equivalently, arbitrary maximally entangled states) of two qubits. We have analyzed the states $\varrho_{\mathcal{E}}=\mathcal{E}\otimes\mathcal{I}[\psi_s]$. We have used the concurrence and purity of these states to classify local channels $\mathcal{E}$. In particular, using the concurrence-versus-purity diagrams we have specified the region induced by local unital channels. This region does not cover the whole set of physically realizable states (see Fig. 2). Local unital channels induce states that are represented by points in the region of the C-P diagram that is bounded from below by the line C_D and from above by the line C_W. We have shown that even the most general (including nonunital) local channels acting just on a single qubit of the original singlet state cannot generate MEMS. This means that the upper bound specified by the $C_{\rm MEMS}$ line cannot be achieved by the action of the local channel of the form $\varrho_{\mathcal{E}}=\mathcal{E}\otimes\mathcal{I}[\psi_s]$. The specific achievable upper bound for nonunital maps is to be determined. It definitely is above the line C_W (except the values of the concurrence $C=1$ and $C=0$ when it coincides with the bound on unital maps). From our analysis it follows that for unital maps the lower bound of the achievable region is determined by the line C_D. We conjecture that this is also a general lower bound for nonunital maps. This conjecture is supported by our numerical analysis, but we have no rigorous proof yet.

From our previous analysis an interesting observation follows. Specifically, if the state $\varrho_{\mathcal{E}}=\mathcal{E}\otimes\mathcal{I}[\psi_s]$ has a purity larger than $1/2$, then the concurrence has to be nonzero; correspondingly, the state is entangled. On the other hand, if the purity is less than $1/3$, then the state is separable.

In our paper we have analyzed neither the action of the bilocal channels ($\mathcal{E}_1\otimes\mathcal{E}_2$) nor the action of the nonlocal maps $\mathcal{E}_{12}$. It is clear that using nonlocal maps any point of the C-P diagram below the MEMS line $C_{\rm MEMS}$ can be achieved; i.e., the state $\varrho_{\mathcal{E}}=\mathcal{E}_{12}[\psi_s]$ can have an arbitrary value of the concurrence and purity that is in the region specified by the bound $C_{\rm MEMS}$. On the other hand, bilocal operations generate states $\varrho_{\mathcal{E}}=\mathcal{E}_1\otimes\mathcal{E}_2[\psi_s]$ that are represented by the points in a region of the C-P diagram that is restricted from above and from below. Specific boundaries are not known. From particular examples one can conclude that if the two local transformations are nonunital maps representing the exponential decay, then one can achieve states with zero concurrence but maximal purity. We will analyze these bounds elsewhere.

We have studied in detail time evolutions described by Markovian semigroups. In particular, we have shown that the decoherence process is represented by a line that forms the lower bound C_D of the achievable region for unital maps. In our parametrization the time $t=0$ is represented by the point $C=1$ and $P=1$, while the point of the entanglement destruction ($C=0$) is achieved at some "entanglement-breaking" time t_{sep} that is infinite—see discussion below.

The depolarization process saturates the upper bound (C_W) of the region of maps induced by unital maps. Here the entanglement-breaking time is finite and equals $t_{sep}=T\ln 3$. For dynamics that are described by nonunital maps (e.g., homogenization processes) the entanglement-breaking times can be both finite as well as infinite (as in the case of the exponential decay).

We have paid attention only to evolutions governed by semigroups—i.e., by Markovian processes. A general rule is that for this type of time evolutions the associated C-P line must be nonincreasing, because local action cannot create entanglement. This property prevents loops in C-P from being in the diagram, but still allows that the purity increases while the concurrence is decreasing (see Fig. 6). For non-Markovian evolutions it is possible to observe even loops; however, the question of more general dynamics is out of the scope of this paper and will be discussed elsewhere.

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