Quantum averages of weak values

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We re-examine the status of the weak value of a quantum mechanical observable as an objective physical concept, addressing its physical interpretation and general domain of applicability. We show that the weak value can be regarded as a *definite* mechanical effect on a measuring probe specifically designed to minimize the back reaction on the measured system. We then present an interesting framework for general measurement conditions (where the back reaction on the system may not be negligible) in which the measurement outcomes can still be interpreted as *quantum averages of weak values*. We show that in the classical limit, there is a direct correspondence between quantum averages of weak values and posterior expectation values of classical dynamical properties according to the classical inference framework.

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I. INTRODUCTION

In previous publications [1–6], an objective description of a quantum system in the time interval between two complete measurements has been proposed in terms of *two* state vectors, together with an interesting type of physical quantity, the "weak value" of a quantum-mechanical observable. Specifically, for a system drawn from an ensemble preselected in the state $|\psi_1\rangle$ and postselected in the state $|\psi_2\rangle$, the weak value for the observable $\hat{A}$ is defined as

$$A_w \equiv \frac{\langle \psi_2|\hat{A}|\psi_1\rangle}{\langle \psi_2|\psi_1\rangle}, \tag{1.1}$$

where the real part is the quantity of primary physical interest (and to which the term "weak value" shall henceforth apply unless otherwise noted). The suggestion was motivated operationally by the fact that both real and imaginary parts of weak values can be linked to conditional measurement statistics predicted by standard quantum mechanics for the general class of "weak measurements," defined so as to minimize the disturbance to the system as a result of a diminished interaction with the measuring instrument. Under these conditions, joint weak measurements of two noncommuting observables can be made with negligible mutual interference, thus ensuring that the simultaneous assignment of weak values to all elements of the observable algebra is operationally consistent.

The usefulness of this description has been demonstrated, both theoretically and experimentally, in a number of applications in which interesting aspects of quantum processes have been uncovered when analyzed in terms of weak values. These include photon polarization interference [7–11], barrier tunneling times [12–14], photon arrival times [15,16], anomalous pulse propagation [17–19], correlations in cavity QED experiments [20], complementarity in "which-way" experiments [21,22], nonclassical aspects of light [23,24], communication protocols [25], and retrodiction "paradoxes" of quantum entanglement [26–28].

A certain amount of skepticism [29–34] has nevertheless prevailed regarding the physical status of weak values, particularly in the light of the unconventional range of values that is possible according to (1.1). Indeed, the real part of A_w, describing the "pointer variable" response in a weak measurement, may lie outside the bounds of the spectrum of $\hat{A}$. Manifestly "eccentric" weak values, as are negative kinetic energies [35,36] or negative particle numbers [26,28], are not easily reconciled with the physical interpretation that is traditionally attached to the respective observables. Less intuitive yet is when $\hat{A}$ stands for a projection operator, in which case the weak value suggests "weak probabilities" taking generally nonpositive values [13,22,37]. Such bizarre interpretations call for a sharper clarification of what physical meaning should be attached to the weak value of an observable.

Another item of skepticism surrounding the physical significance of weak values has to do with their general domain of applicability. It seems reasonable to demand of any new physical concept that it be applicable to a wide variety of situations outside the restricted context in which it is defined operationally. Although progress has been made in this direction [5,38], convincing evidence of the general validity of the concept of the weak value is still lacking.

With these questions in mind, the aim of this paper is twofold: First, we address the physical meaning of weak values by showing that there exists an unambiguous interpretation of the real part of the weak value as a *definite* mechanical effect of the system on a measuring probe that is specifically designed to minimize the dispersion in the back reaction on the system. Second, based on this interpretation, we present a framework for the analysis of general von Neumann measurements, in which the measurement statistics are interpreted as *quantum averages of weak values* (QAWV).

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We believe this framework is physically intuitive and provides compelling evidence for the ubiquity of weak values in more general measurement contexts. In particular, we show that for arbitrary system ensembles, the expectation value of the reading of any von-Neumann–type measurement is an average of weak values over a suitable posterior probability distribution. We furthermore show how QAWV framework has a natural correspondence in the classical limit with the posterior analysis of measurement data according to the classical inference framework. Thus, we can establish a correspondence between weak values and what in the macroscopic domain are regarded as objective classical dynamical variables.

The paper is structured as follows: In Sec. II, we motivate the idea of averaging weak values by discussing the connection between preselected and pre- and postselected statistics in arbitrary von-Neumann–type measurements. In Sec. III we present the operational definition of the weak value as a definite mechanical effect associated with infinitesimally uncertain unitary transformations. The QAWV framework is then introduced in Sec. IV for arbitrary strength measurements. We provide an illustration in Sec. V, where we discuss a number of measurement situations in which the framework gives a simple characterization of the outcome statistics. Finally, we establish in Sec. VI the classical correspondence of the QAWV framework. Some conclusions are given in Sec. VII.

II. PRE- AND POSTSELECTED MEASUREMENT STATISTICS, EIGENVALUES, AND WEAK VALUES

The conventional interpretation of a quantum mechanical expectation value, such as $\langle\psi|\hat{A}|\psi\rangle$ for an observable $\hat{A}$, is as an average of the eigenvalues of $\hat{A}$ over a probability distribution that is realized in the context of a complete strong measurement of $\hat{A}$. Our main suggestion in this paper is that for a wide class of generalized conditions on the von Neumann measurement of $\hat{A}$, the statistics of measurement outcomes can alternatively be related to underlying statistics of a different quantity, the weak value of $\hat{A}$, which is to be regarded as a definite physical property of an unperturbed quantum system in the time interval between two complete measurements. We shall therefore begin by discussing in this preliminary section the connection between pre- and postselected measurement statistics of arbitrary strength von-Neumann measurements, and from this discussion show an instance in which averages of weak values more aptly describe the posterior breakup of the measurement outcome distribution.

In the von-Neumann measurement scheme [39], the device is some external system, described by canonical variables $\hat{q}$ and $\hat{p}$, with $[\hat{q},\hat{p}]=i$ ($\hbar\equiv 1$). The system-device interaction is designed so that the measurement result is read-off from the effect on some designated device "pointer variable," which we take to be $\hat{p}$. For a measurement of the system observable $\hat{A}$ at the time $t=t_i$, this interaction is modeled by the impulsive Hamiltonian

$$\hat{H}_m = -\delta(t-t_i)\hat{A}\hat{q}. \tag{2.1}$$

(Note that a possible coupling constant can always be absorbed by canonically redefining q and p.) The effect of the measurement is then described by the unitary operator $\hat{U}=e^{i\hat{A}\hat{q}}$. Since we will only be concerned with the effect of this interaction from times immediately before to immediately after t_i, we shall henceforth assume the all additional free evolution is already contained in the states.

We first consider the pointer variable statistics from an ensemble defined by pure initial conditions on the system and the apparatus, described by states $|\psi_1\rangle$ and $|\phi\rangle$, respectively. For later convenience, we shall term this ensemble the *preselected measurement ensemble* (PME) Ω_1. Further, we introduce the notation $\prec$ or $\succ$ to denote times immediately before or immediately after the measurement time t_i. Now, for the PME Ω_1, the effect of the measurement interaction is easily described by the Heisenberg picture transformation induced by the evolution operator $e^{i\hat{A}\hat{q}}$:

$$\hat{p}_{\succ} = \hat{p}_{\prec} + \hat{A}. \tag{2.2}$$

Since the initial system plus the apparatus state is separable, the final statistics of the pointer variable are easily obtained. Assuming $\langle p\rangle_\phi \equiv \langle\phi|p|\phi\rangle = 0$, the first two moments are given by

$$\langle p\rangle_{\Omega_1^{\succ}} = \langle\psi_1|\hat{A}|\psi_1\rangle \tag{2.3}$$

$$\langle \Delta p^2\rangle_{\Omega_1^{\succ}} = \langle \Delta p^2\rangle_\phi + \langle\psi_1|\Delta\hat{A}^2|\psi_1\rangle. \tag{2.4}$$

More generally, from the spectral decomposition of $\hat{A}$, one has the probability distribution

$$\mathcal{P}(p|\Omega_1^{\succ}) = \sum_a \langle\psi_1|\hat{\Pi}_a|\psi_1\rangle \mathcal{P}(p-a|\phi), \tag{2.5}$$

where $\mathcal{P}(p|\phi)=|\langle p|\phi\rangle|^2$, and $\hat{\Pi}_a$ is the projector onto the system eigenspace with eigenvalue a. In this description, a "strong" or projective measurement corresponds to the limit $\Delta p\to 0$ [i.e., $\mathcal{P}(p|\phi)\to\delta(p)$], in which case the pointer distribution mimics the spectral distribution of the Born interpretation, $\langle\psi_1|\hat{\Pi}_a|\psi_1\rangle$. Note however, that Eqs. (2.3) and (2.5) are valid regardless of the width of the initial pointer distribution, or in other words, regardless of whether the conditions for a projective measurement are satisfied or not. So even when the spectrum of $\hat{A}$ cannot be resolved, the preselected measurement statistics can still be interpreted in very much the same way as the standard von-Neumann measurement statistics; namely, *as if* on every single trial, the pointer is displaced in proportion to one of the eigenvalues of $\hat{A}$ with the eigenvalues distributed randomly throughout the sample according to $\langle\psi_1|\hat{\Pi}_a|\psi_1\rangle$.

Now suppose that after time t_i, a postselection is performed on the system, and we wish to concentrate on the subset of measurement outcomes arising only from those systems that ended up in some specific state, $|\psi_2\rangle$ in this postselection. This final condition defines for us a suben-

semble Ω_{12} of the PME Ω_1, which we call a *pre- and post-selected measurement ensemble* (PPME). Measurement statistics from this ensemble can be obtained from the conditional final state of the apparatus [1]

$$|\tilde{\phi}_{12}^{\succ}\rangle = \frac{1}{\sqrt{\mathcal{P}_{12}(\phi)}}\langle\psi_2|e^{i\hat{A}\hat{q}}|\psi_1\rangle|\phi\rangle, \tag{2.6}$$

where the normalization $\mathcal{P}_{12}(\phi)$ is shorthand for the probability $\mathcal{P}(\psi_2|\psi_1\phi)$ of obtaining $|\psi_2\rangle$ given the measurement interaction, or equivalently the expected relative size of the ensemble Ω_{12}. From the state $|\tilde{\phi}_{12}^{\succ}\rangle$, the corresponding pointer variable distribution is given by $\mathcal{P}(p|\Omega_{12}^{\succ}) = |\tilde{\phi}_{12}^{\succ}(p)|^2$.

Let us briefly discuss some relations between the PME and PPME statistics. Suppose the postselection involves a complete measurement of some nondegenerate observable $\hat{B}$, with eigenstates $\{|b\rangle\}$. A pooling of the data from all the subensembles $\{\Omega_{1b}\}$ of Ω_1 must then yield the preselected distribution [Eq. (2.5)], in other words

$$\mathcal{P}(p|\Omega_1^{\succ}) = \sum_b \mathcal{P}_{1b}(\phi)\mathcal{P}(p|\Omega_{1b}^{\succ}), \tag{2.7}$$

where $\mathcal{P}_{1b}(\phi)$ is the relative size of each PPME. Two important consequences follow from this decomposition: First, since the PME expectation value of the pointer $\langle p\rangle_{\Omega_1^{\succ}}$ breaks up similarly into $\langle p\rangle_{\Omega_1^{\succ}} = \Sigma_b \mathcal{P}_{1b}(\phi)\langle p\rangle_{\Omega_{1b}^{\succ}}$, then from (2.3) and assuming $\langle\phi|p|\phi\rangle = 0$, the weighted average of the PPME pointer expectation values has to yield the standard expectation value of $\hat{A}$. Thus, one has the sum rule

$$\langle\psi_1|\hat{A}|\psi_1\rangle = \sum_b \mathcal{P}_{1b}(\phi)\langle p\rangle_{\Omega_{1b}^{\succ}}. \tag{2.8}$$

A second consequence of (2.7) is a "covering" condition satisfied by the individual PPME distributions,

$$\mathcal{P}(p|\Omega_1^{\succ}) \geqslant \mathcal{P}_{1b}(\phi)\mathcal{P}(p|\Omega_{1b}^{\succ}), \tag{2.9}$$

for all values of p and all final outcomes b. This imposes a constraint on how rare a PPME Ω_{1b} has to be whenever the corresponding $\mathcal{P}(p|\Omega_{1b}^{\succ})$ is peaked somewhere in the tail region of $\mathcal{P}(p|\Omega_1^{\succ})$.

The relevance of (2.8) and (2.9) is that indeed the weight of a PPME measurement outcome distribution need not lie within the "normal" region of the expectation defined by the bounds of the spectrum of $\hat{A}$ (Fig. 1), contrary to what one would have naively expected given the generality of the spectral expansion of the PME distribution (2.5). This is, of course, not obvious under strong measurement conditions. If the apparatus wave function in p is expanded as a superposition of shifted wave functions

$$\tilde{\phi}_{12}^{\succ}(p) \propto \sum_a \langle\psi_2|\hat{\Pi}_a|\psi_1\rangle\phi(p-a), \tag{2.10}$$

then the resolution of the spectrum under strong measurement conditions is equivalent to demanding that the overlap between two shifted functions $\phi(p-a)$ and $\phi(p-a')$ for all $a \neq a'$ be negligible, in which case the pointer variable is concentrated within the range of the spectrum of $\hat{A}$. However, away from strong measurement conditions, $\mathcal{P}(p|\Omega_{12}^{\succ})$ will involve interference terms between the shifted wave functions $\phi(p-a)$ with coefficients $\langle\psi_2|\hat{\Pi}_a|\psi_1\rangle$ that are not generally real nor positive-definite, preventing the resolution of the individual shifted peaks and allowing for destructive interference effects that may place the weight of $\tilde{\phi}_{12}^{\succ}(p)$ beyond the spectrum of $\hat{A}$.

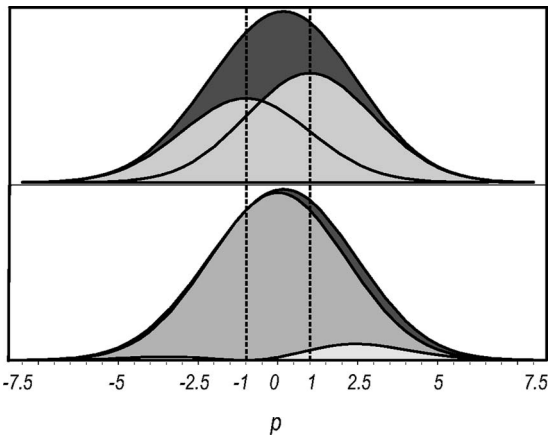

FIG. 1. Decomposition of event distribution (dark shading) for a $\hat{\sigma}_z$ measurement from a PME defined by a Gaussian apparatus state with $\Delta q = \pi$ and an initial eigenstate of $\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{n}_1$ with eigenvalue +1, $\boldsymbol{n}_1 = (\sin\theta, 0, \cos\theta)$, and $\theta = 11\pi/24$. Top, decomposition according to Eq. (2.5); bottom, actual decomposition in terms of event distributions from the two PPME's defined by a post selection measurement of $\hat{\sigma}\cdot\boldsymbol{n}_2$, where $\boldsymbol{n}_2 = (-\sin\theta, 0, \cos\theta)$. The event distribution for the unlikely final outcome $\hat{\sigma}\cdot\boldsymbol{n}_2 = +1$ (lightest shading) is overwhelmingly concentrated beyond the range of the spectrum of σ_z, indicated by dotted vertical lines.

For a wide class of wave functions of the apparatus [1], weak values emerge from the limiting behavior of these interference effects in a limit complementary to that of strong measurement conditions ($\Delta p \to 0$) namely when q, the conjugate to the pointer variable, satisfies $\Delta q \to 0$ (hence $\Delta p \to \infty$). In particular, if $\langle q\rangle_\phi = 0$, one obtains the weak value as the limiting conditional expectation value

$$\lim_{\Delta q\to 0}\langle p\rangle_{\Omega_{12}^{\succ}} \to \mathrm{Re}\frac{\langle\psi_2|\hat{A}|\psi_1\rangle}{\langle\psi_2|\psi_1\rangle}, \tag{2.11}$$

together with the limit $\mathcal{P}_{12}(\phi) \to |\langle\psi_2|\psi_1\rangle|^2$, as if no measurement had taken place. Hence, in this limit the posterior breakup (2.7) of the PME pointer distribution is essentially that of a mixture of distributions, each of which is centered at the weak value defined by its corresponding final state in the postselection and weighted by the corresponding (unperturbed) transition probability. Thus we have an instance in which the expectation value of $\hat{A}$ is more appropriately interpreted operationally as an average of weak values than as an average of eigenvalues. That this is, in fact, possible is guaranteed by a simple sum rule for weak values that is easily verified:

$$\sum_b |\langle\psi_b|\psi_1\rangle|^2 \mathrm{Re}\frac{\langle\psi_b|\hat{A}|\psi_1\rangle}{\langle\psi_b|\psi_1\rangle} = \langle\psi_1|\hat{A}|\psi_1\rangle, \qquad (2.12)$$

stating that the weighted average of the weak values defined by a complete postselection is the standard expectation value of $\hat{A}$ as required by the sum rule (2.8). To ensure that the average falls within the spectral bounds, weak values lying outside the spectrum of $\hat{A}$ must be weighted by correspondingly small probabilities. The rule of thumb, *"eccentric weak values are unlikely,"* therefore captures succinctly the mechanism by which standard expectation values can be viewed as averages of weak values.

In the present discussion, the average of weak values described by Eq. (2.12) is a *classical* average, as it arises from the mixing of weak measurement distributions conditioned on the distinguishable outcomes of the postselection. The QAWV framework presented in Sec. IV extends these results to pre- and postselected statistics away from weak measurement conditions, which can be interpreted in terms of *quantum* averages of weak values. The fact that eccentric weak values are unlikely is key to the consistency of the framework.

III. MECHANICAL INTERPRETATION OF WEAK VALUES

Implicit in the suggestion that standard expectation values can be interpreted (at least under certain conditions) as averages of weak values, is the idea that weak values are in some sense "sharp" physical properties. We shall therefore expand on this notion of "sharpness" by giving an operational sense in which the weak value can indeed be regarded as a definite mechanical property of a system that is known to belong to an ensemble defined by complete pre- and postselections.

The functional dependence on q of the transition amplitude $\langle\psi_2|e^{iq\hat{A}}|\psi_1\rangle$ in (2.6) furnishes the necessary elements to build a description of the PPME statistics based on a picture of "action and reaction," in which, if the variable q is sharply defined, then (a) the measured system is subject to a sharply-defined unitary transformation generated by $\hat{A}$, and (b) the measuring apparatus suffers a sharply defined response given by the weak value of $\hat{A}$. This elementary picture serves the basis for the more general QAWV framework discussed in the following section.

Let us look at the polar decomposition of $\langle\psi_2|e^{iq\hat{A}}|\psi_1\rangle$, which we choose to express as

$$\langle\psi_2|e^{i\hat{A}q}|\psi_1\rangle = \sqrt{\mathcal{P}_{12}(q)}e^{iS_{12}(q)}, \qquad (3.1)$$

where

$$\mathcal{P}_{12}(q) \equiv \left|\langle\psi_2|e^{i\hat{A}q}|\psi_1\rangle\right|^2 \qquad (3.2)$$

(see also Refs. [18,40,41]) gives the transition probability from $|\psi_1\rangle$ to $|\psi_2\rangle$, but mediated by an intermediate unitary transformation $e^{i\hat{A}q}$. Thus, the variable q can be regarded as the parameter of a back reaction on the system, generated by the operator $\hat{A}$, inducing the transformation of the initial state

$$|\psi_1\rangle \overset{q}{\rightarrow} |\psi_1(q)\rangle \equiv e^{i\hat{A}q}|\psi_1\rangle \qquad (3.3)$$

(alternatively, the reaction can be viewed as the inverse transformation $e^{-i\hat{A}q}$ on the final state $|\psi_2\rangle$). On the other hand, the phase factor in (3.1) can be viewed as the generator of a certain reaction of the system on the apparatus corresponding to a specific rotation parametrized by q: viewed as a unitary operator on the apparatus degrees of freedom, $e^{iS_{12}(\hat{q})}$ induces in the Heisenberg picture the generally nonlinear canonical transformation of the pointer operator

$$\hat{p}_> = e^{-iS_{12}(\hat{q})}\hat{p}e^{iS_{12}(\hat{q})}|_< \equiv \hat{p}_< + \mathcal{A}_{12}(\hat{q}), \qquad (3.4)$$

where $\mathcal{A}_{12}(q) \equiv S'_{12}(q) = d/dq\ \mathrm{Im}\ \ln\langle\psi_2|e^{i\hat{A}q}|\psi_1\rangle$. A straightforward derivation then shows that $\mathcal{A}_{12}(q)$ is indeed a weak value

$$\mathcal{A}_{12}(q) = \mathrm{Re}\frac{\langle\psi_2|\hat{A}e^{i\hat{A}q}|\psi_1\rangle}{\langle\psi_2|e^{i\hat{A}q}|\psi_1\rangle} = \mathrm{Re}\frac{\langle\psi_2|\hat{A}|\psi_1(q)\rangle}{\langle\psi_2|\psi_1(q)\rangle}, \qquad (3.5)$$

namely the weak value of $\hat{A}$ for the rotated state $|\psi_1(q)\rangle$ and the final state $|\psi_2\rangle$. Equation (3.4) therefore shows that for a definite value of q, there is an associated definite reaction on the measuring device pointer variable by the weak value for the corresponding pair of states $(|\psi_1(q)\rangle, |\psi_2\rangle)$.

More precisely, note that for the general pointer variable statistics for the PPME Ω_{12}, Eq. (2.10), we can equivalently express the final apparatus wave function $\tilde{\phi}^>_{12}(p)$ as the Fourier integral

$$\tilde{\phi}^>_{12}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dq \sqrt{\frac{\mathcal{P}_{12}(q)}{\mathcal{P}_{12}(\phi)}}\phi(q)e^{-i[pq-S_{12}(q)]}. \qquad (3.6)$$

Let us now suppose that q is constrained to lie exclusively within a finite range around some value $q=q_i$, by taking $\phi(q)$ to be the "window" function of width ε centered at $q=q_i$.

$$W_{q_i,\varepsilon}(q) = \begin{cases} \dfrac{1}{\sqrt{\varepsilon}}, & |q-q_i| < \dfrac{\varepsilon}{2}, \\ 0, & |q-q_i| \geqslant \dfrac{\varepsilon}{2} \end{cases}. \qquad (3.7)$$

In this case the wave function in the p representation is a modulated "sinc" function

$$W_{q_i,\varepsilon}(p) = \sqrt{\frac{2}{\varepsilon\pi}}\frac{\sin\left(\dfrac{\varepsilon p}{2}\right)}{p}e^{-ipq_i}, \qquad (3.8)$$

of characteristic width $\sim 1/\varepsilon$. Now let ε be small enough that variations of $\mathcal{P}_{12}(q)$ and $\mathcal{A}_{12}(q)$ are negligible within the interval $|q-q_i| < \varepsilon/2$. Thus, we can approximate $\mathcal{P}_{12}(\phi) \simeq \mathcal{P}_{12}(q_i)$, and perform the Fourier integral in the "group velocity approximation," i.e., by expanding the phase about q_i to first order and replacing $\mathcal{P}_{12}(q)$ by $\mathcal{P}_{12}(q_i)$; this yields

$$\tilde{\phi}^{>}_{12}(p) \simeq e^{i\Gamma_{12}(q_i)} W_{q_i,\varepsilon}(p - \mathcal{A}_{12}(q_i)), \tag{3.9}$$

where we define $\Gamma_{12}(q) \equiv S_{12}(q) - q\,\mathcal{A}_{12}(q)$. Hence, in the limit $\varepsilon \to 0$, where the apparatus wave function approaches an eigenstate of $\hat{q}$ with eigenvalue q_i, the final wave function for the pointer becomes (up to a phase) the initial wave function *rigidly shifted by a definite weak value*, the weak value $\mathcal{A}_{12}(q_i)$ for the rotated state $|\psi_1(q_i)\rangle$ and the final state $|\psi_2\rangle$.

From the point of view of the system, the limit $\varepsilon \to 0$ can be regarded as an idealization of a situation often encountered in more general contexts, where the evolution of a quantum system is treated as effectively unitary despite the fact that certain parameters of the evolution are actually physical variables of some external (and typically macroscopic) system; for example, a spin rotation, where a macroscopic external magnetic field sets the rotation angle. That such parameters can be treated as classical numbers is a consequence of a negligible uncertainty of the quantum variable of the external system acting as the parameter for the transformation. The interaction with such an external system may thus be idealized as an *infinitesimally uncertain unitary transformation* at a given parameter value. This idealization provides the desired mechanical definition of weak values: The weak value $\mathcal{A}_{12}(q)$ corresponds to a definite conditional reaction of the system on the variable conjugate to the external physical "parameter variable" $\hat{q}$ of an infinitesimally uncertain unitary transformation generated by $\hat{A}$ at parameter value q. The essence of a weak measurement is thus to approach, as close as possible, the ideal conditions of an infinitesimally uncertain transformation.

The above definition presents no ambiguities in the physical interpretation of "eccentric" weak values or in the sometimes unexpected relationships that may arise between the weak values of say, $\hat{A}$ and $\hat{A}^2$ (e.g., negative "weak variances," etc.). To the extent that we associate weak values to infinitesimal unitary transformations, no *a priori* connection between the weak values of two commuting observables should be expected; typically, commuting operators such as $\hat{A}$ and $\hat{A}^2$ generate entirely different types of unitary transformations. Rather, relations between weak values follow from the *linear*, vector space structure of the Lie algebra of Hermitian operators generating infinitesimal transformations. The vector space structure is reflected, for instance, in the fact that for any two initial and final states that are eigenstates of the observables $\hat{A}$ and $\hat{B}$, with eigenvalues a and b, respectively, the reaction to an infinitesimal unitary transformation generated by the linear combination $\hat{C} \equiv \alpha\hat{A} + \beta\hat{B}$ at $q=0$ is the linear combination $\mathcal{C} = \alpha a + \beta b$.

Finally, let us emphasize the significance of the present mechanical interpretation of weak values in connection with certain quantum-mechanical operators, such as kinetic energy or particle numbers [2,26,35], for which any association with negative values would appear to be forbidden. The fact that the reactions associated with weak values will almost always lie within the range of the observable's spectrum is what gives us a reference from which to identify, in those unlikely circumstances where the reaction is "eccentric," what are unique quantum-mechanical effects associated with the role of the observable as a generator of infinitesimal transformations. One would hardly suspect that such effects could indeed be possible given the physical interpretations that we have traditionally attached to the eigenvalues of a quantum-mechanical observable.

IV. QUANTUM AVERAGES OF WEAK VALUES

The framework of QAWV is the extension of the previous analysis to general von-Neumann measurements, with arbitrary pure initial states of the apparatus not necessarily satisfying a "weakness condition." Given a *pure* PPME Ω_{12}, we shall show how the conditional average of measurement outcomes can nevertheless be interpreted as a quantum average of weak values over a suitable distribution. For more general initial and final conditions on the system (as well as more general initial conditions on the apparatus), the corresponding averages can then be obtained by a classical averaging process, similar to that of (2.12), given that any such ensemble can always be broken up into complete pre- and postselected measurement subensembles with appropriate relative weights.

The heuristics of the framework are straightforward: a general apparatus pure state $|\phi\rangle$ entails indefiniteness in the parameter value q driving the back-reaction on the system according to (3.3), so that a generally finite range of system configurations are sampled in the orbit of transformed initial states $|\psi_1(q)\rangle$. Correspondingly, the pointer measurement statistics should reflect the sampling of a certain range of weak values $\mathcal{A}(q)$ associated with this orbit. However, once q is allowed to take arbitrary values, an interesting element in the description comes into play. This has to do with the relative weights associated with the sampled values of q, which reflect a probability reassessment in the light of the additional conditions entailed by the postselection, and which, in particular, will tend to suppress abnormally large weak values. The central idea of the framework is that an arbitrary strength von-Neumann measurement on a pre- and postselected system may be viewed as a superposition of weak measurements at different sampling points q, with a reassessment of the weights of each sample in accordance with Bayes' theorem.

Let us for simplicity consider an initial apparatus function that is real and smooth. This function may then be represented as the limit of a superposition of infinitesimally wide window functions

$$\phi(q) = \lim_{\varepsilon \to 0} \sum_{k=-\infty}^{\infty} \sqrt{\varepsilon}\,\phi(q_k) W_{q_k,\varepsilon}(q), \tag{4.1}$$

centered at the "sampling points" $q_k = k\varepsilon + \delta$ with $k \in \mathbb{Z}$ and $\delta \in [-\varepsilon/2, \varepsilon/2)$. From the results of the previous section, and by linearity, the corresponding final apparatus state wave function may be represented as

$$\tilde{\phi}_{12}^{>}(p) = \lim_{\varepsilon\to 0}\sum_{k=-\infty}^{\infty}\sqrt{\varepsilon\frac{\mathcal{P}_{12}(q_k)}{\mathcal{P}_{12}(\phi)}}\phi(q_k)e^{\Gamma_{12}(q_k)}W_{q_k,\varepsilon}(p-\mathcal{A}_{12}(q_k)). \tag{4.2}$$

The final apparatus state can therefore be viewed as a superposition of weak measurements at the sampling points q_k, but with the initial weights $\phi(q_k)$ replaced by new weights $\sqrt{\mathcal{P}_{12}(q_k)/\mathcal{P}_{12}(\phi)}\phi(q_k)$. In turn, the weights are determined by $\mathcal{P}(q|\Omega_{12})$, the probability distribution function (PDF) for strong measurements of q (performed either before or after the measurement interaction) on the PPME Ω_{12}, which by Bayes' theorem is the posterior distribution for q after a reassessment of the prior PDF $\mathcal{P}(q|\phi)$ by the likelihood $\mathcal{P}_{12}(q)/\mathcal{P}_{12}(\phi)$ of the postselection given the q-dependent rotation of the initial state:

$$\mathcal{P}(q|\Omega_{12}) = \frac{\mathcal{P}_{12}(q)}{\mathcal{P}_{12}(\phi)}\mathcal{P}(q|\phi). \tag{4.3}$$

Note that in accordance with the "eccentric weak values are unlikely" rule of thumb, the likelihood factor $\propto\mathcal{P}_{12}(q)$ will tend to suppress the contributions in the superposition for which the weak value falls outside the spectrum of $\hat{A}$. As we shall illustrate in the coming section, it is this mechanism that ensures, together with quantum-mechanical interference, that the strong measurement distributions peaked at the eigenvalues of $\hat{A}$ can nevertheless be understood as quantum superpositions of weak measurements.

It becomes convenient to capture in compact form the two conceptually different processes involved in the updating of the apparatus state $|\phi\rangle\to|\tilde{\phi}_{12}^{>}\rangle$ as a result of the measurement. The first step, the generally irreversible process of probability reassessment, can be expressed conveniently by defining a fiducial state $|\tilde{\phi}_{12}^{<}\rangle$, which we term the *reassessed initial state* of the apparatus. Defining the state by its wave function in q, it corresponds to the (prior) initial wave function $\phi(q)$ multiplied by the square root of the likelihood factor in (4.3):

$$\tilde{\phi}_{12}^{<}(q) = \sqrt{\frac{\mathcal{P}_{12}(q)}{\mathcal{P}_{12}(\phi)}}\phi(q). \tag{4.4}$$

The other process is the reversible mechanical action of the system on the measurement apparatus generated by the unitary operator $e^{iS_{12}(\hat{q})}$ defined by the polar decomposition (3.1). The final conditional state of the measuring device can then be expressed as a unitary transformation applied to the reassessed state:

$$|\tilde{\phi}_{12}^{>}\rangle = e^{iS_{12}(\hat{q})}|\tilde{\phi}_{12}^{<}\rangle. \tag{4.5}$$

Equivalently, one can compute pointer statistics from the Heisenberg picture transformation (3.4) using the apparatus state $|\tilde{\phi}_{12}^{<}\rangle$. In particular, the conditional PDF of pointer readings can be expressed as a quantum-mechanical analog of a marginal distribution of shifted pointer values,

$$\mathcal{P}(p|\Omega_{12}^{>}) = \langle\tilde{\phi}_{12}^{<}|\delta(p-\hat{p}-\mathcal{A}_{12}(\hat{q}))|\tilde{\phi}_{12}^{<}\rangle, \tag{4.6}$$

in other words, as a *quantum average of weak values*, where the average is taken with respect to the reassessed initial state $|\tilde{\phi}_{12}^{<}\rangle$. As we shall see in Sec. VI, Eq. (4.6) has a natural correspondence in the classical limit; it can be shown to correspond with the marginal posterior PDF for the measurement outcomes of the classical function corresponding to $\hat{A}$ on a classical canonical system specified by initial and final boundary conditions in time.

Using the Heisenberg picture, we finally obtain the pointer reading mean and variance for the PPME Ω_{12}

$$\langle p\rangle_{>} = \langle p\rangle_{<} + \langle\mathcal{A}_{12}\rangle_{<}, \tag{4.7a}$$

$$\langle\Delta p^2\rangle_{>} = \langle\Delta p^2\rangle_{<} + \langle\{\Delta p,\Delta\mathcal{A}_{12}\}\rangle_{<} + \langle\Delta\mathcal{A}_{12}^2\rangle_{<}, \tag{4.7b}$$

where the subscripts $<$ and $>$ stand for expectation values in the state $|\tilde{\phi}_{12}^{<}\rangle$ and $|\tilde{\phi}_{12}^{>}\rangle$, and where the quantum weak value average $\langle\mathcal{A}_{12}\rangle$ and variance $\langle\Delta\mathcal{A}_{12}^2\rangle_{<}$ are directly evaluated using the posterior PDF (4.3). These expressions can be further simplified if the initial apparatus state has a real $\phi(q)$ [42,43], in which case the posterior expectation $\langle p\rangle_{<}$ and the correlation $\langle\{\Delta p,\Delta\mathcal{A}\}\rangle_{<}$ vanish. Then, the first two central moments of (4.6) are indistinguishable from those obtained from classically averaging weak values with a variability defined through the posterior distribution (4.3). Note therefore that a condition for a weak measurement that is more general than the one discussed in the previous section is that we have a sharp *posterior* PDF in q around some value $q=q_*$, in which case the pointer average reflects a measurement of a sharply-defined weak value $\simeq\mathcal{A}_{12}(q_*)$ with a small uncertainty $\langle\Delta\mathcal{A}_{12}^2\rangle_{<}$. Examples of how such effective weak measurements are attained will be given in the next section.

Equation (4.6) provides a statistical characterization of the pointer variable response as a quantum average of weak values, given the most restrictive conditions possible for a pre- and postselected measurement ensemble. Statistics from less restrictive measurement ensembles can then be obtained using standard probability assessments on the PPME Ω_{12} consistent with the specified conditions. In particular, for the preselected measurement ensemble Ω_1 and some specific postselection measurement, the pointer variable distribution (4.6) obeys Eq. (2.7). This correspondence yields a generalization of the sum rule (2.12) to arbitrary measurement strengths, involving both classical and quantum averages

$$\langle\psi_1|\hat{A}|\psi_1\rangle = \sum_b \mathcal{P}_{1b}(\phi)\langle\mathcal{A}_{1b}\rangle_{\Omega_{1b}}, \tag{4.8}$$

and which is easily verified using Eqs. (4.3) and (2.12). Even more generally, since the statistics for any set of less restrictive conditions on the system and/or the apparatus will involve a classical averaging over the states $|\phi\rangle$, $|\psi_1\rangle$, and $|\psi_2\rangle$, the final expectation value of any von-Neumann–type measurement can always be connected to a suitable average of weak values.

V. ILLUSTRATION OF THE QAVW FRAMEWORK

Let us then illustrate how the QAVW framework provides new insight into the measurement statistics of arbitrary-

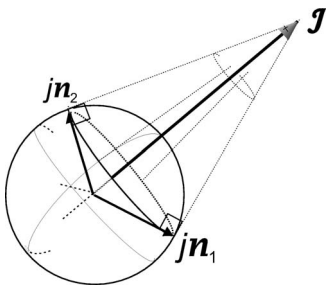

FIG. 2. Geometric construction for the weak spin vector $\mathcal{J}$ for initial and final spin-j coherent states $|\boldsymbol{n}_1;j\rangle$ and $|\boldsymbol{n}_2;j\rangle$.

strength von-Neumann measurements in pre- and postselected ensembles. A particularly graphic example of the interrelationship between the orbit of weak values $\mathcal{A}_{12}(q)$ and the corresponding likelihood function $\propto P_{12}(q)$ is that of spin-component measurements given initial and final spin-j coherent states [44]. Let $\boldsymbol{n}$ be a unit vector with direction parametrized by the polar angles θ and ϕ, and $|j,j\rangle$ the maximal weight J_z eigenstate ($\hat{J}_z|j,j\rangle=j|j,j\rangle$); a spin-$j$ coherent state is then defined as

$$|\boldsymbol{n};j\rangle = e^{-i\hat{J}_z\phi}e^{-i\hat{J}_y\theta}|j,j\rangle, \tag{5.1}$$

and is hence an eigenstate of $\hat{J}_{\boldsymbol{n}}\equiv\hat{\boldsymbol{J}}\cdot\boldsymbol{n}$. Calculations are simplified by the fact that this state can be realized as a product state of $2j$ copies of the spin-$\frac{1}{2}$ coherent state $|\boldsymbol{n};1/2\rangle$. In particular, the transition probability between two coherent states is

$$|\langle\boldsymbol{n}_2;j|\boldsymbol{n}_1;j\rangle|^2 \propto (1+\boldsymbol{n}_2\cdot\boldsymbol{n}_1)^{2j}, \tag{5.2}$$

while the weak value of all spin components are easily captured by a weak spin vector

$$\mathcal{J}\equiv \mathrm{Re}\frac{\langle\boldsymbol{n}_2;j|\hat{\boldsymbol{J}}|\boldsymbol{n}_1;j\rangle}{\langle\boldsymbol{n}_2;j|\boldsymbol{n}_1;j\rangle} = j\frac{\boldsymbol{n}_2+\boldsymbol{n}_1}{1+\boldsymbol{n}_2\cdot\boldsymbol{n}_1}, \tag{5.3}$$

for which the projection onto both $\boldsymbol{n}_2$ and $\boldsymbol{n}_1$ is j (Fig. 2). Note the relation between $|\langle\boldsymbol{n}_2;j|\boldsymbol{n}_1;j\rangle|^2$ and the length $\mathcal{J}$ of the weak spin vector, $|\langle\boldsymbol{n}_2;j|\boldsymbol{n}_1;j\rangle|^2\propto\mathcal{J}^{-4j}$ in consistency with the "eccentric weak values are unlikely" rule.

In a measurement of the spin component $\hat{J}_z$ on a PPME defined by initial and final coherent states $|\boldsymbol{n}_1;j\rangle$ and $|\boldsymbol{n}_2;j\rangle$, the back reaction corresponds to a spin rotation of the initial state about the z axis by the angle $-q$:

$$|\boldsymbol{n}_1;j\rangle\overset{q}{\rightarrow}|\boldsymbol{n}_1(q);j\rangle, \quad \boldsymbol{n}_1(q)=R_z(-q)\boldsymbol{n}_1. \tag{5.4}$$

This reaction in turn entails an orbit for the weak spin vector $\mathcal{J}(q)$ (see Fig. 3), from which the the weak value function $\mathcal{J}_z(q)$ for $\hat{J}_z$ can be obtained by projecting onto the z axis. Note that since the z component of $\boldsymbol{n}_1$ is unaffected by the rotation, $\mathcal{J}_z(q)\propto[1+\boldsymbol{n}_2\cdot\boldsymbol{n}_1(q)]^{-1}$; thus, the likelihood factor satisfies

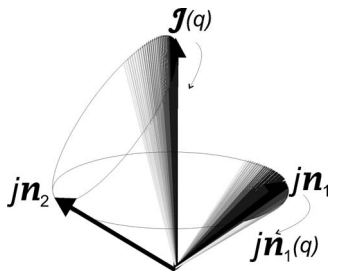

FIG. 3. Orbit of the weak spin vector for a $\hat{J}_z$ measurement, assuming the z axis coincides with the direction of $\mathcal{J}(0)$.

$$\mathcal{P}_{12}(q)\propto\mathcal{J}_z(q)^{-2j}. \tag{5.5}$$

Figure 4 illustrates the correlated behavior of $\mathcal{J}_z(q)$ and $\ln\mathcal{P}_{12}(q)$ for the case $j=20$ and with initial and final spin coherent states with $\boldsymbol{n}_1=(0,1/\sqrt{2},1/\sqrt{2})$ and $\boldsymbol{n}_2=(0,-1/\sqrt{2},1/\sqrt{2})$. For these conditions, the weak value is given by

$$\mathcal{J}_z(q)=j\frac{\sqrt{2}}{1+\sin^2\left(\frac{q}{2}\right)}, \tag{5.6}$$

oscillating between $j\sqrt{2}$ at $q_+=2n\pi$ (full rotations of the initial state), and $j/\sqrt{2}$ at $q_-=(2n+1)\pi$ when $\boldsymbol{n}_1(q)$ coincides with $\boldsymbol{n}_2$. As the figure shows, for $j\gg1$, the likelihood $\mathcal{P}_{12}(q)$ shows essentially an exponential behavior similar to a modular Gaussian distribution; in particular, near values q_+ or q_- (both periodic), for which the magnitude of $\mathcal{J}_z(q)$ is respectively either maximal or minimal on the orbit, we have the approximations for large j

$$\mathcal{P}_{12}(q)\simeq|\mathcal{J}_z(q_\pm)|^{-2j}\exp\left[\pm j\left|\frac{\mathcal{J}_z''(q_\pm)}{\mathcal{J}_z(q_\pm)}\right|(q-q_\pm)^2\right]. \tag{5.7}$$

The exponential suppression of (5.7) near q_+, where the weak value is maximal in magnitude, is generic of the phenomenon of Fourier superoscillations [45–49], exhibited by the amplitude $\langle\boldsymbol{n}_2;j|e^{i\hat{J}_zq}|\boldsymbol{n}_1;j\rangle$ near q_+. This suppression imposes a "robustness" condition on the prior distribution in q if one is to measure eccentric weak values near q_+: not only

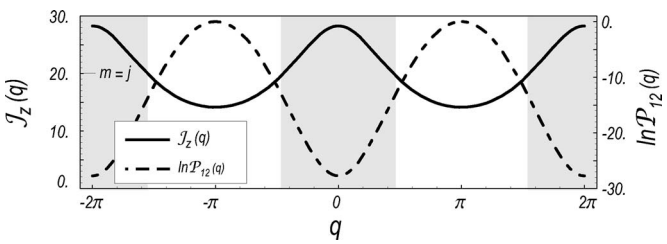

FIG. 4. Weak value $\mathcal{J}_z(q)$ and log-likelihood $\ln\mathcal{P}_{12}(q)$ for $j=20$ and $\hat{J}_z$ measurement with $\boldsymbol{n}_1=(0,1/\sqrt{2},1/\sqrt{2})$ and $\boldsymbol{n}_2=(0,-1/\sqrt{2},1/\sqrt{2})$, as depicted in Fig. 3. Shaded areas represent regions where the weak value lies outside the bounds of the spectrum $[|\mathcal{J}_z(q)|>j]$.

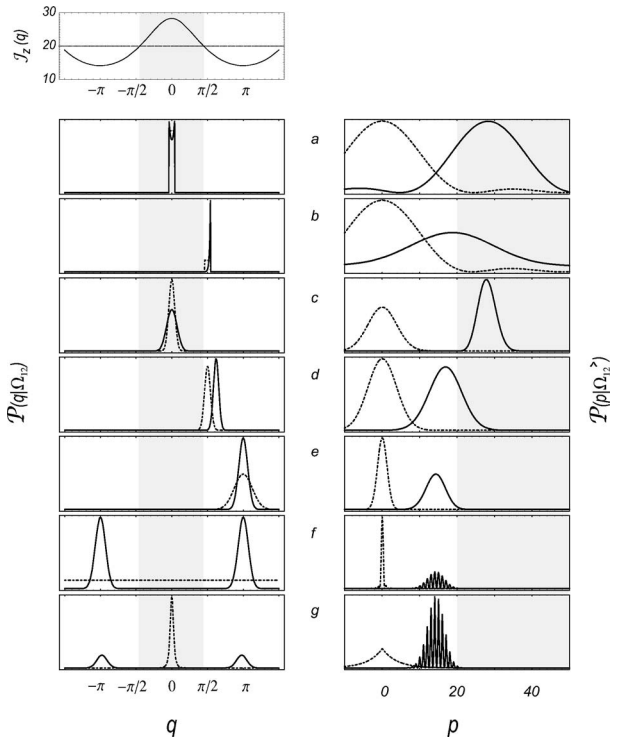

FIG. 5. Interplay between prior distribution in q and the likelihood factor, and resulting consequences on the pointer distribution (see text). (a–g) on the left-hand side show prior (dotted) and posterior (solid) distributions in q, while figures on the right show the corresponding prior (dotted) and final posterior (solid) distributions in p. The top figure shows the curve of the weak value. Shaded regions correspond to regions of "eccentric" effects.

must the prior distribution be "sharp" around $q=q_+$, but additionally it must show a sufficiently fast fall-off to overcome the exponential rise in likelihood.

On the basis of the generic correlated behaviors of the likelihood function and the weak value, the PPME pointer statistics for a relatively wide range of von-Neumann measurement conditions—ranging from weak to strong measurements—can easily be described in the QAWV framework using simple sampling profiles. As discussed in the previous section, a sharp *posterior* distribution $\mathcal{P}(q|\Omega_{12})$ about some well-defined "sampling point" q_* satisfies the conditions for a weak measurement. More generally, however, the reassessment by the likelihood factor of the initial apparatus state $|\phi\rangle$ may yield a state $|\phi_{12}^{\leftarrow}\rangle$ for which the wave function in q shows several well-separated narrow peaks, each satisfying weak measurement conditions. In this case, the PPME pointer distribution will be the result of a coherent superposition of weak measurement pointer wave functions, and will therefore exhibit interference fringes. In simple examples, the existence of just two peaks may be all that is needed to produce the statistical distributions associated with strong measurement conditions (i.e., with maxima at the eigenvalues of the measured observable).

Figures 5(a)–5(g) show how such single or multiple weak measurement conditions are attained from prior distributions in q, of various shapes and locations, for the spin-j PPME setting of Fig. 4 (weak value ranging between $j/\sqrt{2}$ to $\sqrt{2}j$), with real wave functions for the initial state of the apparatus. Starting with Figs. 5(a) and 5(b), we illustrate the likelihood effects on an initial robust state of the apparatus given by a narrow window function in q of the form given by Eq. (3.7), with two different locations q_i. Such profiles guarantee that q, and hence the average weak value, will always lie within a specific interval; thus, the effect of the likelihood factor will primarily be a distortion in the shape of the pointer distribution, with a minimum effect on the expectation value of p.

In Figs. 5(c)–5(e), we show the likelihood effects on robust Gaussian priors of variance σ_i^2 at different locations. Here, the prior sampling region may be significantly altered while still preserving a Gaussian profile with relatively narrow width. For general initial and final states, these effects can be described by performing a Gaussian approximation of the posterior distribution around its maximum q_*, determined by the equation

$$q_* = q_i - 2\sigma_i^2 \,\mathrm{Im}\frac{\langle\psi_2|\hat{A}e^{i\hat{A}q_*}|\psi_1\rangle}{\langle\psi_2|e^{i\hat{A}q_*}|\psi_1\rangle}, \tag{5.8}$$

showing that the imaginary part of the complex weak value can be interpreted as a "bias function" for the posterior sampling point. The resulting pointer distribution will be approximately a Gaussian centered at $p=\mathcal{A}_{12}(q_*)$ with a corrected width determined by the Gaussian approximation. Two interesting effects are then worth noting from these examples: First, as illustrated in Fig. 5(d), if the bias function is large at the prior sampling point q_i, the posterior sampling point q_* may lie in the tail region of the prior distribution. Thus, even if the prior distribution is quite narrow, the sampled weak value $\mathcal{A}_{12}(q_*)$ may differ significantly from the weak value $\mathcal{A}_{12}(q_i)$ at the prior sampling point. The second effect has to do with appreciable alterations of the widths as illustrated in Figs. 5(c) and 5(e) if the prior sampling point q_i is set at a minimum (maximum) of the likelihood function (cases for which $q_*=q_i$ in the Gaussian approximation), the respective posterior distributions in q will be widened (narrowed) with respect to the prior; correspondingly, the pointer distributions may be narrowed (widened) with respect to the prior distribution $\mathcal{P}(p|\phi)$. In particular, it follows that Gaussian measurement conditions probing the most "eccentric" weak value on the orbit will generically show a *squeeze* of the prior pointer distribution—a surprising effect if the statistics are viewed as the result of sampling eigenvalues.

Turning finally to Figs. 5(f) and 5(g), we show the effects on two quite dissimilar nonrobust priors centered at $q=0$: a wide window function of width $\varepsilon=3\pi$, and a narrow Lorentzian of half-width $\Gamma=\pi/24$ [comparable to the prior widths in Figs. 5(a) and 5(c)], both encompassing the maximum likelihood regions around $q=\pm\pi$ with either no suppression or insufficiently slow tail suppression of the likelihood factor. The resulting posterior distributions in q are then both qualitatively very similar and similar in turn to the likelihood factor $\propto\mathcal{P}_{12}(q)$ within the region $q\in[-3\pi/2,3\pi/2]$, which from (5.7) is approximately the sum of two equally shaped narrow Gaussians at $q=\pm\pi$. Thus, conditions are achieved

for the superposition of two weak measurements at $q_*=\pm\pi$, both sampling in this case the least eccentric weak value on the orbit, $\mathcal{J}_z(\pm\pi)=j/\sqrt{2}$. The two peaks in these cases are, in fact, quite similar to the single peak from the Gaussian profile of Fig. 5(e) at $q=\pi$; thus, even while the prior pointer distributions in 5(e) and 5(f) differ substantially in their shapes, the resulting PPME pointer distributions for all three cases share essentially the same envelope, with the last two cases showing interference fringes from the superposition of the two weak measurement sampling points. This interference pattern can then be connected to the spectral distribution expected from a strong measurement: given weak value and likelihood curves symmetric about $q=0$, and a posterior distribution in q with two similarly shaped narrow peaks at locations $q=\pm q_*$, the resulting PPME pointer distribution will be the PPME pointer distibution for the single peak weak measurement at q_*, but modulated by the term

$$\cos^2(pq_* - \delta(q_*)), \quad \delta(q) = \frac{1}{2}\int_{-q}^{q} dq' \mathcal{A}_{12}(q'), \quad (5.9)$$

describing the interference pattern. For the situation depicted in Fig. 5, the phase shift is easily obtained from Eq. (5.6) and is given by

$$\delta(q) = 2j \tan^{-1}\left[\sqrt{2}\tan\left(\frac{q}{2}\right)\right]. \quad (5.10)$$

For $q_*\to\pi$, we have $\delta(q_*)\to 2\pi j$; hence, interference patterns similar to those of Figs 5(f) and 5(g) will show maxima at integer values of p (corresponding to integer values of j), or at half-integer values of p for half-integer j, consistently with spectrum of $\hat{J}_z$.

The foregoing suggests a fairly general picture underlying the transition from weak to strong measurement conditions for fixed initial and final conditions, as the width of the prior distribution in q is varied. Illustrating this passage with a Gaussian prior of variable width σ_i centered at $q=0$ (Fig. 6) for the same $\hat{J}_z$ measurement, we find the onset of a transitional behavior at a critical value of σ_i [Fig. 6(b)] where the Gaussian approximation fails. Beyond this critical value, the exponential rise of the likelihood factor dominates the prior on both sides, thus producing two symmetrically opposed peaks, the locations of which gradually move towards $q=\pm\pi$ as σ_i is increased. This transitional behavior is reflected in the resulting pointer distribution by the emergence of an interference pattern with increasingly closer fringes, modulated by an envelope that gradually shifts with the sampled weak value $\mathcal{A}_{12}(q_*)$ from the eccentric to the normal region of expectation. The pattern eventually settles at the characteristic shape of the strong measurement distribution when the location of the two peaks reaches $q=\pm\pi$, only becoming sharper with increasing σ_i when the tails of the Gaussian prior "activate" the next likelihood peaks at $q=\pm 3\pi$, $\pm 5\pi$, etc.

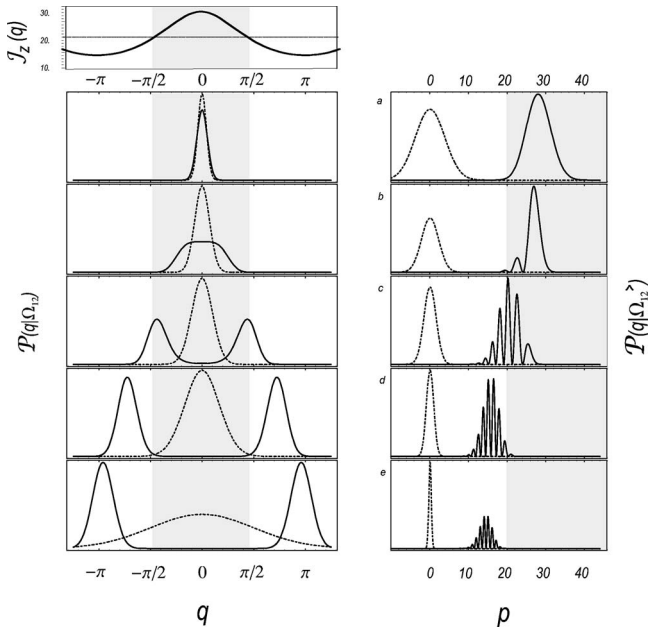

FIG. 6. A weak to strong measurement transition illustrated by priors of increasing width in q around $q=0$. Graphs follow the same convention as those of Fig. 5.

VI. CLASSICAL CORRESPONDENCE OF QAWV FRAMEWORK

The connection between macroscopic "classical" properties and weak values has already been suggested in the literature [1,50,51]. In this section we give further evidence of this connection by showing the correspondence of the QAWV framework in the classical limit. In particular, we show that in the semiclassical limit, the necessary conditions for a precise measurement of a classical dynamical quantity A according to classical mechanics are at the same time the conditions which in the quantum description guarantee a weak measurement of the corresponding observable $\hat{A}$ yielding the same numerical outcome.

Let x be the configuration variable of a classical system, with free dynamics described by the Lagrangian $L_o(\dot{x},x,t)$. For simplicity, we concentrate on a measurement of a function $A(x)$ of the configuration variable x alone, with a measurement Lagrangian of the form

$$L_M(q,x,t) = \delta(t-t_i)A(x)q, \quad (6.1)$$

coupling the system and an external classical apparatus with pointer variable p and canonical conjugate q. To connect with the results of Sec. IV, we interpret the pre- and postselection as the fixing of initial and final boundary conditions on the system trajectory: $x_1\equiv x(t_1)$ and $x_2\equiv x(t_2)$ with $t_2>t_i>t_1$.

Let us assume for simplicity throughout that only one solution is possible for the Euler-Lagrange equations. For nonzero q, the trajectory of the system will differ from its free trajectory due to a modification of the equations of motion by an additional q-dependent impulsive force $F_M=\delta(t-t_i)A'(x)q$ arising from the back reaction of the apparatus on the system. Then, since the actual trajectory will be some

function $x_{12}(t;q)=x(t;x_1,x_2,q)$ of the boundary conditions and q, the quantity $A[x(t_i)]$ will generally depend on q as well. In analogy with our previous notation, define the function

$$\tilde{\mathcal{A}}_{12}(q) \equiv A(x_{12}(t;q)). \tag{6.2}$$

As one can show from the equations of motion, the classical action for the total Lagrangian $L_T=L_o+L_M$ evaluated on the trajectory $x_{12}(t;q)$,

$$\tilde{S}_{12}(q) \equiv \int_{t_1}^{t_2} dt\, L_T[x_{12}(t;q)], \tag{6.3}$$

serves as a generating function for $\tilde{\mathcal{A}}_{12}(q)$, i.e., $\tilde{\mathcal{A}}_{12}(q)=\tilde{S}'_{12}(q)$. Thus, from the equations of motion for the apparatus, we find that the pointer variable suffers the impulse p at the time t_i

$$p_> = p_< + \tilde{\mathcal{A}}_{12}(q) = p_< + \tilde{S}'_{12}(q), \tag{6.4}$$

in direct correspondence with Eq. (3.4).

We now turn to the probabilistic aspects of the measurement. Allowing for uncertainties in the initial state (i.e., the point in phase space) of the apparatus, we describe our knowledge with a prior PDF $\mathcal{P}(qp|I^<)$ for the state of the apparatus before the measurement, where I denotes all available prior information. We also assume that initial conditions on the system are irrelevant for this prior assessment of probabilities so that $\mathcal{P}(qp|Ix_1^<)=\mathcal{P}(qp|I^<)$. Since the variable q enters the equations of motion of the system, knowledge of the final condition x_2 becomes relevant for inferences about q at the time of the measuring interaction, and will, therefore, determine a reassessment of prior probabilities. We must therefore compute the posterior PDF $\mathcal{P}(qp|Ix_1x_2^<)$ for the apparatus, conditioned on the endpoints of the system trajectory, at the time *before* the interaction. The dynamics of the measurement can then be described by the Liouville evolution generated by $\tilde{S}_{12}(q)$, i.e.,

$$\mathcal{P}(qp|Ix_1x_2^>) = \exp\left[-\tilde{\mathcal{A}}_{12}(q)\frac{\partial}{\partial p}\right]\mathcal{P}(qp|Ix_1x_2^<). \tag{6.5}$$

Using Bayes' theorem, we find that

$$\mathcal{P}(qp|Ix_1x_2^<) = \frac{\mathcal{P}(x_2|x_1q)}{\mathcal{P}(x_2|Ix_1)}\mathcal{P}(qp|I^<), \tag{6.6}$$

where we have used the fact that q is the only relevant apparatus variable entering the dynamics of the system, thus yielding a likelihood factor $\mathcal{P}(x_2|x_1q)$ analogous to $P_{12}(q)$ in the quantum case. Finally, evolving to the time after the measurement through Eq. (6.5) and marginalizing, we obtain for the pointer variable distribution after the measurement:

$$\mathcal{P}(p|Ix_1x_2^>) = \langle \delta(p-p'-g\tilde{\mathcal{A}}_{12}(q'))\rangle_< \tag{6.7}$$

where the dummy variables q' and p' are averaged over the reassessed initial phase space PDF for the apparatus $\mathcal{P}(q'p'|Ix_1x_2^<)$. This distribution is in complete analogy with Eq. (4.6) if averages over $\mathcal{P}(qp|Ix_1x_2^<)$ are identified with averages over the reassessed state $|\tilde{\phi}_{12}^<\rangle$ and if $\tilde{\mathcal{A}}_{12}(q)$ is identified with the q-dependent weak value $\mathcal{A}_{12}(q)$. With this identification, Eq. (4.7) for the associated moments can be used for both the classical or quantum descriptions. Furthermore, the terms $\langle p\rangle_<$ and $\langle\{\Delta p,\Delta\mathcal{A}\}\rangle_<$ in (4.7) can also be eliminated in the classical case by requiring that the prior phase space distribution factors as $\mathcal{P}(qp|I^<)=\mathcal{P}(q|I^<)\mathcal{P}(p|I^<)$ with the expectation value of p vanishing over $\mathcal{P}(p|I^<)$.

We can now show that under appropriate semiclassical conditions on a corresponding quantum system, the above analogy is not only formal but rather constitutes a true numerical correspondence between classical and quantum averages. For this, we need to calculate the so-far unspecified likelihood factor $\mathcal{P}(x_2|x_1q)$ in Eq. (6.6), which plays the role of $\mathcal{P}_{12}(q)$ in the state reassessment of Eq. (4.4). In the classical description, the probability of being at x_2 at the time t_2 is proportional to the integral $\int d\pi\,\delta[x_2-x(t_2;x_1,\pi,q,t_1)]$ over all possible initial momenta π of the system, yielding

$$\mathcal{P}(x_2|x_1q) \propto \left|\frac{\partial \pi_1}{\partial x_2}\right|, \tag{6.8}$$

where $\pi_1=\pi(t_1;x_1,x_2,q)$ is the value of the initial momentum as determined from the boundary conditions. This initial momentum can be obtained from a variation of the classical action, $\pi_1=-\partial_{x_1}\tilde{S}_{12}(q)$ [52], so that

$$\mathcal{P}(x_2|x_1q) \propto \left|\frac{\partial^2\tilde{S}_{12}(q)}{\partial x_1\,\partial x_2}\right|, \tag{6.9}$$

(known as Van Vleck determinant [53] from its extension to higher dimensions). Correspondence with the quantum description can now be established by calculating the quantum-mechanical propagator $\langle x_2|\hat{U}(t_2,t_1;q)|x_1\rangle$ for the corresponding quantum system, with $\hat{U}(t_2,t_1;q)$ being the time evolution operator associated with the classical Lagrangian $L_o+L_M(q)$. As is easily verified, this is the relevant amplitude for the von-Neumann measurement of $A(\hat{x})$ at the time t_i with the given boundary conditions. Under appropriate semiclassical conditions [53] (e.g., small times, large masses, slowly varying potentials, etc.), the propagator reduces to the semiclassical or WKB form

$$\langle x_2|\hat{U}(t_2,t_1;q)|x_1\rangle \overset{WKB}{\to} \frac{1}{(2\pi i)^{1/2}}\sqrt{\left|\frac{\partial^2\tilde{S}_{12}(q)}{\partial x_1\,\partial x_2}\right|}\,e^{i\tilde{S}_{12}(q)}, \tag{6.10}$$

where $\tilde{S}_{12}$ is the classical action of Eq. (6.3). Consequently, under semiclassical conditions, the weak value $\mathcal{A}_{12}(q)$ of $A(\hat{x})$ at the time t_i coincides with the classical $\tilde{\mathcal{A}}_{12}(q)$; similarly, the likelihood factor in the reassessment of the initial state of the apparatus [Eq. (4.4)] is the square root of the likelihood factor $\propto|\partial_1\partial_2\tilde{S}_{12}(q)|$ involved in the reassessment probabilities in the classical description.

Thus, assuming the conditions ensuring $\langle p\rangle_{<}=0$, the final posterior mean value of p will be given both in the classical and quantum descriptions by the average value $\langle \mathcal{A}\rangle_{<}$ over the respective posterior distributions in q, which can be made to coincide. This allows us to claim a stronger correspondence between the classical and quantum descriptions when the system satisfies semiclassical conditions: for the same prior distributions in q, the classical and quantum expectation values and variances of A *are numerically equal* and hence, in particular, the final pointer expectation values are equal. It follows that the minimum dispersion conditions on the variable q which in a classical description are required for a precise measurement of A (i.e., $\Delta q\to 0\Rightarrow \Delta A\to 0$), are at the same time the conditions which in the quantum description will guarantee a *weak* measurement of $\hat{A}$ yielding the same numerical value. This correspondence strongly suggests that indeed, what we call macroscopic "classical" properties, are, in fact, weak values.

Let us elaborate on this assertion: The use of classical mechanics to describe macroscopic systems or other quantum systems exhibiting classical behavior relies on the fact that individual measurements may be devised so that (a) the effect on the measurement device accurately reflects the numerical value of the classical observable being measured, (b) no appreciable disturbance is produced on the system as a result of the measurement interaction, and (c) the effect on the measurement device is statistically distinguishable (i.e., the signal to noise ratio is large). The three conditions can be stated as follows: (a) $\Delta\mathcal{A}/\mathcal{A}\ll 1$, (b) $\langle q\rangle=0$, $\Delta q\to 0$, and (c) $\Delta p/\mathcal{A}\gg 1$. In the quantum description, conditions (a) and (b) are weak measurement conditions and can be attained asymptotically by making the posterior uncertainty $\Delta q_{>}$ tend to zero, with the posterior average fixed at $q=0$; however, condition (c) cannot be upheld in the limit $\Delta q\to 0$ since $\Delta p\to\infty$ due to the uncertainty principle. Equivalently, conditions (a) and (b) cannot be fulfilled if condition (c) is to be satisfied by demanding $\Delta p\to 0$ as in the case of an ideally strong measurement.

While it is therefore impossible to satisfy the three conditions either in the absolute strong or weak limits, relatively weak measurement conditions can nevertheless be found as a compromise in the uncertainty relations so that conditions (a), (b), and (c) are simultaneously satisfied for all practical purposes when classical-like physical quantities are involved. Indeed, for such quantities one expects $\mathcal{A}$ to be in a sense "large" relative to atomic scales, or more precisely, to scale extensively with some scale parameter λ growing with the size or "classicality" of the system (such the mass, or the number of atoms). One can then choose a scaling relation for Δq, i.e., $\Delta q\sim\lambda^{-\gamma}$ so that

$$\frac{\Delta p}{\mathcal{A}}\sim\frac{\Delta\mathcal{A}}{\mathcal{A}}\ll 1, \tag{6.11}$$

in which case conditions (a), (b), and (c) can be satisfied in the limit $\lambda\to\infty$. Assuming that $\mathcal{A}'$ scales as $\mathcal{A}$, then with the aid of the uncertainty relation $\Delta p\sim 1/\Delta q$ and $\Delta\mathcal{A}\simeq\mathcal{A}'\Delta q$, we find that this is possible in the quantum description if Δq can be made to scale as

$$\Delta q\sim\lambda^{-1/2}, \tag{6.12}$$

in which case

$$\frac{\Delta p}{\mathcal{A}}\sim\frac{\Delta\mathcal{A}}{\mathcal{A}}\sim\lambda^{-1/2}. \tag{6.13}$$

As was recently shown [54], this is precisely the scaling relation of the optimal compromise for measurements of classical collective properties (such as center of mass position or total momentum) of a large number ($\sim\lambda$) of independent atomic constituents.

VII. CONCLUSION

In this paper we have advanced the claim that weak values of quantum-mechanical observables constitute legitimate physical concepts providing an objective description of the properties of a quantum system known to belong to a completely pre- and postselected ensemble. This we have done by addressing two aspects, namely the physical interpretation of weak values, and their applicability as a physical concept outside the weak measurement context.

Regarding the physical meaning of weak values, we have shown that the weak value corresponds to a definite mechanical response of an ideal measuring probe the effect of which, from the point of the system, can be described as an infinitesimally uncertain unitary transformation. We have stressed how from this operational definition the weak value of an observable $\hat{A}$ is tied to the role of $\hat{A}$ as a generator of infinitesimal unitary transformations. We believe that this sharper operational formulation of weak values in terms of well-defined mechanical effects clarifies the sense in which weak values describe new and surprising features of the quantum domain. Regarding the applicability of the concept of weak values in more general contexts, we have shown that arbitrary-strength von-Neumann measurements can be analyzed in the framework of quantum averages of weak values, in which dispersion in the apparatus variable driving the back-reaction on the system entails a quantum sampling of weak values. The framework has been shown to merge naturally into the classical inferential framework in the semiclassical limit.

It is our hope that the framework introduced in the present paper may serve as a motivation for a refreshed analysis of the measurement process in quantum mechanics.

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